Symmetric four-dimensional polytope and visualization method in four, eight, and sixteen dimensions using Hopf maps

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Inspired by, and using methods of optimization derived from classical three-dimensional electrostatics we have found a four-dimensional polytope, new to our knowledge, with a high degree of symmetry in terms of the lengths of sides—64 of the 80 vertices have twelve nearest neighbors with the same four nearest neighbor distances, and the other 16 vertices have ten nearest neighbors with distances that are two of the four nearest neighbor distances for the set of 64 vertices. We give and illustrate a simple geometric method to visualize this configuration and other configurations in four, eight, and sixteen dimensions.

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Using an optimization method inspired by ones ourselves and others have used (see Refs. [1–12], and references therein) for a problem in three-dimensional electrostatics—Thomson's [13] problem of finding the minimum energy of N unit point charges on the surface of a unit conducting sphere—we have found a configuration with 80 vertices (Fig. 1 and Ref. [14]) with 64 vertices with 12 nearest neighbors of 0.7624, 0.6707, 0.7654, and 0.6661 distances, and 16 vertices with 10 neighbors of 0.7654 and 0.6661 distances. To our knowledge [15,16], this polytope has not been discussed previously. Study of polytopes and optimization methods in higher dimensions can be useful in multidimensional statistical mechanics problems. As well, study of configurations in higher dimensions can provide insight into two- and three-dimensional configurations, as in the case where tessellations of higher dimensional figures has given insight into quasicrystals in lower dimensions, see Ref. [17] and references therein.

We found the configuration looking at the slightly artificial, in four dimensions, but potentially useful (see below), problem of finding the minimum energy configuration of N charges (points) on the surface of the hypersphere (S^3) $x^2+y^2+z^2+w^2=1$ in four dimensions with the energy function $\Sigma_{i\neq j}1/r_{ij}$, where r is the Euclidean distance between two points 1 and 2 $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2+(w_1-w_2)^2}$. As most surely in three dimensions [3], in four or more dimensions the number of good local mimima for this problem grows exponentially with N, and thus we cannot be certain that for $N=80$ or other N that we have found the global minima. Nevertheless, even good local minima may be interesting or important configurations. In this initial work we have found our best local minimum for a given N by starting from 100 random initial starting configurations and then used a standard conjugate gradient optimization. We have looked at $N=2$ to 200. The other N for which we found possible global minima configurations with polytopes with a high degree of symmetry in terms of the lengths of sides are $N=5$, 8, 24, and 120 for which we found the simplex (4D equivalent of the tetrahedron), the 16-cell (the crosspolytope or 4D equivalent of the octahedron), the 24-cell, and the 600-cell (4D equivalent of the icosahedron), respectively, four of the six completely regular Platonic solids in four dimensions. We did not find the other two regular polytopes, i.e., $N=16$ the tesseract (or hypercube, 4D equivalent of the cube) and $N=600$ the 120-cell (4D equivalent of the dodecahedron). Their geometries are not energy minima, similarly to what happens with the cube and the dodecahedron in 3D Thomson's problem (Ref. [4], and references therein). Using a method related to ours other higher dimensional polytopes have been found [18]. The study of higher dimensional polytope by our construction may give physical and intuitive insight into configuration of points in three dimensions.

Visualization of the first Hopf map. Now, using the idea of the Hopf maps (see below) from $S^3 \to S^2$ and $S^7 \to S^4$ we give

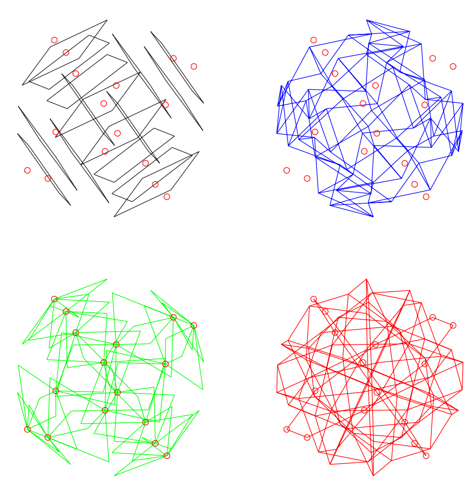

FIG. 1. (Color online) A symmetric four-dimensional polytope with 80 vertices. Four views showing each a different type of nearest neighbor bond (different distance). Dots correspond to a vertex with ten nearest neighbors. These figures show a simple parallel projection from 4D to 2D, i.e., z and w coordinates are discarded, then each point is plotted accordingly to its x and y coordinates. All of the views have been rotated to show a symmetric pattern in 2D.

a new method to visualize configurations in 4, 8, and 16 dimensions and illustrate this with a picture of our novel 80 point configuration as well as the so-called 24-cell configuration in four dimensions, and the E_8 lattice in eight dimensions. These last two configurations are what is known as the kissing configurations in four and eight dimensions, respectively.

The n-dimensional kissing problem asks how many n-dimensional nonoverlapping unit spheres can be placed touching a central unit n-dimensional sphere? This question has applications to making efficient codes and other problems [19]. The two-dimensional sphere is the disk of points $x^2+y^2=1$, and by using N identical circular coins the reader can easily convince oneself that the kissing number in two dimensions (K_2) is six. Similarly, the one-dimensional sphere is a line segment, and $K_1=2$. Already in three dimensions the problem becomes much more interesting. Indeed, the seventeenth century featured a dispute between Isaac Newton who believed that $K_3=12$ and David Gregory who thought that $K_3=13$. Perhaps not surprisingly, Newton was correct but it took more than two and one half centuries to prove this [20]. Using clever linear programming arguments it was proven some decades ago that $K_8=240$ and $K_{24}=196560$ [21,22] and that $K_4=24$ or 25. Recently, a proof that $K_{24}=24$ using much more extensive and subtle use of linear programming has been presented [23], however, the proof has not yet been fully vetted [24]. Though, more recently a complete proof that $K_4=24$ has been posted [25]. There exists a map known as the (first) *Hopf* map between the surface of a three-dimensional sphere and the surface of a four-dimensional sphere, which essentially constructs the surface of a four-dimensional sphere by considering there to be a circle of points at every point on the surface of a three-dimensional sphere. Here we emphasize that the Hopf map also gives a very intuitive way of appreciating the $N=24$ kissing configuration for S^3—known as the 24-cell and then use the second and third Hopf maps to give intuitive descriptions of the E_8/K_8 lattice, and the so called Λ_{16} lattice, currently the best known kissing configuration in 16 dimensions.

The surface of a four-dimensional sphere (a three-dimensional locus or manifold also known as S^3) is defined as the points $x^2+y^2+z^2+w^2=1$ (as the surface of a three-dimensional sphere (S^2) is defined as the points $x^2+y^2+z^2=1$). The Hopf map is one between the points on the surface of a four-dimensional sphere and the pair of complex numbers (w,z) with $|w|^2+|z|^2=1$

$$(w,z) \to (2wz^*,|z|^2-|w|^2) \text{ in } \mathbb{C}\times\mathbb{R}=\mathbb{R}^3. \tag{1}$$

One easily checks that

$$\begin{aligned}|2wz^*|^2+(|z|^2-|w|^2)^2 &= 4|w|^2|z|^2+(|z|^2-|w|^2)^2\\ &= (|z|^2+|w|^2)^2=1.\end{aligned} \tag{2}$$

So this does map to the ordinary sphere. If one fixes a point of the ordinary sphere say (a,t) where a is complex, t is real and $|a|^2+t^2=1$, then its fiber, i.e., the set of all points which map to it, is a circle

$$\left(\frac{ae^{i\theta}}{\sqrt{2(1+t)}},e^{i\theta}\sqrt{(1+t)/2}\right). \tag{3}$$

Further details, discussions and proof of the Hopf map from S^2 to S^3 is given in Ref. [26].

The only known configuration on S^3 with twenty-four kissing spheres is the so-called 24-cell. In this convex four-dimensional polytope all of the faces are octahedra. The standard way of representing the 24-cell is by the coordinates

$$\left(\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2},0,0\right),\ \left(\pm\frac{\sqrt{2}}{2},0,\pm\frac{\sqrt{2}}{2},0\right),\ \left(\pm\frac{\sqrt{2}}{2},0,0,\pm\frac{\sqrt{2}}{2}\right),$$

$$\left(0,\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2},0\right),\ \left(0,\pm\frac{\sqrt{2}}{2},0,\pm\frac{\sqrt{2}}{2}\right),\ \left(0,0,\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2}\right).$$

This configuration is also known as the D_4 lattice. By rotating on S^3 one can also easily see that these same twenty-four centers of the kissing spheres can be obtained by lifting, via the (first) Hopf map from six points on S^2 arranged with one at each pole and four arranged at ninety degree angles around the equator.

Place one point on each pole of a three dimensional sphere and four equally spaced points on the equator, i.e., the vertex of an octahedron. These points are at the antipodal points of the three axes of S^2. These points can be expressed as (a,t), as stated above

$$(0,1),\ (0,-1),\ (1,0),\ (-1,0),\ (i,0),\ (-i,0).$$

Then four points can be placed on each circle via the Hopf map to give a total of twenty four points:

$$(e^{i\theta},0),\ \theta=\frac{\pi}{4}(2k+1),\quad k=0,1,2,3,$$

$$(0,e^{i\theta}),\ \theta=\frac{\pi}{4}(2k+1),\quad k=0,1,2,3,$$

$$\left(\frac{e^{i\theta}}{\sqrt{2}},\frac{e^{i\theta}}{\sqrt{2}}\right),\ \theta=\frac{\pi}{2}k,\quad k=0,1,2,3,$$

$$\left(-\frac{e^{i\theta}}{\sqrt{2}},\frac{e^{i\theta}}{\sqrt{2}}\right),\quad \theta=\frac{\pi}{2}k,\quad k=0,1,2,3.$$

$$\left(i\frac{e^{i\theta}}{\sqrt{2}},\frac{e^{i\theta}}{\sqrt{2}}\right)\equiv\left(\frac{e^{i(\theta+\pi/2)}}{\sqrt{2}},\frac{e^{i\theta}}{\sqrt{2}}\right),\quad \theta=\frac{\pi}{2}k,\quad k=0,1,2,3.$$

It is easy to see that these twenty-four points on S^3 (the surface of a unit four-dimensional sphere) are the same as the points of a 24-cell.

The 24-cell is shown in Fig. 2. The number of different colored polygons indicate how many circles there are for a configuration (for the 24-cell there are 6). The number of sides of the polygons indicate how many points on each circle (for the 24-cell each of the six circles has four points on it). The orientation of the circles with respect to each other as the whole configuration is rotated helps one visualize the relationship of points on different circles.

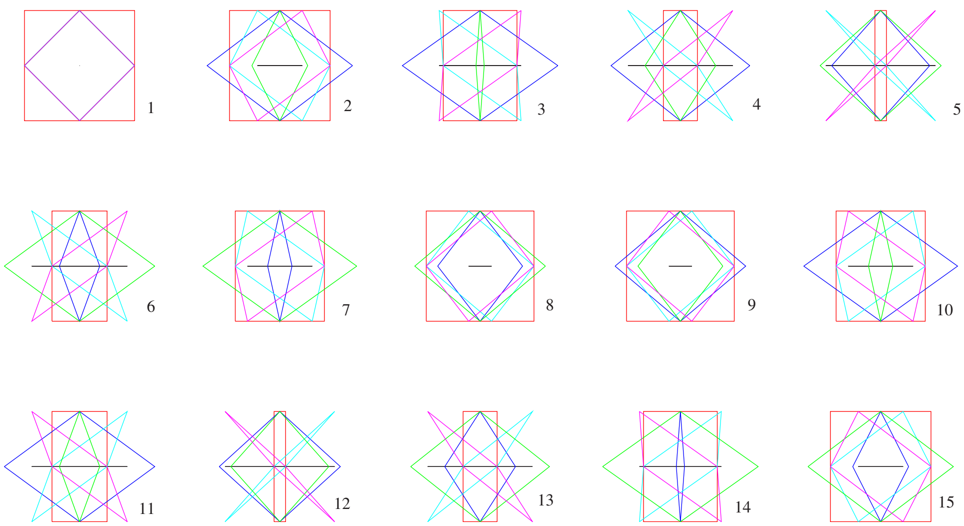

FIG. 2. (Color online) 24-cell. Parallel projection to a 2D plane. Several views during an $i2\pi/15$ rotation in 4D are plotted. Lines join neighbor points on the same circle on S^3, in the online version each circle has a different color. The symmetry of the configuration is evident. The antipodal construction is illustrated as the circles (in S^3) are ninety or one hundred eighty degrees from each other.

Also, the Hopf map gives an extremely simple and somewhat intuitive way to think about the 24-cell: For one point on each equator and four spaced at ninety degrees along the equator then each point in S^2 is a distance of ninety degrees away from its nearest neighbor. Now associate with each of these six points a *Hopf circle* placing four points again spaced ninety degrees. The symmetry of the configuration becomes self-evident.

Points on the same circle are ninety degrees—or distance $\sqrt{2}$ apart from each other. Each point has eight nearest neighbors—two points each from the four nonantipodal circles—that are sixty degrees or distance one apart.

We can also use the Hopf map to nicely illustrate our 80 point symmetric configuration (Fig. 3). We see that the configuration is made up of ten circles each with eight points. This polytope can be built using Hopf's map: Place ten points on the ordinary sphere (S^2) as thus. In (complex) $\times$(real) coordinates pick $(0,1)$=the north pole, $(e^{i\pi m/2}\sqrt{1-t^2},t)$ for $m=0$, 1, 2, 3 and some $0<t<1$, that is four equally spaced points forming the base of a pyramid with vertex the north pole. $(e^{i\pi(2m+1)/4}\sqrt{1-t^2},-t)$ for $m=0,1,2,3$ and the same t, and $(0,-1)$=the south pole. These last five are just the symmetric points in the lower hemisphere after a 45° turn so the two bases of the pyramids are as far from aligned as possible.

This configuration of ten points on S^2 is known as a capped anticube, and also, interestingly, is the minimum energy configuration for a $1/r$ potential of unit point charges on S^2 (Thomson's problem of charges on a sphere for $N=10$, see Ref. [1], and references therein).

Now, suppose you lift these points to S^3. Let $r=\sqrt{(1+t)/2}$. Then we get 10 circles $(0,e^{i\theta})$, $(e^{i\pi m/2}\sqrt{1-r^2}e^{i\theta},re^{i\theta})$ $m=0,1,2,3$, $(e^{i\pi(2m+1)/4}re^{i\theta},\sqrt{1-r^2}e^{i\theta})$ $m=0,1,2,3$, $(e^{i\theta},0)$. Staggering eight points on each in the correct fashion takes a little care, but the correct choice is $(0,e^{2\pi ik/8})$, $(e^{i\pi m/2+i\pi(2k+1)/8}\sqrt{1-r^2},re^{i\pi(2k+1)/8})$, $(e^{i\pi(2m+1)/4+i\pi(2k+1)/8}r,\sqrt{1-r^2}e^{i\pi(2k+1)/8})$, $(e^{i2\pi k/8},0)$, where in each case k runs $0,\ldots,7$. The nearest neighbor distances are $\sqrt{2-\sqrt{2}}$, $\sqrt{2(1-r^2)}$, $\sqrt{2-2r\cos(\pi/8)}$, and $\sqrt{2-(2+\sqrt{2})r\sqrt{1-r^2}}$.

In our 80 vertex polytope in Fig. 3, which is the best known minimum energy configuration for $E=\Sigma_{i>j=1}^{80}1/d$ (where d here is a four dimensional Euclidean distance) $r=0.8423$ and we can see that there will be four nearest neighbor distances in the problem 0.7654, 0.7623, 0.6661, 0.6707. With this Hopf map perspective we can easily see why there are 64 points (points lifted from all circles but the poles) with 12 nearest neighbor distances of 0.7654, distance to two

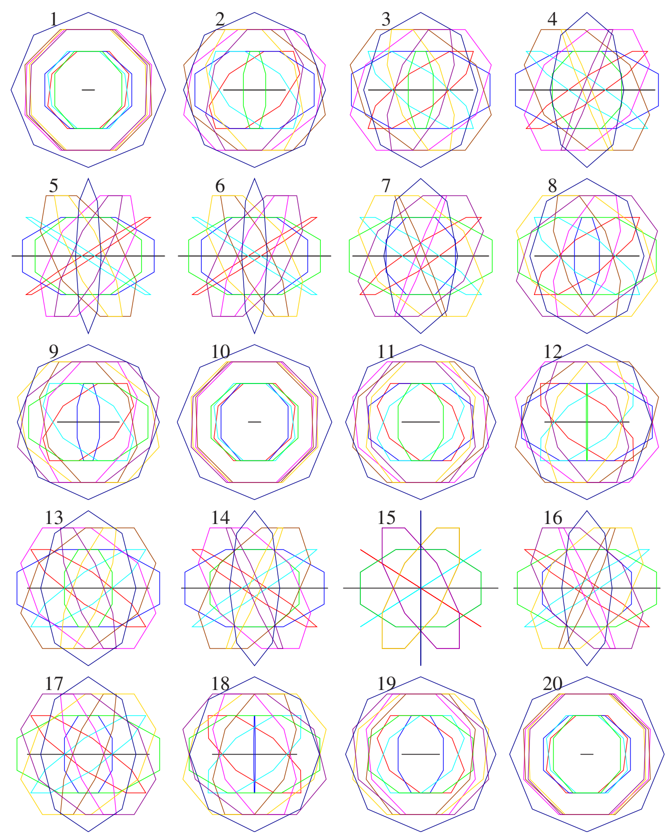

FIG. 3. (Color online) Parallel projection to a 2D plane of several views of the 80 point polytope during a rotation by $i2\pi/20$ around a plane in 4D are plotted. Lines join neighbor points on the same circle on S^3, in the online version each circle has a different color.

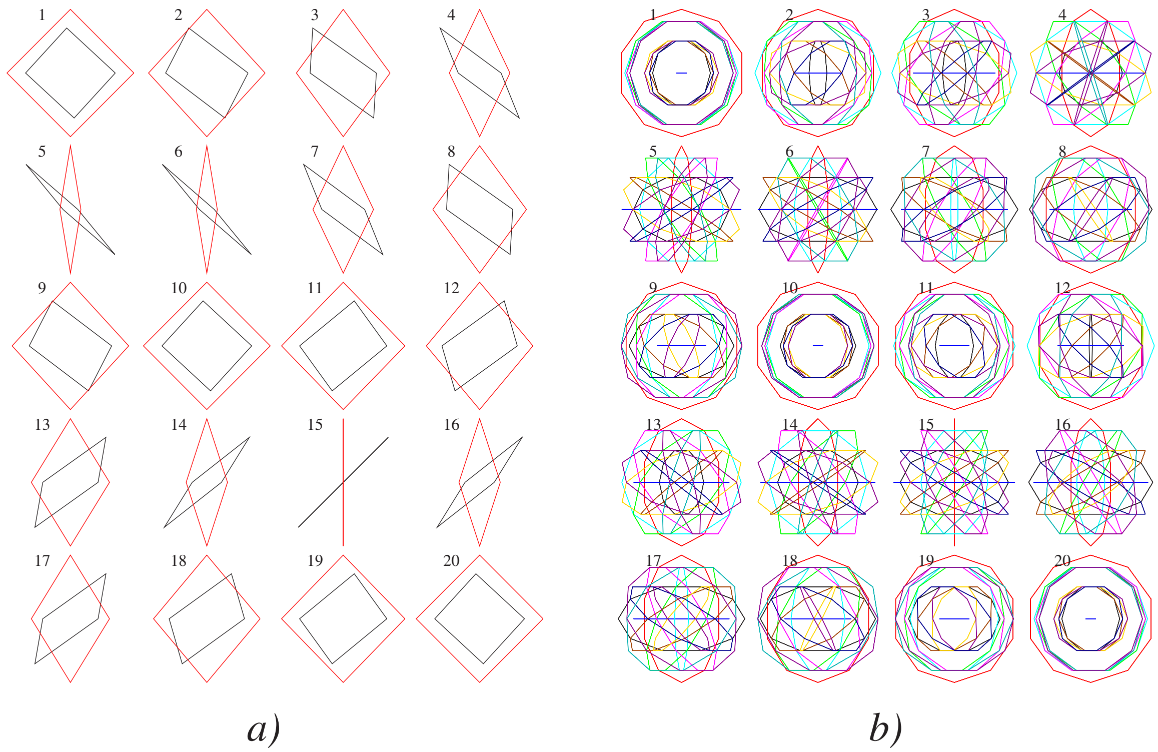

FIG. 4. (Color online) Parallel projection to a 2D plane of several views of the cross polytope (a) and the 600-cell (b) during a rotation by $i2\pi/20$ around a plane in 4D are plotted. Lines join neighbor points on the same circle on S^3, in the online version each circle has a different color.

points on the same circle and to two points on other circles but the poles, 0.7623, distance to two points on other circles of the base of the same pyramid, 0.6661, distance to two points on a pole, 0.6707, distance to four points on circles placed in the base of the other pyramid and similarly 16 points—those lifted from the poles on S^2 with ten nearest neighbors of distances 0.7654, 0.6661, the former being the distance of a point lifted from the north (south) pole to two points on the same circle and the latter the distance to eight points lifted from four points on the base of a pyramid with vertex the north (south) pole.

The question arises as to whether the symmetric 80 point configuration is a global minimum or potentially global minimum energy configuration for other potentials. We find that it is in general not. Indeed, while for a potential $1/r^{1.5}$ at least for 200 runs the analog of the 80 point configuration, adjusting r to minimize the energy is still a global minimum—though this configuration may easily be beaten with more runs—for even $1/r^2$ it is not with the optimized 80 point configuration ($r=0.8428$) having $E=2287.40969$, and a random configuration having energy $E=2287.40506$.

It is not too surprising that analogs of the 80 point symmetric configuration are not global minima for multiple potential functions. In recent studies Cohn, Kumar, and colleagues (Refs. [27,28]), and references therein, find that there are very few universal optimal configurations—configurations which are global minima for all completely monotonic potential functions. In four dimensions the only known ones they find are the simplex, cross polytope [Fig. 4(a)] and the 600-cell [Fig. 4(b)]. None of the other Platonic solids in four dimensions are universal global minima. In particular, by our calculations for $1/r$ and calculations of Cohn and colleagues for other potentials of the form $1/r^n$ [29,27], the 24-cell appears to be the global minimum for $N=24$ points. However, for some exotic potentials the 24-cell is no longer the configuration for 24 points of minimum energy, e.g., the potential function $(1+t)^8$ where t is the cosine angle between the position unit vector. i.e., two points at a distance r have $t=1-r^2/2$. The configuration shown in Fig. 5 [29] is the best known configuration for this potential. This configuration has a Hopf map consisting of eight fibers from S^2 with 3 points per fiber $(e^{i2\pi k/3},0)$, $(0,e^{i2\pi k/3})$, $(e^{i2\pi k/3}\sin\theta, e^{i2\pi k/3}\cos\theta)$, $(e^{i2\pi k/3}\cos\theta, e^{i2\pi k/3}\sin\theta)$, $(e^{i2\pi k/3}\sin\theta, e^{i2\pi(k+1)/3}\cos\theta)$, $(e^{i2\pi k/3}\cos\theta, e^{i2\pi(k+1)/3}\sin\theta)$, $(e^{i2\pi(k+1)/3}\sin\theta, e^{i2\pi k/3}\cos\theta)$, $(e^{i2\pi(k+1)/3}\cos\theta, e^{i2\pi k/3}\sin\theta)$, where $k=0,1,2$ and $\theta=2.5920367\ldots$.

Visualization of the second Hopf map. In addition to the first Hopf map from S^3 to S^2, there is also a second Hopf map from S^7 to S^4 (and a third Hopf map from S^{15} to S^8). Intuitively the second (third) Hopf map uses quaternions (octonions) to accomplish the map. Using the Cayley-Dickson construction for the normed division algebras, $\mathbb{A}_0=\mathbb{R}$ (real), $\mathbb{A}_1=\mathbb{C}$ (complex), $\mathbb{A}_2=\mathbb{H}$ (quaternions), and $\mathbb{A}_3=\mathbb{O}$ (octonions), we can make $\mathbb{A}_n$ from $\mathbb{A}_{n-1}$ for $n=1,2,3$. In this construction, an element in $\mathbb{A}_n$ is made of a pair of elements $a,b\in\mathbb{A}_{n-1}$ with the multiplication

$$(a,b)(c,d) = (ac - db^*, a^*d + cb) \tag{4}$$

and the conjugation in $\mathbb{A}_n$

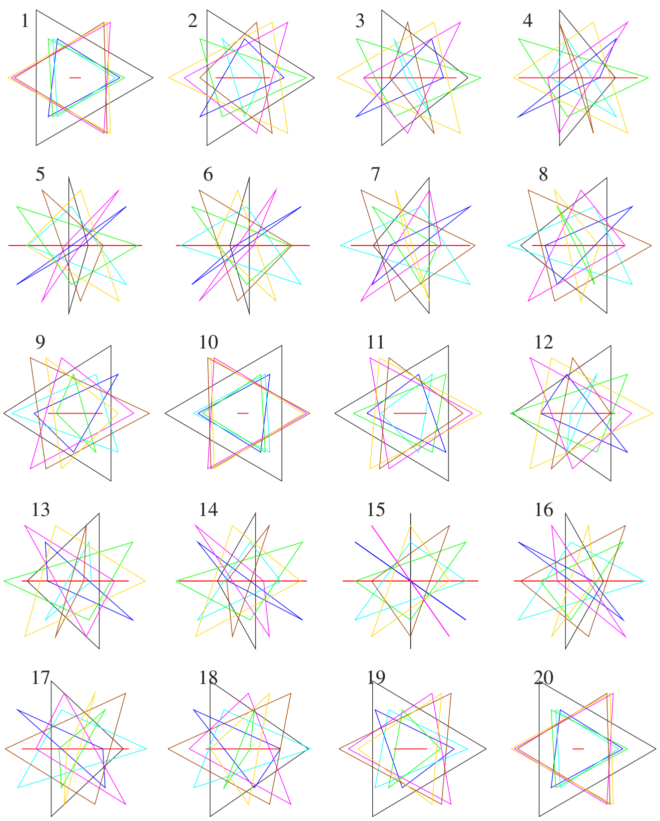

FIG. 5. (Color online) Parallel projection to a 2D plane of several views of the 24 point polytope [29] that has a lower energy than the 24-cell (Fig. 2) for some potential functions during a rotation by $i2\pi/20$ around a plane in 4D are plotted. Lines join neighbor points on the same circle on S^3, in the online version each circle has a different color.

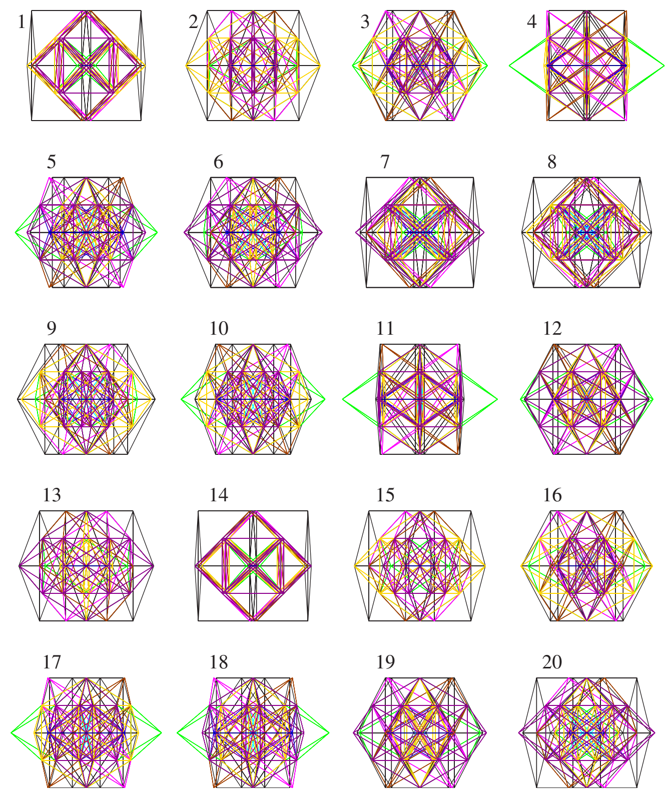

FIG. 6. (Color online) E_8. Parallel projection to a 2D plane of several views during a rotation by $i2\pi/20$ around a plane in 8D are plotted. Lines join neighbor points on the same circle on S^7, in the online version each circle has a differente color. Again we see the symmetry inherent in this configuration, and the antipodal construction being highlighted as the circles (S^7) are ninety or 180° apart.

$$(a,b)^* = (a^*, -b). \tag{5}$$

It is also possible to continue this procedure to get $\mathbb{A}_4$ (sedenions) but it is no longer a division algebra. Hopf maps can be compactly defined as a map h_1 from $\mathbb{A}_n \otimes \mathbb{A}_n$ to $\mathbb{A}_n \cup \{\infty\}$ followed by a second map h_2 from $\mathbb{A}_n \cup \{\infty\}$ to S^{2^n} (stereographic projection), for $n=0,1,2,3$ [31]. h_1 and h_2 can be stated as

$$h_1:(a,b) \to c = ab^{-1}, \tag{6}$$

where $a,b \in \mathbb{A}_n$ and $|a|^2+|b|^2=1$, so the point (a,b) is actually a point of S^m, being $m=2^{n+1}-1$:

$$h_2:c \to X_i(i=1,\ldots,2^n+1), \quad \sum_{i=1}^{2^n+1} X_i^2 = 1. \tag{7}$$

Now, Dixon (see Ref. [30], and references therein) seems to have been the first to have appreciated not only that the first Hopf map can be used to generate the 24-cell from points on S^2, but that E_8 is generated from ten 24-cells lifted from S^4, and Λ_{16} is generated from 18 E_8 lattices lifted from S^8. Dixon's discussion was mostly topological and algebraic. Here we give a remarkably simple and intuitive geometric construction for E_8 and Λ_{16}. We note (see Ref. [31], and references, therein) that such Hopf maps that make constructions may have utility in quantum computers and communication. After initially posting some results as a preprint [32] we greatly appreciate correspondence from Henry Cohn who upon going through Dixon's somewhat opaque notation pointed out that he did in fact give the same constructions for E_8 and Λ_{16} that we do. Dixon gives these with a topological flavor. By taking a geometric viewpoint and approach we are able to give easy enumeration and understanding of other features of the lattices such as numbers of nearest neighbors. Also, we give figures of the configurations to allow one to "picture" or "put a face" on these beautiful structures. S^7 is the surface of an eight-dimensional sphere. The kissing configuration of eight-dimensional spheres on the surface of an eight-dimensional sphere as mentioned is known to be 240 points arranged in the E_8 lattice. The kissing configuration in five dimensions (K_5) which is points arranged on the surface of S^4 is thought to be 40 points arranged in an D_5 lattice, but there is no proof of this. Initially we wondered if we could use the second Hopf map to lift from the forty kissing points on D_5 six points each onto S^7 and obtain the E_8 lattice/K_8 configuration. We have not been able to do this, however, we noticed that by taking again the ten antipodal points from the axes on S^4, $(\pm1,0,0,0,0)$, $(0,\pm1,0,0,0)$, $(0,0,\pm1,0,0)$, $(0,0,0\pm1,0)$, and $(0,0,0,0,\pm1)$, and lifting ten 24 cells to S^7 we get the E_8 lattice. This construction is illustrated in Fig. 6. Again, our construction immediately illustrates that as for the 24-cell points are separated by 60°, 90°, 120°, or 180°. Our construction also intuitive explains why each point has 56 nearest neighbors: eight on its own S^3 circle and six (two per orthogonal axis of a three-dimensional object) on each of the eight other nonantipodal circles on S^4.

Similarly, by lifting from the 16 antipodal points of the axes of S^8 an E_8 lattice one gets the Λ_{16} lattice. This construction again illustrates why Λ_{16} is a kissing configuration with points separated by angles 60°, 90°, 120°, or 180°, and why each point on Λ_{16} has 280 nearest neighbors=56 on *its own* S^7 circle+14 (two points per orthogonal axis of a seven-dimensional object) $\times$ 16, where 16 is the number of nonantipodal S^7 circles on S^8. This construction suggests, but of course in no way proves, that Λ_{16} may be a configuration of maximum kissing number. Our construction may give a helpful picture in proving, or disproving this, or in attacking other problems such as finding the maximal packing configurations in 4, 8, or 16 dimensions. Study of higher-dimensional polytope by our construction may give physical and intuitive insight into configuration of points in three dimensions. Geometric and intuitive insight into higher dimensional polytopes may be useful in studying multidimensional statistical mechanics problems.

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