Fractal dimension in dissipative chaotic scattering

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The effect of weak dissipation on chaotic scattering is relevant to situations of physical interest. We investigate how the fractal dimension of the set of singularities in a scattering function varies as the system becomes progressively more dissipative. A crossover phenomenon is uncovered where the dimension decreases relatively more rapidly as a dissipation parameter is increased from zero and then exhibits a much slower rate of decrease. We provide a heuristic theory and numerical support from both discrete-time and continuous-time scattering systems to establish the generality of this phenomenon. Our result is expected to be important for physical phenomena such as the advection of inertial particles in open chaotic flows, among others.

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I. INTRODUCTION

The effect of weak dissipation on chaotic Hamiltonian systems has been an interesting topic [1–4]. In a closed Hamiltonian system with a mixed phase space where Kol'mogorov-Arnol'd-Moser (KAM) islands and chaotic seas coexist, a small amount of dissipation can convert the islands into sinks, or attractors. Due to the hierarchical structure of KAM islands in the original Hamiltonian system, the dissipation-induced attractors occur at all scales and their basins of attraction are intermingled in a complicated way. This leads to a multistability and unpredictability of the final state for given initial conditions [1]. In an open Hamiltonian system where the phenomenon of interest is scattering, weak dissipation can also have some consequences [2], particularly for nonhyperbolic chaotic scattering [5,6] where KAM islands and nonattracting chaotic invariant sets coexist. For such a system, it has been known that the particle-decay law is algebraic due to the "stickiness" effect of KAM islands [7], but weak dissipation typically causes particles to decay exponentially from the scattering region [2]. Another consequence is the appearance of multiple attractors and the rising of Wada basin boundaries [4,8,9].

For a scattering system, a quantity of physical interest is scattering functions, which give the dependences of some physical variables after the scattering on some input variables (e.g., the impact parameter) before the scattering. Scattering functions can be experimentally measured, from which information about the interior of the scattering system can be inferred. For a chaotic scattering system, a scattering function typically contains an uncountably infinite number of singularities [10]. Of interest is thus the fractal dimension of the set of singularities. For nonhyperbolic scattering, it has been known that the algebraic decay law leads to unity dimension values [5]. When the decay law becomes exponential, as caused by dissipation, the fractal dimension should assume values less than unity [2]. A question of interest, which to our knowledge has not been addressed in the chaotic-scattering literature, is how the fractal dimension of the set of singularities in a typical scattering function varies as a parameter characterizing the amount of dissipation is increased from zero.

In this paper, we investigate the variation of the fractal dimension in nonhyperbolic chaotic scattering as the underlying system becomes increasingly more dissipative. As a dissipation parameter, say, ν, is systematically increased from zero, the fractal dimension decreases from unity, as expected. A somewhat unexpected finding is a crossover phenomenon, where the rate of decrease of the dimension is relatively large initially, but as the dissipation parameter passes through a critical value ν_c, the rate is reduced significantly and becomes nearly zero for $\nu > \nu_c$. This crossover behavior appears quite general, as has been found numerically for two different model systems of nonhyperbolic chaotic scattering, one described by a map and another by flow. An examination of the phase-space structure reveals that the onset of the crossover behavior is due to the appearance of *physically accessible attractors* in the system, i.e., attractors that can be accessed from initial conditions of finite precision far away from the scattering region. Since the phase-space resolution is finite in any realistic physical situation, we expect the crossover phenomenon to be relevant.

In Sec. II, we provide numerical evidence for the crossover behavior of the fractal dimension as a function of some dissipative parameter. In Sec. III, we give heuristic arguments to explain the crossover behavior. A brief discussion is presented in Sec. IV.

II. NUMERICAL EVIDENCE FOR CROSSOVER BEHAVIOR IN FRACTAL DIMENSIONS

A. Discrete map

Our first numerical example illustrating the crossover phenomenon is the following two-dimensional map [2]:

$$x' = \lambda[x - (x+y)^2/4 - \nu(x+y)],$$

$$y' = \lambda^{-1}[y + (x+y)^2/4], \tag{1}$$

where $\lambda > 1$ is an energy parameter and ν is a dissipation parameter. The conservative version of the map (ν=0) has

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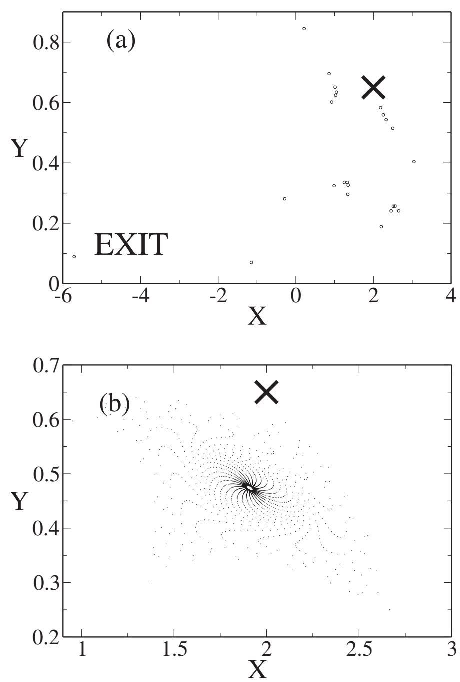

FIG. 1. Single trajectories from different initial conditions for $\lambda = 4$ and $\nu = 5 \times 10^{-3}$ (the cross indicates the initial condition). (a) From $(x_0, y_0) = (2, 0.65)$ the trajectory escapes after 24 iterations. (b) From $(x_0, y_0) = (1.99, 0.65)$ the trajectory falls into a fixed-point attractor.

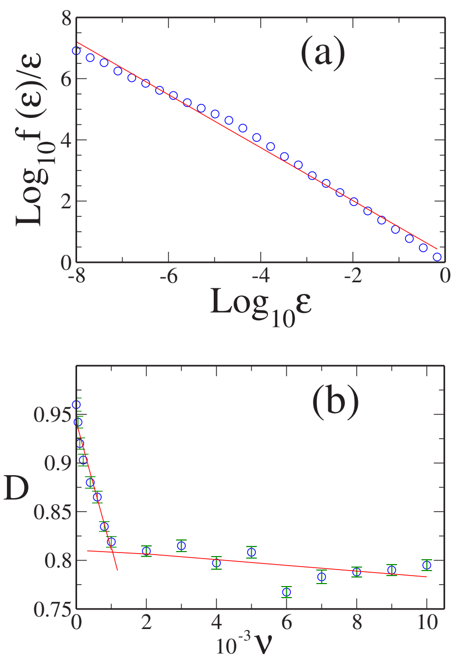

FIG. 2. (Color online) For the map system, Eq. (1), with $\lambda = 4$, (a) algebraic scaling of $f(\varepsilon)/\varepsilon$ with ε for $\nu = 6 \times 10^{-4}$. The fractal dimension is estimated to be $D = 0.87 \pm 0.01$ at the 95% confidence level. (b) Dependence of the dimension on the dissipation parameter ν. A crossover phenomenon can be seen to occur near $\nu = \nu_c \approx 10^{-3}$.

been used to establish that fact that, in nonhyperbolic chaotic scattering, the fractal dimension of the set of singularities in a scattering function is unity [5]. In particular, it can be shown that for $\lambda \leqslant 6.5$, the system dynamics is nonhyperbolic while it is hyperbolic for $\lambda \geqslant 6.5$. The dissipative version of the map ($\nu \gtrsim 0$) is a convenient model for studying the effect of weak dissipation on chaotic scattering [2].

Figures 1(a) and 1(b) show, for $\nu = 5 \times 10^{-3}$, two trajectories from two nearby initial conditions. In Fig. 1(a) the trajectory starts from $(x_0, y_0) = (2, 0.65)$ and escapes after 24 iterations. If we change the initial condition slightly to $(x_0, y_0) = (1.99, 0.65)$, as in Fig. 1(b), the trajectory falls into a fixed-point attractor. These suggest a sensitive dependence on initial conditions, the hallmark of chaotic scattering.

To perform the dimension calculation, we use the uncertainty algorithm [11]. In particular, we choose a line segment defined by $y_0 = -2$ from which trajectories are launched toward the scattering region about $(x, y) = (0, 0)$. For a given initial condition $\mathbf{x}_0$ on the line segment, a perturbed initial condition $\mathbf{x}_0 + \varepsilon$ can be chosen, where ε is the amount of perturbation. If the two trajectories from the initial conditions escape the system (say, when $\sqrt{x^2 + y^2} > 10$) in same number of iterations or if both trajectories approach the same attractor, the two initial conditions are *certain* with respect to the perturbation ε. Otherwise, if the trajectories escape the system in different numbers of iterations or if the trajectories approach distinct attractors, the initial conditions are *uncertain* with respect to the perturbation ε. Among a large number of initial-condition pairs, the fraction of uncertain initial conditions $f(\varepsilon)$ scales algebraically with ε as $f(\varepsilon) \sim \varepsilon^{1-D}$, or $f(\varepsilon)/\varepsilon \sim \varepsilon^{-D}$, where D is the fractal dimension [11] of the set of singularities in a scattering function defined on the initial line segment. Figure 2(a) shows, for $\nu = 6 \times 10^{-4}$, the algebraic scaling of $f(\varepsilon)/\varepsilon$ with ε. We obtain $D = 0.87 \pm 0.01$ at the 95% confidence level. [In the actual computation of $f(\varepsilon)$, we increase the number of random initial conditions until the number of uncertain initial conditions reaches a prescribed value (say, 500).] The variation of D as a function of the dissipation parameter ν is shown in Fig. 2(b). We observe that the dimension decreases rapidly from unity as ν is increased from zero and a crossover phenomenon occurs for $\nu = \nu_c \approx 10^{-3}$, after which the dimension decreases much more slowly for $\nu > \nu_c$.

This crossover phenomenon is due to the presence of new attractors in the scattering system. For convenience, we call *region A* the region below the critical value of $\nu < \nu_c$ and *region B* the region above $\nu > \nu_c$. Region *A* in Fig. 2(b) exhibits a rapid linear decreasing in the fractal dimension when ν is increased from zero. This is due to the metamorphic transition of the scattering dynamics from algebraic to exponential in the survival probability due to weak dissipation [2]. In region *B* the dimension decreases slowly, the origin of which can be attributed to the occurrence of some dominant attractors in the scattering region. To provide evidence, we calculate the basins of attraction for three different values of ν, as shown in Figs. 3(a)–3(c) for $\nu = 10^{-4}$, $\nu = 5 \times 10^{-3}$, and $\nu = 2 \times 10^{-2}$, respectively. We see that, for relatively small values of ν [Fig. 3(a)], there is only one attractor that is numerically detectable, which corresponds to the fixed point at the center of a dominant KAM island in the conservative case. As ν is increased, multiple attractors occur [e.g., the periodic attractors denoted by crosses in Figs. 3(b) and 3(c)].

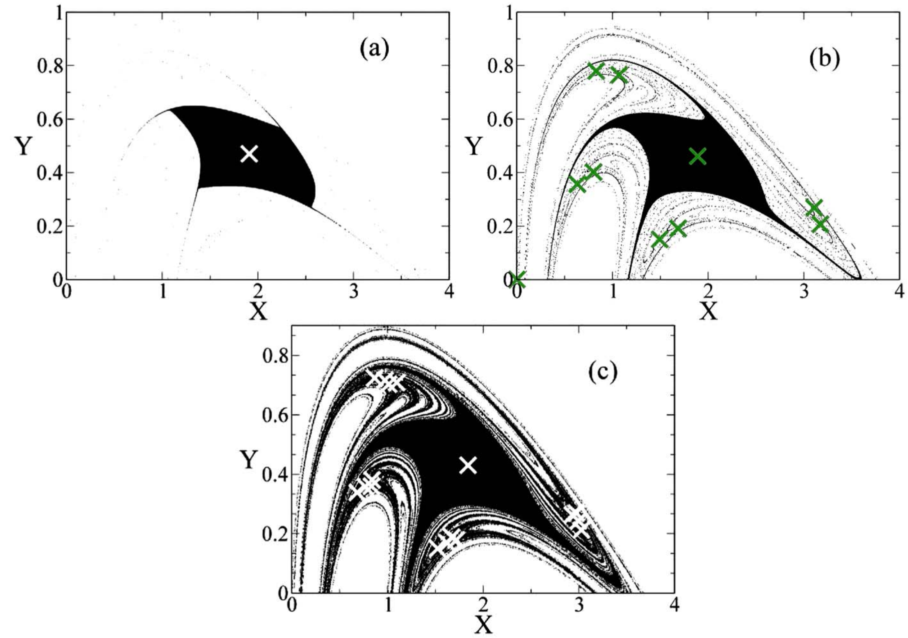

FIG. 3. (Color online) For $\lambda=4$, basin of attraction (denoted in black) of the fixed-point attractor located in the center of the dominant KAM island (denoted by the white cross) for (a) $\nu=10^{-4}$, (b) $\nu=5\times10^{-3}$, and (c) $\nu=0.02$. Crosses in (b) and (c) denote new attractors in the system.

These new attractors attract a significant fraction of the scattering trajectories, leading to a significantly slower decrease in the fractal dimension.

B. Continuous-time system

To illustrate the generality of the crossover phenomenon, we now consider a continuous-time system, the dissipative Hénon-Heiles system [4]. The Hamiltonian version of the system was first proposed in 1964 to address the question of whether there exist more than two constants of motion in the dynamics of a galaxy model [12], which correspond to bounded and unbounded orbits. The Hamiltonian is given by

$$H=\frac{1}{2}(\dot{x}^2+\dot{y}^2)+\frac{1}{2}(x^2+y^2)+x^2y-\frac{1}{3}y^3, \tag{2}$$

which defines the motion of a particle of unit mass under the two-dimensional potential $V(x,y)=\frac{1}{2}(x^2+y^2)+x^2y-\frac{1}{3}y^3$. Depending on its energy, a particle can be trapped in a region near $(x,y)=(0,0)$ or escape to infinity. The escaping energy E_e can be shown to be $1/6$, above which particles can escape [4,9,12]. Weak dissipation can be conveniently introduced in the model by adding terms proportional to the particle velocity in the equations of motion [4], as follows:

$$\ddot{x}+x+2xy+\alpha\dot{x}=0,$$

$$\ddot{y}+y+x^2-y^2+\beta\dot{y}=0, \tag{3}$$

where α and β are dissipation parameters. Without loss of generality we set $\alpha=\beta=\mu$.

In the presence of dissipation, attractors can arise in the scattering region. In particular, there can be coexisting fixed-point attractors [4]. Thus, for $E>E_e$, both escaping and trapped trajectories can be found. For instance, Fig. 4(a) shows an escaping trajectory for $E=0.19$ and $\mu=8\times10^{-4}$, where the initial condition is $(x_0,y_0)=(0,0.76)$ (marked by cross). When the initial condition is changed slightly to $(x_0,y_0)=(0,0.75)$, the resulting trajectory approaches asymptotically to the fixed-point attractor, as shown in Fig. 4(b), where the plus sign encloses the location of one of the fixed-point attractors.

The fractal dimension of the set of singularities in a scattering function can be calculated by using the same uncertainty algorithm. Figure 5(a) shows, for $E=0.19$ and $\mu=4\times10^{-5}$, the algebraic scaling of $f(\varepsilon)/\varepsilon$ with ε, where initial conditions are chosen from the line segment outside the scattering region defined by $(x_0,y_0)=(0,1)$ and $\Theta\in(0,2\pi)$, where Θ is the shooting angle with respect to the x axis. We obtain $D=0.925\pm0.005$. Figure 5(b) shows D versus the dissipation parameter μ. A crossover behavior similar to that in

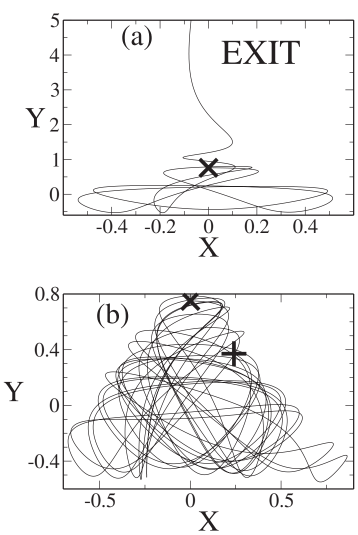

FIG. 4. For $E=0.19$ and $\mu=8\times10^{-4}$ in the dissipative Hénon-Heiles system, two types of trajectories, escaping (a) versus trapping (b), which can be generated by using slightly different initial conditions.

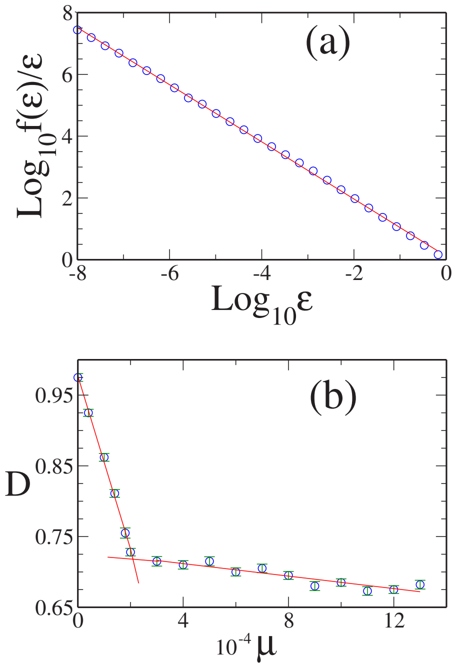

FIG. 5. (Color online) For $E=0.19$ in the dissipative Hénon-Heiles system, (a) algebraic scaling between $f(\varepsilon)/\varepsilon$ and ε for $\mu=4\times10^{-5}$. The fractal dimension is estimated to be $D=0.925\pm0.005$ at the 95% confidence level. (b) Fractal dimension D versus the dissipation parameter μ. The crossover behavior similar to that observed in the map system occurs at $\mu_c\approx2\times10^{-4}$.

the map system is observed, where the dimension decreases rapidly from unity for $\mu<\mu_c\approx2\times10^{-4}$ and the rate of decrease is much smaller for $\mu>\mu_c$.

Figures 6(a)–6(c) show the basins of scattering destinations and of the fixed point attractor at the center of the scattering region for $E=0.19$ and for $\mu=10^{-4}$, 8×10^{-4}, and 5×10^{-3}, respectively. The sets of black (black), red (dark gray), and yellow (light gray) dots denote initial conditions resulting in trajectories that escape through exits 1, 2, and 3, respectively, and the white regions inside the plotted structure denote the set of points falling into the attractor. In Fig. 6(a) the effects of the attractor are insignificant in the sense that almost all trajectories escape through one of the channels (in fact only about 0.92% of the examined trajectories fall into the attractor). The rapid decrease of D with μ in this regime is thus mainly due to the abrupt transition from algebraic to exponential decay in the scattering dynamics. In Figs. 6(b) and 6(c), the fractions of attracting trajectories are about 14.8% and 40%, respectively, indicating a significant effect of the attractors on the scattering dynamics. Due to the structural stability of attractors and due to their dominant influence on the scattering dynamics, increasing the dissipation further will not affect the dynamics in a significant way. We thus observe a much slower decrease in D.

III. FRACTAL-DIMENSION THEORY

We now provide a heuristic theory to explain the crossover phenomenon. Previous research has established that chaotic scattering is due to a nonattracting chaotic set (i.e., chaotic saddle) in the phase-space region where interactions responsible for scattering occur [10]. Both the stable and the unstable manifolds of the chaotic saddle are fractals [13]. Scattering particles are typically launched from a line segment straddling the stable manifold outside the scattering region. The set of singularities is the set of intersections of the stable manifold with the line segment, which can effectively be regarded as a Cantor set. The fractal dimension of the set can therefore be analyzed by using simple geometrical models.

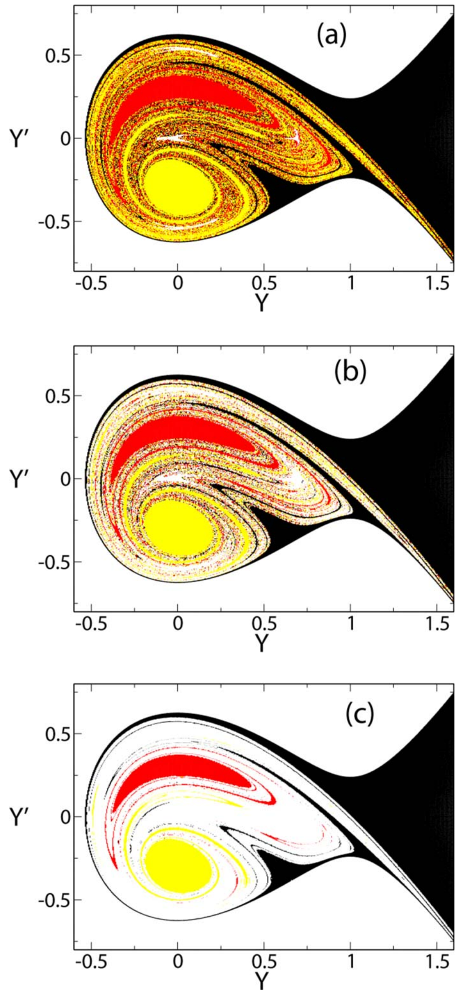

FIG. 6. (Color online) Basins of scattering particles for $E=0.19$ and (a) $\mu=10^{-4}$, (b) $\mu=8\times10^{-4}$, and (c) $\mu=5\times10^{-3}$. Black (black), red (dark gray), and yellow (light gray) color denote the particles escaping through channels 1, 2, or 3, respectively, and the set of white points inside the color region means the particles falling into the attractor.

For nonhyperbolic chaotic scattering, the fractal dimension of the set of singularities in the scattering function is $D=1$ [5]. This is a direct consequence of the underlying algebraic-decay law and can be seen intuitively by considering a zero-Lebesgue-measure Cantor set that has $D=1$. Start with the unit interval [0,1]. Remove the open middle third interval. From each of the two remaining intervals remove the middle fourth interval. Then from each of the four remaining intervals remove the middle fifth and so on. At the nth stage of the construction, there are $N=2^n$ subintervals, each of length $\epsilon_n = [2/(n+2)]2^{-n}$. The total length of all subintervals, $\epsilon_n N \sim n^{-1}$, goes to zero *algebraically* as $n \to \infty$. In order to cover the set with intervals of size ϵ_n, the required number of intervals is $N(\epsilon) \sim \epsilon^{-1}(\ln \epsilon^{-1})^{-1}$. The box-counting dimension of the set is then $D = \lim_{\epsilon\to 0} \ln N(\epsilon)/\ln \epsilon^{-1} = 1$. Note that D is the exponent of the dependence $N(\epsilon) \sim 1/\epsilon^D$, to which the weak logarithmic dependence does not contribute. However, it is the logarithmic term which is responsible for ensuring that the Lebesgue measure is zero: $\epsilon N(\epsilon) \sim (\ln \epsilon^{-1})^{-1} \to 0$ as $\epsilon \to 0$. In a more general setting, if at each stage a fraction $\eta_n = a/(n+c)$, where a and c are constants, is removed from the middle of each of the 2^n remaining intervals, then $N(\epsilon) \sim (1/\epsilon)[\ln(1/\epsilon)]^{-a}$. In this case, the slope of the curve $\ln N(\epsilon)$ versus $\ln \epsilon^{-1}$, which is $d \ln N(\epsilon)/d(\ln \epsilon^{-1})$, is always less than 1 for small ϵ, but it approaches 1 logarithmically as $\epsilon \to 0$. Thus, the result $D=1$ still holds. A practical implication is that for fractals whose general characters are similar to those for this example, an accurate numerical estimation of the dimension requires going to very small scales and, as such, any numerical estimation of the dimension over a finite range of scales will be an underestimate. As the scale is decreased, the numerically determined value of the dimension increases toward 1.

In a chaotic-scattering system, particles are launched from a line segment straddling the stable manifold of the chaotic saddle. There is then an interval of input variables which lead to trajectories that remain in the scattering region for at least a duration of time, say, T_0. By time $2T_0$ a fraction η of these particles leave. If the initial conditions of these escaping particles are all located in the middle of the original interval, there are then two equal-length subintervals of the input variable which lead to trajectories that remain for at least time $2T_0$. By time $3T_0$ an additional fraction η of the particles, whose initial conditions are located in the middles of the two subintervals remaining at time $2T_0$, escape. There are then four subintervals, particles initiated from which can remain in the scattering region for time at least $3T_0$ and so on. The resulting set is a Cantor set of Lebesgue measure zero on which particles never escape. The box-counting dimension of the Cantor set is given by

$$D = \frac{\ln 2}{\ln[(1-\eta)/2]^{-1}}.$$

In the conservative case, if the scattering is nonhyperbolic, because of the algebraic decay: $P(t) \sim t^{-z}$, the fraction η is no longer a constant: it varies at each stage of the construction of the Cantor set. At the nth stage (n large), the fraction η_n is approximately given by $\eta_n \approx -T_0 P^{-1} dP/dt \approx z/n$, which yields a Cantor set with dimension 1, where the quantity a in the mathematical construction of the Cantor set corresponds to the algebraic-decay exponent z. For conservative hyperbolic chaotic scattering, particles escape exponentially from the scattering region: $P(t) \sim e^{-\gamma t}$, where the decay rate γ is related to the fraction η as $\gamma = T_0^{-1} \ln(1-\eta)^{-1}$.

When there is a small amount of dissipation, the fraction η is no longer a constant of time [3]. Intuitively we can write

$$\eta_n = \frac{a}{n+b} - \delta, \tag{4}$$

where a and b are constants and δ is the fraction remaining at each stage due to the dissipation-induced small attractors in the scattering region. For weak dissipation we can assume $0 < \delta \ll a/b$. The fractal dimension is then given by

$$D = \ln 2/\ln\left(\frac{2}{1+\delta}\right). \tag{5}$$

As $\delta \to 0$, we have $D \to 1$ but $dD/d\delta = 1/\ln 2 \neq 0$.

As the dissipation parameter ν is increased from zero, δ also increases from zero, leading to a decrease of the fractal dimension from unity. If ν is not too large (say, $\nu < \nu_c$), attractors are constantly created as ν is increased from zero, i.e., as the system becomes more dissipative. A relatively large rate of increase in δ can then be expected. As the system becomes sufficiently dissipative so that most of the attractors that the system is capable of having have already been created, the increase in δ slows down as ν is increased further (say, for $\nu > \nu_c$). This leads to a much slower decrease in the dimension. A crossover behavior can then be expected when ν passes through ν_c. The actual value of ν_c depends on system details, and it cannot be predicted analytically.

IV. CONCLUSION AND DISCUSSION

In summary, our investigation of the fractal dimension in dissipative chaotic scattering reveals a crossover phenomenon: as the system becomes progressively more dissipative, the physically accessible fractal dimension of the set of singularities in a scattering function first decreases rapidly from unity and then exhibits a much slower rate of decrease. This can be explained by considering an intuitive mathematical construction of a Cantor set, taking into account the generation of attractors in the scattering region. The phenomenon appears universal in dissipative chaotic scattering as the Cantor-set theory is generally applicable and does not assume knowledge of system details. Indeed, the same phenomenon has been found in two quite distinct prototype models, one in discrete time and another in continuous time.

A context of physical interest where our result can be potentially useful is particle advection in open chaotic flows. Particles in systems with escapes can be seen, for instance, as inertial particles of fluids in open flows where dissipation plays the role of the mass of inertial particles. It is well known that the advective dynamics of idealized particles in two-dimensional, incompressible flows can be described as Hamiltonian [14,15]. For instance, consider such a flow characterized by a stream function $\Psi(x,y,t)$. For a particle with zero inertia and zero size, its trajectory in the flow obeys the following equations: $dx/dt = \partial\Psi(x,y,t)/\partial y$ and $dy/dt = -\partial\Psi(x,y,t)/\partial x$, which are the standard Hamilton's equations of motion generated by the Hamiltonian $H(x,y,t) = \Psi(x,y,t)$. That is, the particle velocity $\mathbf{v}(x,y,t) = (dx/dt, dy/dt)$ follows exactly the flow velocity $\mathbf{u}(x,y,t)$, as given by the right-hand side of the equations. This idealized picture changes completely when particles have finite inertia and size. In this realistic case, the particle velocity is generally not the same as the flow velocity and the equations of motion are no longer Hamilton's equations. The resulting dynamical system is no longer Hamiltonian but dissipative instead [16]. Considering that in an open Hamiltonian flow ideal particles coming from the upper stream must necessarily go out of the region of interest in finite time, the formation of attractors of inertial particles is remarkable. Suppose these physical particles are biologically or chemically active. That they can be trapped permanently in some region in physical space is of great interest or concern. Furthermore, the presence of attractors in the interacting region may result in a decrease of the fractal dimension in some scattering function that can be measured downstream.

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