Wave chaos in rotating optical cavities

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We show that, even when the eigenmodes of an optical cavity are wave chaotic, degenerate eigenfrequencies of those modes split into different frequencies due to the rotation of the cavity. The frequency difference is proportional to the angular velocity, although the splitting eigenmodes are still wave chaotic and do not correspond to any unidirectionally rotating waves.

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In 1913 Sagnac pointed out that the path length of clockwise (CW) propagating light in a rotating ring interferometer for one round trip is different from that of counterclockwise (CCW) propagating light; hence the Sagnac interferometer can be used as a rotation rate sensor [1,2].

The conventional theory of ring laser gyroscopes is based on Sagnac's original idea, that is, the difference of the path lengths between counterpropagating light is derived from the special theory of relativity for the ray-dynamical description [1,2]. A more precise electromagnetic wave description of the Maxwell equations, which includes the effect of rotation, is given by the general theory of relativity [1–3]. These two different conventional methods of describing the effect of rotation on the lasing frequency have given the same result because of the following two assumptions. (i) The ray-wave correspondence of the cavity modes: the conventional theory assumes that there exist closed ring trajectories and the cavity modes localize on these ring trajectories in a ring resonator. (ii) The equivalence of the descriptions for the cavity modes by standing waves and propagating waves: the conventional theory assumes that the size of the ring resonator is much larger than the wavelength of the laser, and in this short-wavelength limit the cavity modes are separated into two directions propagating along and transverse to the ring trajectory.

Assumption (ii) means that the cavity modes can always be expressed as unidirectionally rotating waves and that the counterpropagating modes are degenerate. Precisely speaking, however, the cavity modes of nonrotating cavities are standing waves as long as the cavity shape does not have a special symmetry like a circle.

Studies on quantum chaos over the last three decades have shown that there exist so-called wave chaotic cavity modes that do not localize on any ray-dynamical trajectories due to the chaotic property of the ray dynamics in the cavity [4–6]. The ray-dynamical description for ring laser gyroscopes is not applicable for these wave chaotic cavity modes although the wave-dynamical description is still applicable. In this paper, the recently established theory [7,8] of rotating resonant microcavities without assumptions (i) and (ii) is applied to a cavity that exhibits the ray-dynamical properties of the mixed system. It is well known that some eigenfunctions in the mixed system localize on stable periodic trajectories while others are wave chaotic and spread over the chaotic sea in phase space. We show that the degenerate eigenfrequency corresponding to the wave chaotic cavity mode of the nonrotating cavity splits into two frequencies. Their difference is proportional to the rotation rate, although the splitting cavity modes are still wave chaotic and do not have any corresponding CW and CCW propagating modes as well as ray-dynamical counterparts. This result cannot be explained by the conventional Sagnac effect.

First let us introduce the ray-dynamical properties and the wave functions of the eigenmodes when the cavity is not rotating. We will discuss the two-dimensional shape of the resonant microcavity defined by $R(\theta)=R_0(1+\epsilon\cos 4\theta)$ in cylindrical coordinates. The parameters are set as follows: $R_0=6.2866, \epsilon=0.04$, and the refractive index $n=1$.

Ray-dynamical properties can be easily seen by plotting the trajectories on the Poincaré surface of section (SOS). It is convenient to use the Birkhoff coordinates (η, s) for the SOS, where η is the normalized curvilinear distance measured along the edge of the cavity from a certain origin on the edge to the incident point, and s is the sine of the incident angle. The SOS of ray dynamics in the cavity, shown in Fig. 1, includes stable islands and a chaotic sea, indicating that it is a typical mixed system [9]. The stable islands around $s=1/\sqrt{2}$ and $-1/\sqrt{2}$ correspond to CCW and CW rotational ray motions along a ring trajectory, respectively, like the ray motion shown in Fig. 1(a) and 1(b). A typical example of a single trajectory in the chaotic sea is provided in Fig. 1(c).

The wave properties are discussed only in the case of the TM mode of the electromagnetic field oscillating as $\boldsymbol{E}(\boldsymbol{r},t)=(0,0,\psi(\boldsymbol{r})e^{-ickt}+\text{c.c.})$, where c is the velocity of light and the eigenfrequency ω equals ck. The Dirichlet boundary condition is imposed for simplicity. By solving the Helmholtz equation $(\nabla_{xy}^2+n^2k^2)\psi=0$, where n is the refractive index inside the cavity, we obtain the eigenmodes respectively cor-

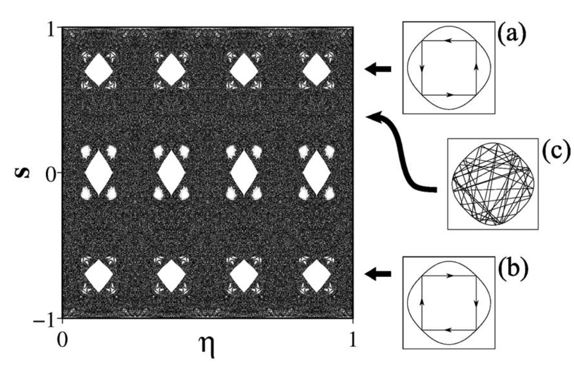

FIG. 1. Poincaré surface of section in Birkhoff coordinates.

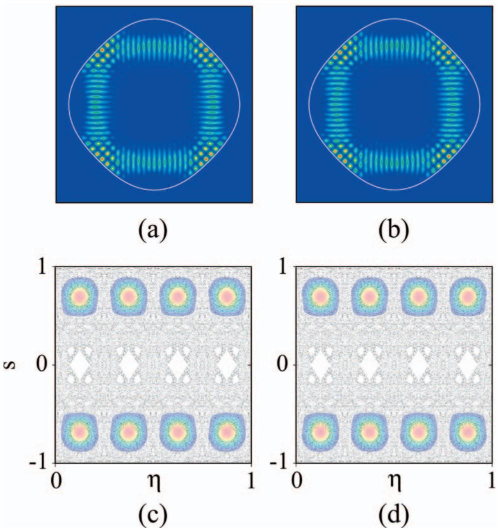

FIG. 2. (Color) (a), (b) Eigenmodes corresponding to the stable islands of the ring trajectory in a nonrotating cavity. Each eigenmode labeled by (a) and (b) has odd (even) parity and even (odd) parity with respect to the x (y) axis, and they are obtained at $nkR_0=50.264\,118$. (c), (d) Husimi representations corresponding to eigenmodes (a) and (b).

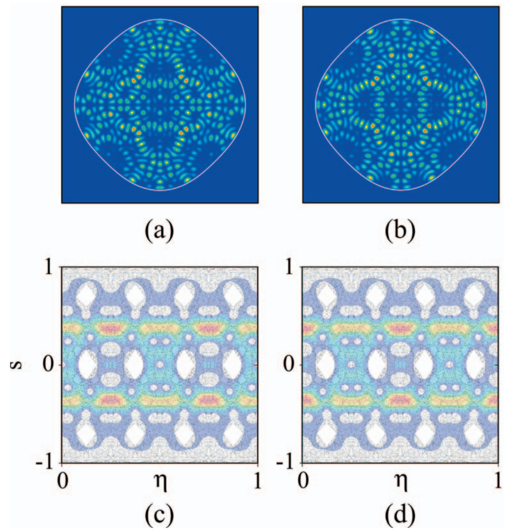

FIG. 3. (Color) (a), (b) Wave-chaotic eigenmodes in a nonrotating cavity. Each eigenmode labeled by (a) and (b) has odd (even) parity and even (odd) parity with respect to the x (y) axis, and they are obtained at $nkR_0=50.248\,944$, which is almost the same value as that of the modes shown in Fig. 2. (c), (d) Husimi representations corresponding to eigenmodes (a) and (b).

responding to the stable islands and the chaotic sea as shown in Figs. 2(a) and 3(a). The eigenmode in the Husimi representation [10] of Fig. 2(c) localizes on both of the stable islands of the ring trajectories at $s=\pm 1/\sqrt{2}$, which means that the eigenmode in the nonrotating cavity has both CW and CCW propagating wave components and is a standing wave along the ring trajectory. On the other hand, Fig. 3(c) shows that the eigenmode in Fig. 3(a) does not localize on the stable islands but is distributed over the chaotic sea. It is well known that no simple ray-wave correspondence exists for such an eigenmode.

Those two modes belong to the same symmetry class that is, respectively, even and odd with respect to the y and x axes because the cavity has C_{4v} symmetry. When these eigenmodes are rotated by $\pi/2$, one can obtain other eigenmodes that are, respectively, even and odd with respect to the x and y axes. Since these modes created by $\pi/2$ rotation are orthogonal to the original ones, the pairs of degenerate eigenmodes in this cavity are obtained as shown in Figs. 2(b) and 3(b). The Husimi representations of the eigenmodes shown in Figs. 2(b) and 3(b) are shown in Figs. 2(d) and 3(d), respectively.

Next we discuss the effect of rotation of the cavity on the above eigenmodes. A theory of the effect of the rotating resonant microcavity on resonances has recently been obtained [7,8]. This theory is applicable to wave-chaotic eigenmodes where the electric fields spread over the entire phase space. Let us briefly review this theory for the case of degenerate eigenmodes.

According to the general theory of relativity, the electromagnetic fields in a rotating resonant microcavity are subject to the Maxwell equations generalized to a noninertial frame of reference in uniform rotation with the angular velocity vector $\boldsymbol{\Omega}$ [1–3]. By neglecting $O(|\boldsymbol{\Omega}|^2)$ and assuming that the two-dimensional (2D) resonant microcavity is perpendicular to the angular velocity vector $\boldsymbol{\Omega}$, we obtain the following equation for the eigenmodes of the rotating cavity:

$$(\nabla_{xy}^2 + n^2k^2)\psi - 2ik(\boldsymbol{h}\cdot\boldsymbol{\nabla})\psi = 0, \tag{1}$$

where $\boldsymbol{h}=(\boldsymbol{r}\times\boldsymbol{\Omega})/c$ and the 2D resonant cavity is rotating clockwise in the xy plane, i.e., $\boldsymbol{\Omega}=[0,0,\Omega(>0)]$.

By applying the perturbation theory for degenerate states in quantum mechanics, the degenerate eigenmodes ψ_0 and ψ_1 of the nonrotating cavity are superposed to reproduce the solutions of Eq. (1) as

$$\psi_\pm = 1/\sqrt{2}\psi_0 \pm i/\sqrt{2}\psi_1, \tag{2}$$

where the degenerate wave number k_0 splits into two up to the first order of $|\Omega/c|$,

$$k_\pm = k_0 \pm \frac{1}{n^2}\left|\int\int_D d\boldsymbol{r}\ \psi_0(\boldsymbol{h}\cdot\boldsymbol{\nabla})\psi_1\right|. \tag{3}$$

Using the relation

$$\left|\int\int_D d\boldsymbol{r}\ \psi_0(\boldsymbol{h}\cdot\boldsymbol{\nabla})\psi_1\right| = \left|\int\int_D d\boldsymbol{r}\ \psi_\pm i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\psi_\pm\right|\frac{\Omega}{n^2}, \tag{4}$$

we can obtain the frequency difference $\Delta\omega\ [\equiv c(k_+ - k_-)]$ proportional to the angular velocity,

$$\Delta\omega = 2|\langle m\rangle|\frac{\Omega}{n^2}, \tag{5}$$

where $\langle m\rangle$ is the average angular momentum of the superposing eigenmodes (2),

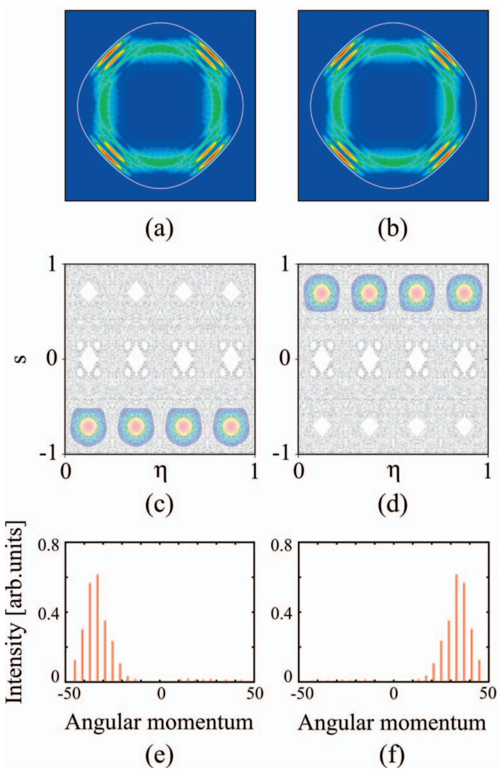

FIG. 4. (Color) (a), (b) Eigenmodes corresponding to the stable islands of the ring trajectory in a rotating cavity with $R_0\Omega/c=6.28\times10^{-5}$. (c), (d) Husimi representations corresponding to eigenmodes (a) and (b). (e), (f) Angular momentum spectra of eigenmodes (a) and (b).

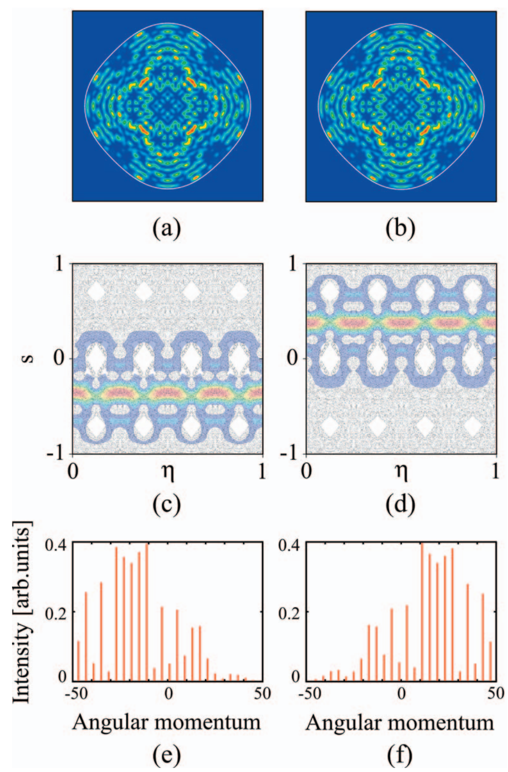

FIG. 5. (Color) (a), (b) Wave-chaotic eigenmodes in a rotating cavity with $R_0\Omega/c=6.28\times10^{-5}$. (c), (d) Husimi representations corresponding to eigenmodes (a) and (b). (e), (f) Angular momentum spectra of eigenmodes (a) and (b).

$$\langle m\rangle = i\int\int_D d\boldsymbol{r}\,\psi_{\mp}\left(y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}\right)\psi_{\pm}. \tag{6}$$

Equation (5) is a more generalized expression than the conventional formula of the Sagnac effect shown in Refs. [1,2].

Applying this theory to the degenerate eigenmodes where the electric fields localize on the ring trajectories as shown in Figs. 2(a) and 2(b) yields almost the same results as the conventional theory for ring laser gyroscopes, because the superposition (2) of these degenerate eigenmodes of the nonrotating cavity produces split eigenmodes which propagate almost CW and CCW along the ring trajectories in the rotating cavity. The eigenmodes become completely CW and CCW propagating modes in the short-wavelength limit because it has been demonstrated that the wave functions constructed by Gaussian beams unidirectionally propagating along the ring trajectory are the eigenmodes of the cavity in the short-wavelength limit [11].

However, when we apply this theory to degenerate eigenmodes where the electric fields are wave chaotic as shown in Figs. 3(a) and 3(b), we cannot expect the superposed solutions (2) to be CW and CCW propagating modes. The cavity mode ψ_0 in Fig. 3(a) can be expressed by superposition of the Bessel functions as

$$\psi_0=\sum_{m=0}^{\infty}a_{4m+1}J_{4m+1}(nk_0r)\sin(4m+1)\theta + \sum_{m=0}^{\infty}a_{4m+3}J_{4m+3}(nk_0r)\sin(4m+3)\theta, \tag{7}$$

because the wave function ψ_0 is odd (even) with respect to the $x(y)$ axis. Then, the expression for the cavity mode ψ_1 in Fig. 3(b) can be obtained by $\pi/2$ rotation of ψ_0,

$$\psi_1=\sum_{m=0}^{\infty}-a_{4m+1}J_{4m+1}(nk_0r)\cos(4m+1)\theta + \sum_{m=0}^{\infty}a_{4m+3}J_{4m+3}(nk_0r)\cos(4m+3)\theta, \tag{8}$$

which has the same eigenfrequency as that of ψ_0 and is even (odd) with respect to the x (y) axis. Accordingly, we have alternative expressions of the degenerate eigenmodes (2),

$$\psi_{\pm}=\mp\frac{i}{\sqrt{2}}\sum_{m=0}^{\infty}a_{4m+1}J_{4m+1}(nk_0r)\exp[\pm i(4m+1)\theta] \pm\frac{i}{\sqrt{2}}\sum_{m=0}^{\infty}a_{4m+3}J_{4m+3}(nk_0r)\exp[\mp i(4m+3)\theta]. \tag{9}$$

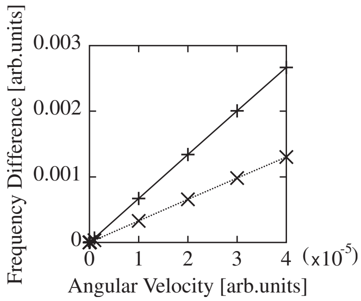

FIG. 6. Frequency difference $\Delta\omega$ between the eigenfrequencies of modes shown in Fig. 4 versus angular velocity (Ω/c) (solid line). Frequency difference between those of modes in Fig. 5 versus angular velocity (dashed line).

It is important that $\psi_{\pm}$ should be the eigenmode of the rotating cavity, but it contains both CW and CCW propagating waves. Consequently, it is impossible to discuss the difference in path lengths between CW and CCW propagating light. Nevertheless, according to Eq. (5), one can obtain the frequency splitting $\Delta\omega$ proportional to the angular velocity even on chaotic eigenmodes. Even in the short-wavelength limit, the frequency splitting of the chaotic eigenmodes would occur; the superposed eigenfunctions $\psi_{\pm}$ do not have symmetric distributions of the angular momentum components because $\psi_{\pm}^{*} \neq \psi_{\pm}$ according to Eq. (9) and the average momentum $\langle m \rangle$ does not become zero. Therefore, we conclude that the frequency splitting is a more general effect of rotation of the resonant cavity on its eigenmodes. Only in the special case that the eigenmodes localize on the ray-dynamical ring trajectories can this frequency splitting be related to the Sagnac effect, which is the difference in path lengths between the CW and CCW propagating modes.

To confirm the above conclusion from perturbation theory, we actually solved Eq. (1) numerically in a rotating cavity with angular velocity Ω. The wave functions ψ can be expanded on the basis of the Bessel functions J_m as $\psi = \Sigma_{m=-\infty}^{\infty} c_m J_m(nk_m r) e^{im\theta}$, where $k_m^2 = n^2k^2 - 2k(\Omega/c)m$ (see Ref. [7]). The expanded coefficients c_m and the eigen wave number k are obtained by substituting ψ in Eq. (1) and solving it under Dirichlet boundary conditions. We obtained the eigenmodes in the rotating cavity as shown in Figs. 4 and 5.

Figures 4(a) and 4(b) display the wave functions of the eigenmodes corresponding to the stable island of the ring trajectory in a rotating cavity. In Figs. 4(c) and 4(d), we see the Husimi representations of the eigenmodes of Figs. 4(a) and 4(b), and the distribution of the angular momentum components $|c_m|^2$ of the eigenmodes are shown in Figs. 4(e) and 4(f), respectively. These figures show that the eigenmodes become almost unidirectionally propagating waves along the ring trajectory in a rotating cavity, whereas they are standing waves along the ring trajectory in a nonrotating cavity. Thus, the frequency splitting is proportional to the angular velocity, as shown in Fig. 6. The slope of the line agrees with that calculated using the difference in path lengths of the ring trajectory. Accordingly, the conventional theory of the Sagnac effect can be reproduced in this case.

Figures 5(a) and 5(b) show the wave-chaotic eigenmodes, and the Husimi representations corresponding to each eigenmode appear in Figs. 5(c) and 5(d). Unlike the case where the eigenmodes localize on a stable island, the distributions of the wave-chaotic modes in the Husimi representations clearly show that the eigenmodes do not become unidirectionally propagating modes corresponding to stable islands. Moreover, the angular momentum spectra in Figs. 5(e) and 5(f) show that the wave-chaotic eigenmodes have both CW $(m<0)$ and CCW $(m>0)$ propagating wave components even when the cavity is rotated. However, the eigenfrequency actually splits into two, and this frequency splitting is proportional to the angular velocity as shown in Fig. 6. According to Eq. (5), the slope of the line is decided by the average angular momentum obtained from the momentum distributions shown in Figs. 5(e) and 5(f). From these results, one can confirm that the frequency splitting due to rotation can be observed even on eigenmodes that do not split into CW and CCW propagating wave modes.

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