PHYSICAL REVIEW E **76**, 016104 (2007)

Synchronization in time-varying networks: A matrix measure approach

Maoyin Chen
Department of Automation, Tsinghua University, Beijing 100084, China
(Received 11 January 2007; published 11 July 2007)

Synchronization in complex networks has attracted lots of interest in various fields. We consider synchronization in time-varying networks, in which the weights of links are time varying. We propose a useful approach—i.e., the *matrix measure approach*—to derive some analytically sufficient conditions for synchronization in time-varying networks. These conditions are less conservative than many existing synchronization conditions. Theoretical analysis and numerical simulations of different networks verify our main results.

DOI: 10.1103/PhysRevE.76.016104 PACS number(s): 89.75.−k, 82.40.Bj

I. INTRODUCTION

In the past decades, an increasing interest has been focused on complex networks with different topologies [1–26]. A complex network is a large set of interconnected nodes, in which a node is a fundamental unit with specific contents. Examples of complex networks include the Internet, the World Wide Web, electrical power grids, food webs, cellular and metabolic networks, etc. From the human brain to the Internet to human society, complex networks are prominent candidates to describe sophisticated collaborative dynamics in many sciences [1,2]. So far, the dynamics of complex networks has been extensively investigated, with special emphasis on the interplay between the complexity in the overall topology and the local dynamical properties of the coupled nodes.

As a typical kind of dynamics, synchronization in complex networks has been a research topic in many fields [1–26]. In 1998, Pecora and Carroll proposed the *master-stability-function* (MSF-) based method to study the synchronization in complex networks [3]. For a network of coupled chaotic oscillators, the stability of the synchronous manifold can be transformed into the stability of the *master stability equation* (MSE). This method computes the Lyapunov exponents of the variational equations numerically [3–5,7,10,16]. From the viewpoint of engineering, Lyapunov's direct method can also be used to study the synchronization in networks by constructing a Lyapunov function, which decreases along trajectories and gives analytical criteria for local or global synchronization [11–13,25,26]. The *connection-graph-stability* (CGS-) based method combines the Lyapunov direct method with graph theoretical reasoning, and this method also provides some analytical synchronization conditions [18–20]. *Matrix theory* (MT) can be successfully utilized to analyze synchronization in networks, which is based on the eigenvalue analysis [8,17].

Although there are a large amount of works in the field of complex networks, the great majority of research activities have been focused on static networks, whose topology and coupling configuration in the network are time invariant [3–10,14–17]. However, in many real-world networks, such as biological, epidemiological, and social networks, it is reasonable to assume that the network topologies can evolve over time. For this kind of networks, the weights of links between two connected nodes are time varying, which results in variations of the network topology and coupling configuration over time [20–26]. For these time-varying networks, it is not easy to study the synchronization. Compared with the rich works with respect to static networks, the results on synchronization in time-varying networks are less. Therefore, this paper tries to consider synchronization in time-varying networks. We propose a useful approach—i.e., the *matrix measure approach* (MMA)—to give some analytical sufficient conditions for synchronization in time-varying networks. This approach is based on the concept of the matrix measure. We demonstrate that the proposed synchronization conditions are less conservative than many existing synchronization conditions.

The rest of this paper is organized as follows. In the next section, we give a time-varying network model and some necessary analysis. In Sec. III, we discuss analytical conditions for the synchronization in time-varying networks using the MMA. We derive some analytical sufficient conditions for the synchronization. In Sec. IV, we extend the MMA to a general class of time-varying networks, in which two connected nodes are partially coupled. We show that the proposed synchronization conditions are less conservative than many existing conditions in Sec. V. In Sec. VI, we simulate different time-varying networks to show the effectiveness of the proposed conditions. We give our conclusions in the last section.

II. A COMPLEX NETWORK MODEL AND NECESSARY ANALYSIS

Consider a set of N linearly coupled identical nodes, with each node being an n-dimensional continuous dynamical system, in the form

$$\dot{x}_i = F(x_i) + \sum_{j=1}^{N} w_{ij}(t)(x_j - x_i), \quad 1 \le i \le N, \tag{1}$$

where $x_i=(x_{i1},\ldots,x_{in})^T \in \mathbb{R}^n$ is the coordinate vector and $F(x_i)$ is a function which governs the dynamics of each individual node. Let $W(t)=(w_{ij}(t))\in\mathbb{R}^{N\times N}$ represent the adjacency matrix of the network at time instant t, in which $w_{ij}(t)$ is defined as follows: if there is a connection from node j to node i, then $w_{ij}(t)\neq 0$; otherwise, $w_{ij}(t)=0$, and the diagonal elements of matrix $W(t)$ are defined by $w_{ii}(t)=0$.

From the network (1), the time-varying matrix $W(t)$ can be symmetric (or asymmetric), which stands for undirected time-varying networks (or directed time-varying networks). Network (1) is said to be in a synchronous state $x_1=x_2=\cdots=x_N$ if $\lim_{t\to\infty}[x_i(t)-x_j(t)]=0$ for $1\leq i\neq j\leq N$. Note that the time-varying weights of links can lead to the variations of network topology and coupling configuration over time. From Refs. [3–26], the stability of the synchronous state is equivalent to the stability of the linear synchronous manifold $\Xi=\{x_1=x_2=\cdots=x_N\}$. Here x_{i_0} $(1\leq i_0\leq N)$ can be regarded as the reference direction of the synchronous manifold Ξ. Without loss of generality, let x_1 be the reference direction.

Let the errors be $X_{1j}=x_j-x_1$ for $2\leq j\leq N$. Hence our aim is to derive analytical conditions to make the limit $\lim_{t\to\infty}X_{1j}=0$ as time t tends to infinity. After a simple calculation, we get

$$\begin{bmatrix}\dot{X}_{12}\\ \dot{X}_{13}\\ \vdots\\ \dot{X}_{1N}\end{bmatrix}=\begin{bmatrix}F(x_2)-F(x_1)\\ F(x_3)-F(x_1)\\ \vdots\\ F(x_N)-F(x_1)\end{bmatrix}+(S_1(t)\otimes I_n)\cdot\begin{bmatrix}X_{12}\\ X_{13}\\ \vdots\\ X_{1N}\end{bmatrix},\quad (2)$$

where the symbol $\otimes$ is the Kronecker product and matrix $S_1(t)$ is described by

$$S_1(t)=\begin{bmatrix}-(w_{12}+\sum_{j=1}^N w_{2j}) & w_{23}-w_{13} & \cdots & w_{2N}-w_{1N}\\ w_{32}-w_{12} & -(w_{13}+\sum_{j=1}^N w_{3j}) & \cdots & w_{3N}-w_{1N}\\ \vdots & \vdots & \ddots & \vdots\\ w_{N2}-w_{12} & w_{N3}-w_{13} & \cdots & -(w_{1N}+\sum_{j=1}^N w_{Nj})\end{bmatrix}.\quad (3)$$

The above procedure can be seen in Refs. [17,18]. In Ref. [17], matrix $S_1(t)$ is called the *diffusive synchronization stability matrix* (DSSM). In this paper we show that synchronization in time-varying networks also depends on this kind of matrix as well as the concept of the matrix measure.

Before we give our main results, we first introduce some necessary analysis of Eq. (2). Belykh *et al.* proposed the CGS-based method to consider the synchronization of coupled chaotic oscillators based on the concept of the *auxiliary system* [18–20]. In order to get the synchronization condition, we also utilize this concept [18–20]:

$$\dot{X}_{1j}=\left[\int_0^1 DF(\beta x_j+(1-\beta)x_i)d\beta-A\right]X_{1j},\quad j=2,3,\ldots,N,\quad (4)$$

where $DF(\cdot)$ denotes the Jacobian matrix and the matrix $A=\mathrm{diag}(a_1,a_2,\ldots,a_n)$ with $a_h\geq 0$ for $h=1,2,\ldots,n$.

Further, we make the following assumption.

Assumption [18–20]. Suppose the *auxiliary system* (4) is globally stabilized in the sense of a Lyapunov function $V_{1j}(t)=\frac{1}{2}X_{1j}^T X_{1j}$. It means that the derivative $\dot{V}_{1j}(t)$ along the solution of the *auxiliary system* (4) is strictly less than zero if $X_{1j}\neq 0$. That is,

$$\dot{V}_{1j}(t)=X_{1j}^T\left[\int_0^1 DF(\beta x_j+(1-\beta)x_i)d\beta-A\right]X_{1j} < 0,\quad \text{if } X_{1j}\neq 0.\quad (5)$$

Although Eqs. (4) and (5) are two crucial requirements, many examples of linearly coupled dynamical systems satisfy these conditions and global synchronization arises with increasing coupling. However, a few examples of coupled systems for which this is not the case were reported [9,18]. Among them is a lattice of x-coupled Rösler systems, in which the stability of the synchronization regime is lost with an increase of coupling such that the *assumption* (5) cannot be satisfied for this network and global synchronization cannot be achieved. The nature of global synchronization is strongly dependent on the dynamical properties of subsystems (18).

III. MATRIX MEASURE APPROACH

In this section we give some less conservative synchronization conditions for network (1) using the MMA. For completeness, we first introduce the concept of the matrix measure.

The matrix measure of a complex square matrix $B=(b_{ij})\in C^{n\times n}$ is defined as follows [27]:

$$\mu.(B) = \lim_{\varepsilon\to 0^+} \frac{\|I_n + \varepsilon B\| - 1}{\varepsilon}, \tag{6}$$

in which $\|\cdot\|$ is a matrix norm and I_n is the identity matrix.

When matrix norms $\|B\|_1 = \max_j \Sigma_{i=1}^n |b_{ij}|$, $\|B\|_2 = [\lambda \max(B^T B)]^{1/2}$, $\|B\|_\infty = \max_i \Sigma_{j=1}^n |b_{ij}|$, we can obtain the matrix measures [27]

$$\mu_1(B) = \max_j \left\{ \mathrm{Re}(b_{jj}) + \sum_{i=1, i\neq j}^{n} |b_{ij}| \right\}, \tag{7}$$

$$\mu_2(B) = \frac{1}{2} \lambda_{\max}(B^* + B), \tag{8}$$

$$\mu_\infty = \max_i \left\{ \mathrm{Re}(b_{ii}) + \sum_{j=1, j\neq i}^{n} |b_{ij}| \right\}, \tag{9}$$

respectively, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a complex matrix and B^* is the complex-conjugate transpose of a complex matrix. Note that $B^* = B^T$ if B is a real square matrix.

Hence Eq. (2) is equivalent to

$$\begin{bmatrix} \dot{X}_{12} \\ \dot{X}_{13} \\ \vdots \\ \dot{X}_{1N} \end{bmatrix} = A_1 \begin{bmatrix} X_{12} \\ X_{13} \\ \vdots \\ X_{1N} \end{bmatrix} + [A_0 + S_1(t) \otimes I_n] \begin{bmatrix} X_{12} \\ X_{13} \\ \vdots \\ X_{1N} \end{bmatrix}, \tag{10}$$

where $A_1 = \mathrm{diag}(\int_0^1 DF[\beta x_2 + (1-\beta)x_1]d\beta - A, \ldots, \int_0^1 DF(\beta x_N + (1-\beta)x_1)d\beta - A)$ and $A_0 = \mathrm{diag}(A, \ldots, A)$.

Denote the vector $X_1 = [X_{12}^T X_{13}^T \ldots X_{1N}^T]^T$. For Eq. (10), define a non-negative function as

$$V_1(t) = \sum_{j=2}^{N} V_{1j}(t) = \frac{1}{2} \sum_{j=2}^{N} X_{1j}^T X_{1j}. \tag{11}$$

From the *assumption* (5), we can easily get

$$\dot{V}_1(t) \le \frac{1}{2} X_1^T ((A_0^T + A_0) + \{[S_1(t) \otimes I_n]^T + [S_1(t) \otimes I_n]\}) X_1. \tag{12}$$

In order to make the right-hand side of the above inequality be negative, one basic condition is that $(A_0^T + A_0) + [(S_1(t) \otimes I_n]^T + [S_1(t) \otimes I_n)] < 0$ for all time $t > 0$ if $X_1 \neq 0$. If so, the function $V_1(t)$ is called a Lyapunov function. Though there are different types of the above inequality, negativeness of the Lyapunov function is required [11–13,25]. Based on the Lyapunov stability, we conclude that the networks can be in a synchronous state. However, this is too strict, especially for networks with time-variant weights. For these networks, the right-hand side of inequality (12) could be non-negative during some time intervals. It means that in this case the Lyapunov stability cannot be satisfied. Therefore, we should give other conditions for synchronization. Obviously, owing to the positiveness of the function $V_1(t)$, if the limit $\lim_{t\to\infty} V_1(t) = 0$ holds, we still have $X_{1j}^T X_{1j} = 0$ for $2 \le j \le N$. Hence the networks can be in a synchronous state $x_1 = x_2 = \cdots = x_N$. So our aim is to make limit $\lim_{t\to\infty} V_1(t) = 0$.

From Eqs. (7)–(9), we have

$$\begin{aligned} \dot{V}_1(t) &\le \tfrac{1}{2}(\mu_{\{1,2,\infty\}}(A_0^T + A_0) + \mu_{\{1,2,\infty\}}\{[S_1(t) \otimes I_n]^T \\ &\quad + [S_1(t) \otimes I_n]\}) X_1^T X_1 \\ &= (\mu_{\{1,2,\infty\}}(A_0^T + A_0) + \mu_{\{1,2,\infty\}}\{[S_1(t) \otimes I_n]^T \\ &\quad + [S_1(t) \otimes I_n]\}) V_1(t). \end{aligned} \tag{13}$$

No mater which case it is, we get

$$V_1(t) \le \exp\left(\int_0^t (\mu_{\{1,2,\infty\}}(A_0^T + A_0) + \mu_{\{1,2,\infty\}}\{[S_1(\tau) \otimes I_n]^T + [S_1(\tau) \otimes I_n]\}) dt \right) V_1^0, \tag{14}$$

in which V_1^0 is the initial value of $V_1(t)$. If the following limit holds,

$$\int_0^t (\mu_{\{1,2,\infty\}}(A_0^T + A_0) + \mu_{\{1,2,\infty\}}\{[S_1(\tau) \otimes I_n]^T + [S_1(\tau) \otimes I_n]\}) d\tau \to -\infty \quad \text{as } t \to \infty. \tag{15}$$

we have $V_1(t) \to 0$ as time t approaches infinity, which implies $X_1^T X_1 \to 0$ as $t \to \infty$. It means that the networks can be in a synchronous state.

It seems that condition (15) is complicated. But we can simplify the above condition under certain circumstances. From the *auxiliary system* (4), we know that a_i is positive. Let $a = \max\{a_1, a_2, \ldots, a_n\} > 0$. For the matrix measures $\mu_{\{1,\infty\}}$, condition (15) can be simplified into

$$\int_0^t \{2a + \mu_{\{1,\infty\}}[S_1^T(\tau) + S_1(\tau)]\} d\tau \to -\infty \quad \text{as } t \to \infty, \tag{16}$$

since $\mu_{\{1,\infty\}}\{[S_{j_0}(t) \otimes I_n]^T + [S_{j_0}(t) \otimes I_n]\} = \mu_{\{1,\infty\}}[S_{j_0}^T(t) + S_{j_0}(t)]$. For the matrix measure μ_2, condition (15) can be simplified into

$$\int_0^t (2a + \mu_2\{[S_1(\tau) \otimes I_n]^T + [S_1(\tau) \otimes I_n]\}) d\tau \to -\infty \quad \text{as } t \to \infty. \tag{17}$$

If we choose the node x_{i_0} $(1 < i_0 \le N)$ as the reference direction of the manifold Ξ (that is, the errors are defined by $X_{i_0 j} = x_j - x_{i_0}$), the DSSM can be written

$$S_{i_0}(t)=\begin{bmatrix} -\left(w_{i_01}+\sum_{j=1}^{N}w_{1j}\right) & \cdots & w_{1i_0-1}-w_{i_0i_0-1} & w_{1i_0+1}-w_{i_0i_0+1} & \cdots & w_{1N}-w_{i_0N} \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ w_{i_0-1,1}-w_{i_01} & \cdots & -\left(w_{i_0i_0-1}+\sum_{j=1}^{N}w_{i_0-1,j}\right) & w_{i_0-1,i_0+1}-w_{i_0i_0+1} & \cdots & w_{i_0-1,N}-w_{i_0N} \\ w_{i_0+1,1}-w_{i_01} & \cdots & w_{i_0+1i_0-1}-w_{i_0i_0-1} & -\left(w_{i_0i_0+1}+\sum_{j=1}^{N}w_{i_0+1,j}\right) & \cdots & w_{i_0+1,N}-w_{i_0N} \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ w_{N,1}-w_{i_01} & \cdots & w_{Ni_0-1}-w_{i_0i_0-1} & w_{N,i_0+1}-w_{i_0i_0+1} & \cdots & -\left(w_{i_0N}+\sum_{j=1}^{N}w_{Nj}\right) \end{bmatrix}. \quad (18)$$

Correspondingly, the conditions for synchronization can be changed into

$$\int_0^t \{2a+\mu_{\{1,\infty\}}[S_{i_0}^T(\tau)+S_{i_0}(\tau)]\}d\tau \to -\infty \quad \text{as } t\to\infty \quad (19)$$

and

$$\int_0^t (2a+\mu_2\{[S_{i_0}(\tau)\otimes I_n]^T+[S_{i_0}(\tau)\otimes I_n]\})d\tau \to -\infty \quad \text{as } t\to\infty, \quad (20)$$

respectively.

From conditions (16)–(20), if one among them holds, we conclude that the limit of the non-negative function $V_{i_0}(t)$ is zero as t approaches infinity; that is, the time-varying networks can be in a synchronous state. Now we analyze these conditions. For certain j_0 $(1\le j_0\le N)$ and for all time $t>0$, sufficient conditions for (16), (19) and for (17), (20) are given by

$$\mu_{\{1,\infty\}}[S_{j_0}(t)^T+S_{j_0}(t)]<-(2a+\delta_0) \quad (21)$$

and

$$\mu_2\{[(S_{j_0}(t)\otimes I_n]^T+[S_{j_0}(t)\otimes I_n)]\}<-(2a+\delta_0), \quad (22)$$

respectively, in which δ_0 is a sufficient small constant. According to the concept of positive semidefinite ordering [28], we have the fact that $(A_0^T+A_0)+\{[S_{j_0}(t)\otimes I_n]^T+[S_{j_0}(t)\otimes I_n]\}<0$ for all time $t>0$. Hence the function $V_{j_0}(t)$ is a Lyapunov function and time-varying networks can be in a synchronous state in the sense of the Lyapunov stability. However, compared with conditions (16), (17), (19), and (20), conditions (21) and (22) are also strict for arbitrary time t.

From the above analysis, synchronization conditions (15)–(17) and (19)–(22) are dependent on inequality (12). In fact, the matrix $(A_0^T+A_0)+\{[S_1(t)\otimes I_n]^T+[S_1(t)\otimes I_n]\}$ in this inequality is symmetric, and all the eigenvalues are real numbers. Hence inequality (12) can become one of the following two inequalities:

$$\dot V_1(t)\le\frac{1}{2}(\lambda_{\max}(A_0^T+A_0)+\lambda_{\max}\{[S_1(t)\otimes I_n]^T+[S_1(t)\otimes I_n]\})$$

and

$$\dot V_1(t)\le\frac{1}{4}[\lambda_{\max}[2(A_0^T+A_0)]+\lambda_{\max}(2\{[S_1(t)\otimes I_n]^T + [S_1(t)\otimes I_n]\})].$$

The former results in synchronization conditions with respect to the matrix measures $\mu_{\{1,\infty\}}$. The latter leads to synchronization conditions on the matrix measure μ_2. Obviously, these conditions are relative to the time-varying eigenvalues of a time-varying matrix. Though the same idea may be difficultly applied to other matrix norms, synchronization conditions resulting from matrix measures are general in the sense that these conditions can be applied to the time-varying networks, including fast or slowly switching networks, asymmetric networks, and random networks. This will be illustrated in the simulation section. In addition, we will show that these conditions are also less conservative than many existing synchronization conditions from the MSF-based method, the CGS-based method, and the MT-based method.

IV. EXTENSION TO NETWORKS WITH PARTIALLY COUPLED STATES

In the above section, we derive some sufficient conditions for the synchronization in time-varying networks with fully coupled states between connected nodes. Now we further consider the synchronization in time-varying networks where two connected nodes are partially coupled. In this case, the networks are modeled as

$$\dot{x}_i = F(x_i) + \sum_{j=1}^{N} w_{ij}(t)P(x_j - x_i), \quad 1 \le i \le N, \tag{23}$$

where the diagonal matrix $P = \text{diag}(p_1, p_2, \ldots, p_n)$ is the coupling matrix between two connected nodes, where $p_h = 1$, $h = 1, 2, \ldots, s$ and $p_h = 0$ for $h = s+1, \ldots, n$.

Without loss of generality, let x_1 be the reference direction of the synchronous manifold Ξ. For the network (23), the *auxiliary system* is still chosen as Eq. (4), where $A = \text{diag}(a_1, \ldots, a_n)$ with $a_h > 0$ for $h = 1, 2, \ldots, s$ and $a_h = 0$ for $h = s+1, \ldots, n$. Further, the *auxiliary system* (4) is globally stabilized in the sense of the Lyapunov function $V_{1j}(t) = \frac{1}{2} X_{1j}^T H X_{1j}$, in which

$$H = \text{diag}(\underbrace{1, \ldots, 1}_{s}, H_1)$$

with H_1 is a symmetric positive definite. It means that

$$\dot{V}_{1j}(t) = X_{1j}^T H \left[\int_0^1 DF(\beta x_j + (1-\beta)x_i)d\beta - A \right] X_{1j} < 0, \quad \text{if } X_{1j} \neq 0. \tag{24}$$

Denote $H = M^T M$, where M is a nonsingular square matrix. For the manifold Ξ, the dynamics of the errors X_{ij} is still given by Eq. (10), where the term $S_1(t) \otimes I_n$ is replaced by the term $S_1(t) \otimes P$. Define a non-negative function as follows:

$$V_1(t) = \sum_{j=2}^{N} V_{1j}(t) = \frac{1}{2} \sum_{j=2}^{N} X_{1j}^T H X_{1j}. \tag{25}$$

After a simple calculation, we get

$$\begin{aligned} \dot{V}_1(t) \le & \frac{1}{2} X_1^T M^T [(M^{-1})^T A_0^T M^T + M A_0 M^{-1}] M X_1 \\ & + \frac{1}{2} X^T M^T \{(M^{-1})^T [S_1(t) \otimes P]^T M^T \\ & + M[S_1(t) \otimes P] M^{-1}\} M X. \end{aligned}$$

Hence we have

$$\begin{aligned} \dot{V}_1(t) \le (& \mu_{\{1,2,\infty\}}[(M^{-1})^T A_0^T M^T + M A_0 M^{-1}] + \mu_{\{1,2,\infty\}}\{(M^{-1})^T \\ & \times [S_1(t) \otimes P]^T M^T + M[S_1(t) \otimes P] M^{-1})\} V_1(t). \end{aligned} \tag{26}$$

Therefore, if the following limit holds,

$$\begin{aligned} \int_0^t (& \mu_{\{1,2,\infty\}}[(M^{-1})^T A_0^T M^T + M A_0 M^{-1}] + \mu_{\{1,2,\infty\}}\{(M^{-1})^T \\ & \times [S_1(\tau) \otimes P]^T M^T + M[S_1(\tau) \otimes P] M^{-1}\}) d\tau \to -\infty \end{aligned} \tag{27}$$

as time t tends to infinity, we have $V_1(t) \to 0$ as time t approached infinity, which implies $X_1^T X_1 \to 0$ as $t \to \infty$. It means that the networks can be in a synchronous state $x_1 = x_2 = \cdots = x_N$.

Obviously, condition (15) is a special case of condition (27) if the diagonal matrix P is the identity matrix (equivalently, $M = I_n$).

V. COMPARISON WITH OTHER SYNCHRONIZATION CONDITIONS

In this section we show that the MMA-based synchronization conditions are less conservative than some existing synchronization conditions.

Comparison with the MSF-based synchronization condition. As far as we know, the MSF-based synchronization condition is very effective to deal with the synchronization [3–5,7,10,16]. A key problem is to transform the stability of the synchronous manifold into the following *master stability equation*:

$$\dot{\zeta} = [F(t) + (\alpha + i\beta)H(t)]\zeta, \tag{28}$$

where $F(t)$ and $H(t)$ are the Jacobian matrices [in this paper $H(t) = I_n$ or $H(t) = P$] and α and β are parameters. Hence the corresponding largest Lyapunov exponent is the function of two parameters α and β, which is called the *master stability function*. For the nonzero eigenvalues γ_k and the global coupling σ, the networks can be in a synchronous state if $\sigma\gamma_k$ belongs to the *synchronization region*, where the largest Lyapunov exponent of the MSE (28) is negative.

From Refs. [3–5,7,10,16], if we use the MSFs, the linearized equation around the synchronous manifold must be block diagonalized. In fact, the MSF-based networks are basically static networks. Once this kind of network is generated, the total topology and the coupling configuration in the network are time invariant. However, in many real-world networks, such as biological, epidemiological, and social networks, it is reasonable to assume that the network topologies can evolve over time. For this kind of networks, the weights of links between two connected nodes are time varying, which results in variations of the network topology and coupling configuration over time. For these time-varying networks, it is difficult to block-diagonalize the linearized equation around the synchronous manifold, and the MSEs and MSFs are also difficult to be derived. However, the MMA-based synchronization conditions need not require block diagonalization. Further, they provide sufficient analytic conditions for general time-varying networks.

Comparison with the CGS-based method. Belykh *et al.* proposed the CGS-based method to consider the synchronization of coupled chaotic oscillators. This method combines the Lyapunov-function approach with graph-theoretical reasoning. Its main step is to establish a bound on the total length of all paths passing through an edge on the network connection graph [18]. Further, this method can also be applied to the synchronization to asymmetrically coupled networks with node balance, the property that all nodes in the network have equal input and output weight sums [19], as well as the blinking model of the network, where some links are rapidly switched on and off independently of each other [20].

However, this method has its limitations. Except for the requirement of the Lyapunov function, another common

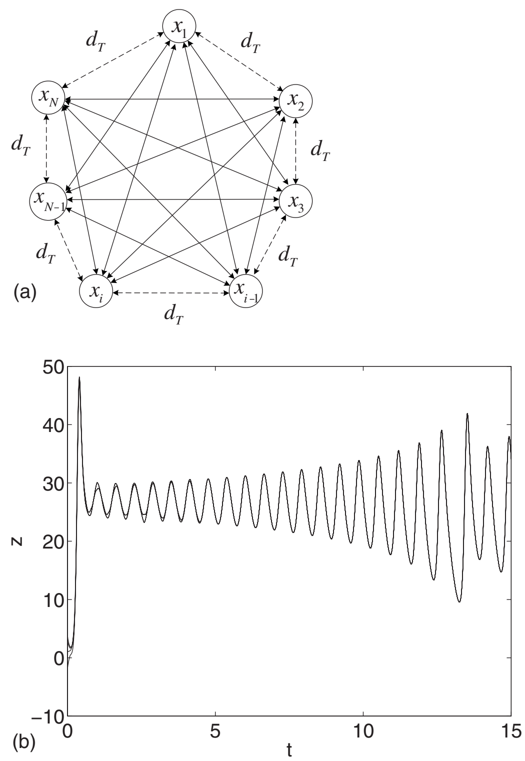

FIG. 1. The global coupled network: (a) the scheme of the network and (b) the simulation result on z states.

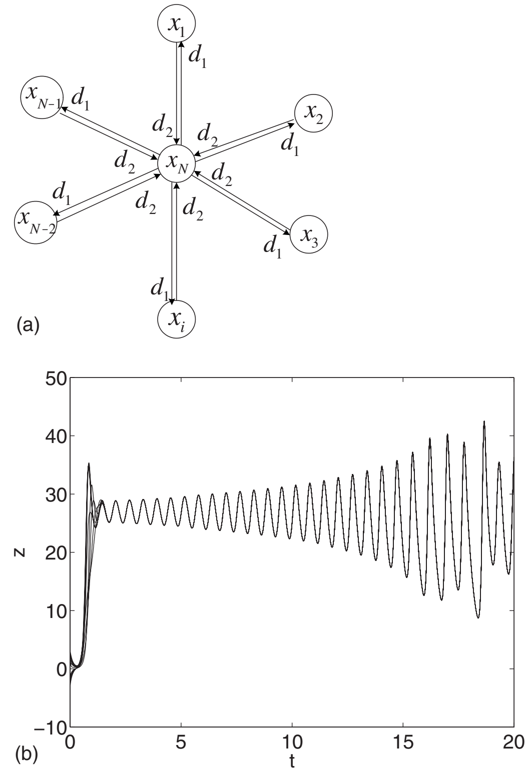

FIG. 2. The star-type coupled network: (a) the scheme of the network and (b) the simulation result on z states.

limitation is that the time-varying weights in networks are assumed to be non-negative [18–20]. In fact, there exist some networks with negative and positive weights. If the weight $w_{ij}(t)<0$ or $w_{ij}(t)>0$, such a coupling element is called *cooperative* or *competitive*, respectively [12]. In addition, some links in the blink model of the network can only be switched fast [20]. If they are switched slowly, the blink model cannot be modeled by the average model, whose stability cannot be ensured by the CGS-based method. The CGS-based method is also difficult to extend to general asymmetrically coupled networks without node balance, the property that some nodes in the network have unequal input and output weight sums. In this paper, the MMA provides sufficient conditions for general time-varying networks with positive and negative weights. These networks include slowly switching networks. Although theorem 1 in Ref. [18] provides a sufficient condition for time-varying networks, its main inequality must depend on the dynamics of nodes in networks (i.e., the difference of states of nodes). But in this paper sufficient conditions only depend on the DSSM of time-varying networks, rather than the dynamics of nodes in networks. So the present approach is less conservative than the CGS-based method.

Comparison with the MT-based method. The MMA-based method is also less conservative than some MT-based methods [8,17]. In Ref. [8], the coupling matrix in the network is time invariant, and the real part of nonzero eigenvalues is assumed to be less than some negative number (in fact, this number is relative to the maximal Lyapunov exponents of chaotic dynamics of an isolated node). But when the weights are time varying, it is difficult to derive the condition for synchronization since the eigenvalues of the coupling matrix are also time varying, and the network cannot be transformed into low-dimensional dynamics. Though the concept of DSSM in Ref. [17] can be used to the asymmetrically coupled networks, the coupling matrix in the network is in fact time-invariant. For the time-varying networks, it is difficult to ensure synchronization even if all the eigenvalues of the DSSM for arbitrary time are negative for arbitrary time. However, in this paper, the MMA can provide some sufficient conditions for synchronization even if the weights in networks are time varying.

Comparison with some existing synchronization conditions in time-varying networks. Some works have already considered the synchronization in time-varying networks, including the switching networks [21–26]. Empirical evidence and numerical simulations indicated that synchronization in time-varying networks can be reached under certain conditions [21–23]. Stilwell *et al.* proposed sufficient conditions for fast-switching synchronization in time-varying networks [24]. In addition, some synchronization conditions for time-varying networks have been proposed from Lyapunov's direct method [25,26]. However, these results are based on either numerical simulations [21–23] or strict conditions (such as fast switching [24] and negativeness of the Lyapunov functions [25,26]). In this paper, sufficient conditions (16), (17), (19), and (20) can be applied to the more general time-varying networks including the slow- or fast-switching networks. Hence the proposed conditions for synchronization are less conservative.

VI. NUMERICAL SIMULATIONS

In this section we will discuss the application of the proposed synchronization conditions. The networks can be global coupled networks, star-type coupled networks, nearest-neighbor coupled networks (specially the chain-type coupled networks), and random coupled networks. In each network, the Lorenz system, given by

$$\dot{x}_1 = 10(x_2 - x_1), \quad \dot{x}_2 = 28x_1 - x_2 - x_1x_3, \quad \dot{x}_3 = -\frac{8}{3}x_3 + x_1x_2,$$

is chosen as the node in networks and two connected nodes are fully coupled through x, y, z states. After sufficient simulations, we can choose the matrix $A = \mathrm{diag}(2,2,2)$, which could make the *auxiliary system* (4) be globally stabilized. It means that the two Lorenz systems can be globally synchronized by the negative feedback $-A[xyz]^T$.

A. Global coupled networks

Suppose all nodes in the network are symmetrically globally coupled. The global coupled network is plotted in Fig. 1(a). In this network, all the links are weighted by $d > 0$, except that the links $x_1 \leftrightarrow x_2 \leftrightarrow \cdots \leftrightarrow x_N \leftrightarrow x_1$ are weighted by a switching rule:

$$d_T = \begin{cases} d, & t \in [2(n-1)T, (2n-1)T), \\ -d, & t \in [(2n-1)T, 2nT), \end{cases} \tag{29}$$

where $n = 1, 2, \ldots$ and $T > 0$. Obviously, the network is time varying and can be switched between positive and negative values. Therefore, the MSF-, CGS-, and MT-based methods are difficult to ensure synchronization in the switching networks. In this case, the symmetric coupling matrix $W(t)$ has the form

$$W(t) = \begin{bmatrix} 0 & d_T & d & \cdots & d_T \\ d_T & 0 & d_T & \cdots & d \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ d & d & \cdots & 0 & d_T \\ d_T & d & \cdots & d_T & 0 \end{bmatrix}_N . \tag{30}$$

The DSSMs for $t \in [2(n-1)T, (2n-1)T)$ and $[(2n-1)T, 2nT)$ are given by

$$S_1^+(t) = d \begin{bmatrix} -N & 0 & \cdots & 0 \\ 0 & -N & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -N \end{bmatrix}_{N-1} \tag{31}$$

and

$$S_1^-(t) = d \begin{bmatrix} -(N-6) & -2 & 0 & \cdots & 0 & 0 & 2 \\ 0 & -(N-4) & -2 & \cdots & 0 & 0 & 2 \\ 2 & -2 & -(N-4) & \cdots & 0 & 0 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 0 & 0 & \cdots & -(N-4) & -2 & 2 \\ 2 & 0 & 0 & \cdots & -2 & -(N-4) & 0 \\ 2 & 0 & 0 & \cdots & 0 & -2 & -(N-6) \end{bmatrix}_{N-1} , \tag{32}$$

respectively. From the matrix measures $\mu_{\{1,\infty\}}$ and condition (16), we can conclude

$$\begin{aligned} \int_0^t \{2a + \mu_{\{1,\infty\}}[S_1^T(\tau) + S_1(\tau)]\} d\tau &= \sum_{n=1}^{\infty} \int_{2(n-1)T}^{2nT} \{2a + \mu_{\{1,\infty\}}[S_1^T(\tau) + S_1(\tau)]\} d\tau \\ &= \sum_{n=1}^{\infty} [(2a - 2Nd)T + (2a + 10d)T] = \sum_{n=1}^{\infty} [4a - (2N-10)d]T \end{aligned}$$

if $N \geq 6$. Obviously, if $d > \frac{2a}{N-5}$, we have

$$\int_0^t \{2a + \mu_{\{1,\infty\}}[S_1^T(\tau) + S_1(\tau)]\} d\tau = -\infty .$$

Hence the networks can be in a synchronous state if $N \geq 6$. For $N < 6$, we utilize the measure μ_2 to consider the synchronization. We choose $N = 4$. After a simple calculation, we get

$$\begin{aligned} \int_0^t (2a + \mu_2\{[S_1(\tau) \otimes I_3]^T + [S_1(\tau) \otimes I_3]\}) d\tau &= \sum_{n=1}^{\infty} \int_{2(n-1)T}^{2nT} (2a + \mu_2\{[S_1(\tau) \otimes I_3]^T + [S_1(\tau) \otimes I_3]\}) d\tau \\ &= \sum_{n=1}^{\infty} [(2a - 8d)T + (2a + 4d)T] = \sum_{n=1}^{\infty} [(4a - 4d)T] = -\infty \end{aligned}$$

if $d>a$. It means that the networks with $N=4$ can also be synchronous state if d is sufficiently large. In this case, we simulate the networks with $N=4$, $T=0.1$ s, and $d=3$. The simulation result with respect to z state of each node is plotted in Fig. 1(b). It shows that the networks can be in a synchronous state.

B. Star-type coupled networks

In this network, only one node is a center node with degree $N-1$, and all the other nodes with degree 1 are connected to this center node. The scheme of this network can be seen in Fig. 2(a). In this figure, the links from node x_N to its connected nodes are weighted by d_1, and the links from $x_i(i=1,2,\ldots,N-1)$ to node x_N are weighted by d_2. In this case, the coupling matrix $W(t)$ and the DSSM $S_1(t)$ have the following forms:

$$W(t)=\begin{bmatrix} 0 & 0 & \cdots & 0 & d_1 \\ 0 & 0 & \cdots & 0 & d_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & d_1 \\ d_2 & d_2 & \cdots & d_2 & 0 \end{bmatrix}_N,$$

$$S_1(t)=\begin{bmatrix} -d_1 & 0 & \cdots & 0 & 0 \\ 0 & -d_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -d_1 & 0 \\ d_2 & d_2 & \cdots & d_2 & -(d_1+(N-1)d_2) \end{bmatrix}_{N-1}. \tag{33}$$

Obviously, if $d_{\{1,2\}}>0$ are time invariant and satisfy $(d_2-2d_1)+2a<0$, we conclude that this kind of asymmetric network can be in a synchronous state according to the matrix measures $\mu_{\{1,\infty\}}$.

Now we consider the case where d_1 is time varying and d_2 is time invariant. For example, we choose $d_1=d[1+1.3\sin(t)]$ and $d_2=d$ where $d>0$ is a parameter. Note that the weight d_1 could be negative or positive, and it can be regarded as the slowly switching between $2.3d$ and $-0.3d$. In this case, we have

$$\mu_{\{1,\infty\}}[S_1^T(t)+S_1(t)]=\max\{-2d_1+d_2,\ -2d_1-Nd_2\}$$
$$=-d+2.6d\sin(t).$$

Since

$$\int_0^\infty(\cdot)dt=\sum_{n=1}^\infty\left[\int_{2(n-1)\pi}^{(2n-1)\pi}(\cdot)dt+\int_{(2n-1)\pi}^{2n\pi}(\cdot)dt\right],$$

we have

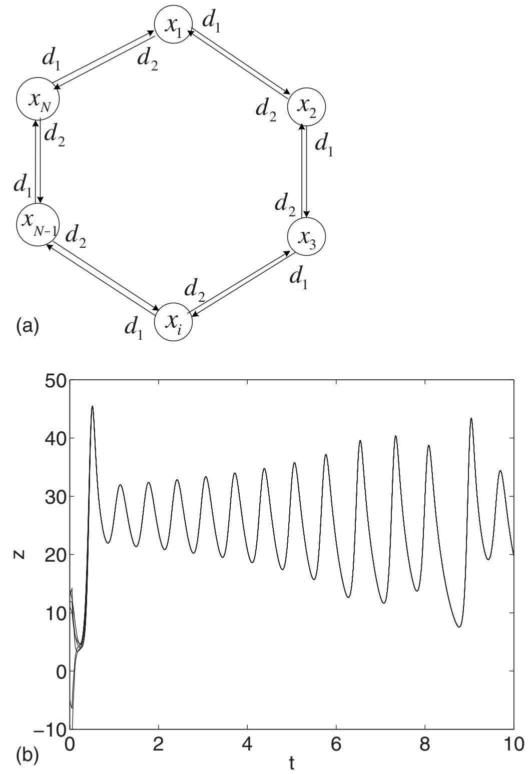

FIG. 3. The chain-type coupled network: (a) the scheme of the network and (b) the simulation result on z states.

$$\int_0^\infty\{2a+\mu_{\{1,\infty\}}[S_1^T(t)+S_1(t)]\}dt=-(d-2a)\int_0^\infty dt$$
$$+2.6d\int_0^\infty\sin(t)dt=-\infty$$

if $d>2a$. We simulate the network with $N=10$ and $d=6$. The simulation result is plotted in Fig. 2(b). It shows that the networks can be in a synchronous state.

C. Nearest-neighbor coupled networks

We consider a simple nearest-neighbor coupled network—i.e., the chain-type coupled network. The scheme of this kind of network is shown by Fig. 3(a). The links in the path $x_1\to x_2\to\cdots x_N\to x_1$ are weighted by d_1, and the links in the path $x_1\to x_N\to\cdots x_2\to x_1$ are weighted by d_2. In this case, the coupling matrix $W(t)$ and the DSSM $S_1(t)$ have the following forms:

$$W(t)=\begin{bmatrix} 0 & d_2 & 0 & \cdots & 0 & 0 & d_1 \\ d_1 & 0 & d_2 & \cdots & 0 & 0 & 0 \\ 0 & d_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & d_2 & 0 \\ 0 & 0 & 0 & \cdots & d_1 & 0 & d_2 \\ d_2 & 0 & 0 & \cdots & 0 & d_1 & 0 \end{bmatrix}_N \tag{34}$$

and

$$S_1(t)=\begin{bmatrix} -(d_1+2d_2) & d_2 & 0 & \cdots & 0 & 0 & -d_1 \\ d_1-d_2 & -(d_1+d_2) & d_2 & \cdots & 0 & 0 & -d_1 \\ -d_2 & d_1 & -(d_1+d_2) & \cdots & 0 & 0 & -d_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -d_2 & 0 & 0 & \cdots & -(d_1+d_2) & d_2 & -d_1 \\ -d_2 & 0 & 0 & \cdots & d_1 & -(d_1+d_2) & d_2-d_1 \\ -d_2 & 0 & 0 & \cdots & 0 & d_1 & -(2d_1+d_2) \end{bmatrix}_{N-1}. \tag{35}$$

Now we consider the synchronization of the network with $N=6$ nodes. Suppose that the weights $d_1=d_2$, and they are switched between values $d^1, d^2, \ldots, d^m$ with periods $\tau_1, \tau_2, \ldots, \tau_m$, respectively, and the total period is $T=\Sigma_{i=1}^m \tau_i$. From condition (17), the above network can be in a synchronous state if

$$\sum_{n=1}^{\infty}\sum_{i=1}^{m}\{2a+\mu_2[(S_1^i\otimes I_n)^T+(S_1^i\otimes I_n)]\}\tau_i=-\infty, \tag{36}$$

where S_1^i is the DSSM of the network during the interval $[(n-1)T+\Sigma_{j=1}^i \tau_{i-1},(n-1)T+\Sigma_{j=1}^i \tau_i)$ for $n=1,2,\ldots$ and $\tau_0=0$. Here we assume that $d_{\{1,2\}}$ are switched between $-1,5,10,15,20$ with period $\tau_i=0.05$ s $(i=1,2,\ldots,5)$. After a simple calculation, we get

$$\sum_{i=1}^{m}\{2a+\mu_2[(S_1^i\otimes I_n)^T+(S_1^i\otimes I_n)]\}\tau_i<-0.05. \tag{37}$$

Hence Eq. (36) is satisfied. From condition (17), the switched chain-type coupled network can be in a synchronous state. The simulation result is shown by Fig. 3(b).

D. Random coupled networks

Now we consider a simple random coupled network. The basic network is a symmetric nearest-neighbor coupled network with $N=8$, and each node is connected to its four neighborhood nodes. In this network, all the links are weighted by $d>0$. Further, for time intervals $[(n-1)T,nT)$ with $T>0$ and $n=1,2,\ldots$, only the same single link is added to the basic network. At the time instant $t=nT$, this single link is switched with certain probability. With probability p we add a link between node x_1 and node x_4, which is weighted by $-d$; with probability $1-p$, we add a link between node x_1 and node x_5, which is weighted by d.

From Eq. (3), we can easily get the DSSM $S_{1,4}$ for the link $x_1\leftrightarrow x_4$ (or $S_{1,5}$ for the link $x_1\leftrightarrow x_5$). For the limited space, here we omit these DSSMs. In this random coupled network, we choose the parameter $d=3$, the probability $p=0.5$, and the constant $T=0.02$ s. Therefore, if condition (17) is satisfied, the random coupled network can be in a synchronous state. When the nodes x_1 and x_4 are connected, we can compute that $\mu_2[(S_{1,4}\otimes I_3)^T+(S_{1,4}\otimes I_3)]=-0.4353$; when the nodes x_1 and x_5 are connected, we can compute that $\mu_2[(S_{1,5}\otimes I_3)^T+(S_{1,5}\otimes I_3)]=-7.7574$. Since this network is in effect a random network, the stability analysis for the random network using the MMA will be reported in other paper. Thesimulation result is shown in Fig. 4(b). From this figure, we conclude that the random coupled network can be in a synchronous state.

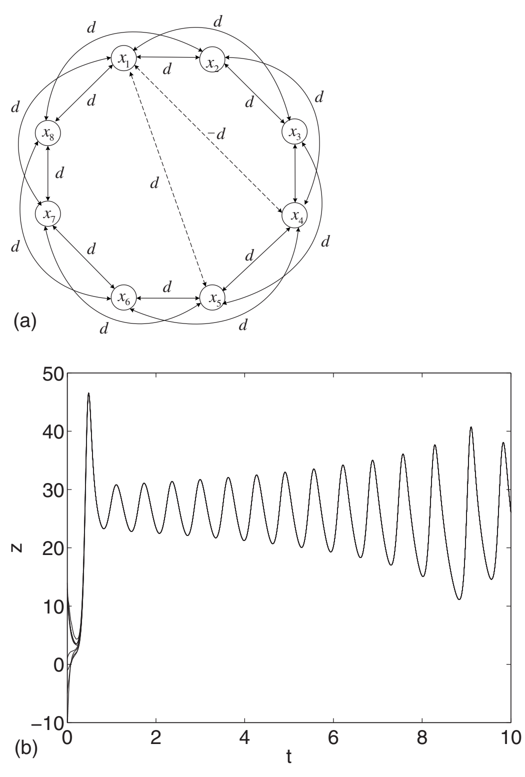

FIG. 4. The random coupled network: (a) the scheme of the network and (b) the simulation result on z states.

VII. CONCLUSION

This paper considers the synchronization in time-varying networks, in which the weights of links are time varying. A useful approach—i.e., the *matrix measure approach*—is proposed to derive less conservative synchronization conditions than many existing synchronization conditions. Theoretical analysis and numerical simulations of different networks verify our main results.

ACKNOWLEDGMENT

This work is supported by the Alexander von Humboldt Foundation.

[1] R. Albert and A. L. Barabasi, Rev. Mod. Phys. **74**, 47 (2002); S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
[2] A. Pikovsky, M. Rosemblum, and J. Kurths, *Synchronization: A universal concept in nonlinear sciences* (Cambridge University Press, Cambridge, England, 2001); S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, and C. S. Zhou, Phys. Rep. **366**, 1 (2002).
[3] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998).
[4] J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. E **50**, 1874 (1994).
[5] M. Barahona and L. M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).
[6] J. Jost and M. P. Joy, Phys. Rev. E **65**, 016201 (2001).
[7] M. Chavez, D.-U. Hwang, A. Amann, H. G. E. Hentschel, and S. Boccaletti, Phys. Rev. Lett. **94**, 218701 (2005); D. U. Hwang, M. Chavez, A. Amann, and S. Boccaletti, *ibid.* **94**, 138701 (2005).
[8] G. Rangarajan and M. Ding, Phys. Lett. A **296**, 204 (2002); Y. Chen, G. Rangarajan, and M. Ding, Phys. Rev. E **67**, 026209 (2003).
[9] J. F. Heagy, L. M. Pecora, and T. L. Carroll, Phys. Rev. Lett. **74**, 4185 (1995); V. N. Belykh, I. V. Belykh, and M. Hasler, Phys. Rev. E **62**, 6332 (2000).
[10] A. E. Motter, C. S. Zhou, and J. Kurths, Europhys. Lett. **69**, 334 (2005); A. E. Motter, C. S. Zhou, and J. Kurths, Phys. Rev. E **71**, 016116 (2005); C. S. Zhou and J. Kurths, Chaos **16**, 015104 (2006).
[11] C. W. Wu and L. O. Chua, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **42**, 775 (1995); **42**, 430 (1995).
[12] C. W. Wu, Phys. Lett. A **346**, 281 (2005); Z. J. Ma, Z. R. Liu, and G. Zhang, Chaos **16**, 023103 (2006).
[13] X. F. Wang and G. Chen, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **49**, 54 (2002); X. Li and G. Chen, *ibid.* **50**, 1381 (2003); J. Lü, X. Yu, G. Chen, and D. Cheng, *ibid.* **51**, 787 (2004).
[14] M. Chen and D. Zhou, Chaos **16**, 013101 (2006); Y. L. Zou, J. Zhu, and G. Chen, Phys. Rev. E **74**, 046107 (2006).
[15] M. Chen, IEEE Trans. Circuits Syst., II: Analog Digital Signal Process. **53**, 1185 (2006).
[16] L. Kocarev and P. Amato, Chaos **15**, 024101 (2005).
[17] A. Stefanski, J. Wojewoda, T. Kapitaniak, and S. Yanchuk, Phys. Rev. E **70**, 026217 (2004).
[18] V. N. Belykh, I. V. Belykh, and M. Hasler, Physica D **195**, 159 (2004).
[19] I. V. Belykh, V. N. Belykh, and M. Hasler, Chaos **16**, 015102 (2006).
[20] I. V. Belykh, V. N. Belykh, and M. Hasler, Physica D **195**, 188 (2004).
[21] I. Leyva, I. Sendina-Nadal, J. A. Almendral, and M. A. F. Sanjuan, Phys. Rev. E **74**, 056112 (2006).
[22] R. Amritkar and C. K. Hu, Chaos **16**, 015117 (2006).
[23] J. D. Skufca and E. Bollt, Math. Biosci. Eng. **1**, 347 (2004).
[24] D. Stilwell, E. Bollt, and D. Roberson, SIAM J. Appl. Dyn. Syst. **5**, 140 (2006).
[25] J. Lü and G. Chen, IEEE Trans. Autom. Control **50**, 841 (2005).
[26] C. W. Wu, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **52**, 282 (2005).
[27] S. I. Nicykescy, *Delay Effects on Stability: A robust control approach*, (Springer-Verlag, London, 2001).
[28] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, England, 1985).