Return times for stochastic processes with power-law scaling

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An analytical study of the return time distribution of extreme events for stochastic processes with power-law correlation has been carried out. The calculation is based on an ϵ expansion in the correlation exponent: $C(t)=|t|^{-1+\epsilon}$. The fixed point of the theory is associated with stretched exponential scaling of the distribution; analytical expressions have been provided in the preasymptotic regime. Also, the permanence time distribution appears to be characterized by stretched exponential scaling. The conditions for application of the theory to non-Gaussian processes have been analyzed and the relations with the issue of return times in the case of multifractal measures have been discussed.

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I. INTRODUCTION

Calculating the return time statistics of rare events in stochastic processes is one of the classical problems in probability theory. The applications are widespread, one of the most direct being the determination of safety margins against catastrophic events such as floods and earthquakes. In other fields, such as statistical mechanics and information theory, the statistics of return times plays an important role, being intimately connected with the way a system loses memory of its initial conditions.

Starting from the work of Döeblin [1], and that of Bellman and Harris on Markov chains [2], one of the known results is that if the system correlations decay sufficiently fast the distribution of the return times of asymptotically rare events will tend to be exponential (see [3] for recent references).

More recently, motivated by the observation that a wide variety of experimental records present long-time correlations (see [4], and references therein), there has been growing interest in the case where the stochastic process is power-law correlated, with an exponent small enough for the correlation time to be infinite. An interesting result is that, in this case, the return time distribution of extreme events appears to be well fitted by a stretched exponential, with an exponent equal to the power in the correlation decay, rather than by a simple exponential [4–7].

Actually, seen in the light of the classical work by Newell and Rosenblatt [8] on the probability of no zero crossing for power-law correlated Gaussian processes, this result is not really surprising. If the rare event is associated with the crossing of a high threshold by a stochastic variable, the return time probability will be the no-crossing probability, for an initial condition in which the variable is right below threshold.

The central idea in [8] is that threshold crossings will be less likely for processes with longer correlations. Comparing with the processes with known return time distribution (typically, a superposition of an Ornstein-Uhlembeck process and a stochastic variable), Newell and Rosenblatt were able to prove that the no zero-crossing probability for y is bounded from above and from below by stretched exponentials.

In principle, the approach in [8] could be extended to the case of a threshold different from zero, allowing one to conclude that the stretched exponential is the correct asymptotic scaling of the return probability for large values of its argument. This leaves open, however, some important questions. It would be of obvious interest to tighten the inequalities in [8], fixing the values of the prefactors in the stretched exponential scaling. It would also be interesting to have some idea of how and when (and perhaps why) the asymptotic regime is reached, and if Gaussianity is really an essential hypothesis.

The purpose of this paper is to present an analytical treatment of these issues, based on renormalized perturbation theory and an ϵ expansion in the correlation exponent: $\langle y(t)y(0)\rangle \sim |t|^{-1+\epsilon}$; this is basically an expansion around the transition to infinite correlation time. The expansion will turn out to work well also for rather large values of ϵ, providing, in the Gaussian case, valid approximate expressions for the return time distributions.

Among other things, the analysis will point out the dominance of transient behaviors, in any range of practical interest for the return times. It will also point out that, perhaps contrary to intuition, the return time distribution is less sensitive to the extreme statistics of the process than to its correlation structure, in particular that of the correlations between scales. This will allow extension of the results to rather generic non-Gaussian processes.

This paper is organized as follows. In Sec. II, the main definitions and results are recalled. In Sec. III, it is shown how the long memory of the process is associated with secular behaviors in the evolution equations for the exit probabilities. Section IV contains the main results of the paper, and analytic expressions for the return time distribution are provided in the form of a renormalized ϵ expansion. Sections V and VI focus again on the relation between the statistics of the stochastic process and that of the return times, extending the results to the case of non-Gaussian processes. Section VII is devoted to the statistics of permanence above threshold. Section VIII contains the conclusions.

II. RETURN, PERMANENCE, AND EXIT

Let us consider a unit variance and zero mean, stationary Gaussian process $y(t)$, with correlation $C(t)=\langle y(t)y(0)\rangle$ de-

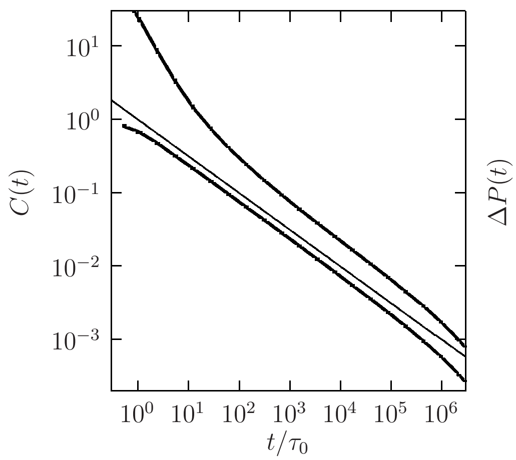

FIG. 1. Scaling of the correlation function $C(t)$ (lower line) and the probability difference $\Delta P(t)=P(y(t)>q|y(0)>q)-P(y>q)$ (upper line), for $\epsilon=0.5$ and $N=24$. The thin line is $t^{-0.5}$. The correlation function is obtained from Eq. (3) using $C(t)=\Sigma_n\langle x_n(t)x_n(0)\rangle$; $\Delta P(t)$ is obtained using this expression for $C(t)$ and Eqs. (11) and (12).

caying similar to a power law at sufficiently long time separations,

$$C(t)=\tilde{A}|t|^{-1+\epsilon}, \quad |t|>\tau_0, \quad 0\leq\epsilon<1. \tag{1}$$

This corresponds to the energy spectrum

$$C_\omega=\int dte^{i\omega t}C(t)\sim\omega^{-\epsilon}, \tag{2}$$

which is a common occurrence in several physical systems [9]. Notice also that the scaling in Eq. (1) is that of the velocity of a superdiffusive particle: $y=\dot{r}$, $\langle|r(t)-r(0)|^2\rangle\sim|t|^{2-\epsilon}$. For $\epsilon<0$, in turn, the correlation time would be $O(\tau_0)$, and $\langle|r(t)-r(0)|^2\rangle\sim|t|$, such as in the case of a Brownian particle. For $\epsilon>1$, Eq. (2) would give the spectrum of a signal with local anomalous diffusive behavior $\langle|y(t)-y(0)|^2\rangle\sim|t|^{-1+\epsilon}$.

The Gaussian process $y(t)$ could be generated numerically, e.g., by the algorithm described in [10], approximating $y(t)$ by a superposition of independent Ornstein-Uhlenbeck processes $x_n(t)$, $n=0,1,\ldots,N$. A correlation such as the one in Eq. (1) could be obtained for the correlation time and variance of x_n by setting

$$\tau_n=\tau_0 2^n \quad \text{and} \quad \sigma^2_{x_n}=\frac{(1-2^{\epsilon-1})2^{(\epsilon-1)n}}{1-2^{(N+1)(\epsilon-1)}}. \tag{3}$$

The lower line in Fig. 1 is the correlation profile obtained with this technique for $\epsilon=0.5$ and $N=24$.

We identify an extreme event by the condition $y>q$, where q is a sufficiently large threshold for the process, and introduce two occurrence times: the permanence time S_q of the variable y above threshold, and its counterpart below threshold, the return time R_q to the event, which coincides with the first exit time from $y<q$, for initial condition $y(0)=q$. The averages $\bar{R}_q$ and $\bar{S}_q$ can be related to the event probability $P(y>q)$ by means of the general relation (Kac theorem [11])

$$\bar{S}_q/\bar{R}_q\simeq P(y>q). \tag{4}$$

We are interested in the occurrence time distributions $P(S_q>t)$ and $P(R_q>t)$. We focus first on the return time distribution $P(R_q>t)$, and we can write

$$P(R_q>t)=P(y(\tau)<q,\tau\in[0,t]|y(0)=q, \tag{5}$$

which coincides with the no-exit probability for the initial condition $y(0)=q$. This probability is the integral from $\tau=0$ to $\tau=t$ of the probability current across q. If $y(t)$ obeyed a stochastic differential equation, the whole problem would reduce to a solution of a Fokker-Planck equation with absorbing boundaries at q [12].

In general, calculation of the current requires knowledge of the profile near q of the conditional probability density function (PDF) $\rho(y(t)|y(\tau)<q,\tau\in[0,t];y(0)=q)$. Exponential scaling of $P(R_q>\tau)$ would be associated with a current $-\gamma P(R_q>t)$, with γ a constant, determined by the limit form of $\rho(y(t)|y(\tau)<q,\tau\in[-\infty,t])$, when the initial condition time is sent to $-\infty$. In the case of a long correlated stochastic process, it will appear that this limit form is associated with zero current and that the approach to the limit is what generates the anomalous scaling of the return time distribution.

III. THE EFFECT OF MEMORY

The problem becomes more tractable if sampling is carried on at discrete times $t_k=k\Delta$; we then indicate $y_k=y(t_k)$ and define $A=\tilde{A}\Delta^{-1+\epsilon}$ so that $C(t_k)=A|k|^{-1+\epsilon}$. Notice that for $\Delta\gtrsim\tau_0$, the statistics would itself become Δ dependent, and, for large Δ, $\bar{S}_q(\Delta)\sim\Delta$; from Kac theorem we would then have $\bar{R}_q(\Delta)\sim\Delta/P(y>q)$. Normalizing times with respect to Δ, the discrete version of Eq. (5) will read

$$P(R_q>n)=P(y_k<q,k=1,\ldots,n|y_0>q),$$

which can be expressed in terms of the conditional probabilities

$$P(y_k<q,k=1,\ldots,n|y_0>q)=\prod_{k=1}^{n}[1-P_k(k)], \tag{6}$$

with $P_k(k)=P(y_k>q|y_l<q,l=1,\ldots,k-1;y_0>q)$ the return probability at time t_k conditioned to being a first return. More in general, we introduce the return probability conditioned to no exits before a time $t_l\leq t_n$:

$$P_l(n)=P(y_n>q|y_j<q,j=1,\ldots,l-1;y_0>q), \tag{7}$$

the return probability conditioned to exits at times $t_{k_1}<t_{k_2}<\ldots t_{k_p}$, and no exits before a time $t_l\leq t_{k_1}$,

$$P_l(n|\mathbf{k})\equiv P_l(n|y_{k_i}>q,i=1,\ldots,p). \tag{8}$$

From here, an exact recursion relation can be derived, describing the evolution of the return probability from the initial condition at $n=0$. Equations (7) and (8) provide us, in fact, with the following identities:

$$P_{n+1}(m) = P_n(m|y_n < q) = [1 - P_n(n)]^{-1} P_n(y_m > q, y_n < q)$$
$$= [1 - P_n(n)]^{-1} [P_n(m) - P_n(y_{n,m} > q)]$$
$$= [1 - P_n(n)]^{-1} [P_n(m) - P_n(n) P_n(m|n)],$$

which can be rearranged to give

$$P_{n+1}(m) = P_n(m) + \hat{P}_n(n)[P_n(m) - P_n(m|n)],$$

where $\hat{P}_n(n)=P_n(n)[1-P_n(n)]^{-1}$. In a certain sense, this is the analog for a discrete non-Markovian process of the Fokker-Planck equation with absorbing boundary conditions discussed previously, and can be iterated to give

$$P_n(m) = P_1(m) + \sum_{l=1}^{n-1} \hat{P}_l(l)[P_l(m) - P_l(m|l)]. \tag{9}$$

The physical meaning of this equation is to quantify how many of the phase points above threshold at times $l=1,\ldots,n-1$ should be subtracted from the probability mass that, without taking into account the condition of first return, would be above threshold at time m.

In order to solve Eq. (9), we need to know the function $P_l(m|l)$, which is essentially the second return probability. Repeating the same steps leading to Eq. (9) with the probability $P_l(m|\mathbf{k})$, we see that Eq. (9) is the first item in an unclosed hierarchy of equations, whose generic element reads

$$P_n(m|\mathbf{k}) = P_1(m|\mathbf{k}) + \sum_{l=1}^{n-1} \hat{P}_l(l|\mathbf{k})[P_l(m|\mathbf{k}) - P_l(m|\mathbf{k}l)], \tag{10}$$

where $\hat{P}_l(l|\mathbf{k})=P_l(l|\mathbf{k})[1-P_l(l|\mathbf{k})]^{-1}$.

A natural strategy could be, at this point, perturbation theory around the "ground states" $P_n^{(0)}(m)=P_1(m)$ and $P_n^{(0)}(m|\mathbf{k})=P_1(m|\mathbf{k})$. Notice that substituting $P_k(k)\to P_k^{(0)}(k)$ in Eq. (6) and sending $n\to\infty$ would lead to exponential scaling of the return time distribution,

$$P(R_q > n) \simeq [1 - P_1(\infty)]^n \simeq \exp(-n/\bar{R}_q),$$

where $\bar{R}_q \simeq 1/P(y>q)$ [compare with Eq. (4)], and we have exploited $P_1(\infty)=P(y>q)>0$.

The lowest-order expressions $P_n^{(0)}(m)=P_1(m)$ and $P_n^{(0)}(m|\mathbf{k})=P_1(m|\mathbf{k})$ can be calculated explicitly from PDFs in the form

$$\rho_1(n) = \frac{1}{(2\pi)^{1/2}\sigma_1(n)} \exp\left(-\frac{[y_n - \mu_1(n)]^2}{2\sigma_1^2(n)}\right), \tag{11}$$

and the similar expression for $\rho_1(n|\mathbf{k})$. For large q, in fact, the conditions $y_0, y_{k_i} > q$ in $P_1(\mathbf{k})$ and $P_1(n|\mathbf{k})$ can be replaced by $y_0, y_{k_i}=q$ and this guarantees Gaussian statistics. The conditional mean and variance in Eq. (11) can be written in the following form (see, e.g., [13], Appendix C):

$$\mu_1(n|\mathbf{k}) = q \sum_{ij} C_i D_{ij},$$

$$\sigma_1(n|\mathbf{k}) = \sum_{ij} C_i D_{ij} C_j, \tag{12}$$

where $C_i=C(n-k_i)$, D_{ij} is the inverse of the matrix $C(k_i-k_j)$, and we have defined $k_0=0$ so that now $i,j=0,1,\ldots,p$.

Unfortunately, we are going to see that the first-order correction $P_n^{(1)}(m)=\sum_{l=1}^{n-1}\hat{P}_1(l)[P_1(m)-P_1(m|l)]$ diverges for $n\to\infty$, and the same occurs with $P_n^{(1)}(m|\mathbf{k})$. As expected, exponential scaling does not appear to be an appropriate guess for the asymptotic behavior of the return time distribution.

Let us prove this. Rearrange indices so that $P_1(l)[P_1(m)-P_1(m|l)]\to\hat{P}_1(m-l)[P_1(m)-P_1(m|m-l)]$ and send $n,m\to\infty$ with $m-n$ finite, so that memory of the initial condition at $t=0$ is lost. We have in this limit

$$P_n^{(1)}(m) = \sum_{l=m-n+1}^{\infty} \hat{P}_1(\infty)[P_1(\infty) - P_1(l)].$$

The PDF $\rho_1(\infty)$ associated with $P_1(\infty)$ is just the equilibrium PDF for y, corresponding to setting in Eq. (11) $\mu=0$ and $\sigma^2=1$. From Eqs. (12) and (1) we find

$$\mu_1(l) = qAl^{-1+\epsilon}, \quad \sigma_1^2(l) = 1 - A^2 l^{-2+2\epsilon}, \tag{13}$$

and, for large l, we can Taylor expand Eq. (11),

$$\rho_1(l) = \left(\mu_1(l)y_l + \frac{1}{2}(y_l^2 - 1)[\sigma_1^2(l) - 1]\right)\rho_1(\infty).$$

Substituting into $P_1(l)=\int_q^\infty \rho_1(l)dy_l$ and calculating the integral by the Laplace method, we find

$$P_1(\infty) - P_1(l) = -(q^2/2)P_1(\infty)Al^{-1+\epsilon} + O(l^{-2-2\epsilon}), \tag{14}$$

where $P_1(\infty)\simeq(2\pi)^{-1/2}q^{-1}\exp(-q^2/2)$; this scaling can be compared with the top curve in Fig. 1. Substituting into $\sum P_1(\infty)[P_1(\infty)-P_1(l)]$, we find that the sum is divergent for $\epsilon\geqslant 0$. Notice that the leading contribution to the scaling in Eq. (14), the one that causes divergence of $P_n^{(1)}(m)$, is the slow decay of the conditional mean $\mu_1(l)$ in Eq. (13).

Let us prove that also $P_n^{(1)}(m|\mathbf{k})$ diverges as the initial condition is sent to $t=-\infty$. The summand in $P_n^{(1)}(m|\mathbf{k})$ [see Eq. (10)] reads

$$\hat{P}_1(l|\mathbf{k})[P_1(m|\mathbf{k}) - P_1(m|\mathbf{k}l)]$$
$$= P_1(m|\mathbf{k})(\hat{P}_1(l|\mathbf{k})/P_1(l|\mathbf{k}))[P_1(l|\mathbf{k}) - P_1(l|m\mathbf{k})].$$

From Eq. (12), for $k_1-l\gg m-k_1$: $\mu_1(l|\mathbf{k}),\mu_1(l|m\mathbf{k})\simeq qC(k_1-l)\Sigma_{ij}D_{ij}\sim(k_1-l)^{-1+\epsilon}$ and we find again $P_1(l|\mathbf{k}),P_1(l|m\mathbf{k})\sim[1+O(|k_1-l|^{-1+\epsilon})]P_1(\infty)$. This leads to the expression

$$\sum_{l=1}^{n-1} \hat{P}_1(l|\mathbf{k})[P_1(m|\mathbf{k}) - P_1(m|\mathbf{k}l)] \sim \sum_{l=1}^{n} l^{-1+\epsilon}, \tag{15}$$

which grows similar to n^ϵ; again, divergence is produced by the slow decay of the conditional mean $\mu_1(l)$.

IV. ϵ EXPANSION

We have seen that a "bare" perturbation expansion of Eqs. (9) and (10) around the zeroth order $P_n^{(0)}(m)=P_1(m)$ and $P_n^{(0)}(m|\mathbf{k})=P_1(m|\mathbf{k})$ leads to infinities. Some kind of renormalization is necessary; however, in order to have a workable theory, we still need, to lowest order, the equations in the hierarchy (10) to remain decoupled. This basically fixes the renormalization procedure.

We separate out of the probabilities in Eqs. (9) and (10) a renormalized part P^{R} and a remnant P^{N}, $P=P^{\mathrm{R}}+P^{\mathrm{N}}$, and expand $P^{\mathrm{R}}=\Sigma_{l=0}^{\infty}\Delta_l P^{\mathrm{R}}$, where $\Delta_0 P^{\mathrm{R}}=P_1$. The first-order renormalization to Eq. (9) reads

$$\Delta_1 P_n^{\mathrm{R}}(m)=\sum_{l=1}^{n-1}[\Delta_1 P_l^{\mathrm{R}}(l)+P_1(l)][P_1(m)-P_1(m|l)]_{\mathrm{sec}},$$

where $[P_1(m)-P_1(m|l)]_{\mathrm{sec}}$ contains the contribution leading to divergence of $P_n^{(1)}(m)$, and, seeking an analogy with quantum field theory (see also [14]), $\Delta_1 P_l^{\mathrm{R}}(l)$ could loosely be seen as a counterterm. Analogous expressions are obtained for $\Delta_1 P_n^{\mathrm{R}}(m|\mathbf{k})$ and Eq. (10). Combining with $P_1(m)$, we obtain the lowest-order equation for the renormalized probability

$$P_n^{\mathrm{R}}(m)=P_1(m)+\sum_{l=1}^{n-1}P_l^{\mathrm{R}}(l)[P_1(m)-P_1(m|l)]_{\mathrm{sec}}. \quad (16)$$

The pth order renormalization can be expressed in the form

$$\Delta_p P_n^{\mathrm{R}}(m)=\sum_{l=1}^{n-1}\{\Delta_p P_l^{\mathrm{R}}(l)[P_l(m)-P_l(m|l)]_{\mathrm{sec}}^{(p-1)} + P_l^{\mathrm{R},p}(l)\Delta_{p-1}[P_l(m)-P_l(m|l)]_{\mathrm{sec}}\}, \quad (17)$$

and we use here a superscript to indicate the order at which each expression is considered: $P^{\mathrm{R},p}=P_1+\Sigma_{l=1}^{p}\Delta_l P^{\mathrm{R}}$ and $[\cdots]_{\mathrm{sec}}^{(p-1)}=\Sigma_{l=0}^{p-1}\Delta_l[\cdots]_{\mathrm{sec}}$. Equation (17) can be rewritten in the more concise form, which generalizes Eq. (16):

$$P_n^{\mathrm{R},p}(m)=P_1(m)+\sum_{l=1}^{n-1}P_l^{\mathrm{R},p}(l)[P_l(m)-P_l(m|l)]_{\mathrm{sec}}^{(p-1)}.$$

Again, analogous expressions hold for $\Delta_p P_n^{\mathrm{R}}(m|\mathbf{k})$ and Eq. (10). From inspection of Eqs. (10) and (17), we see that, in order to renormalize $P_n(m)$ to order p, we have to solve the first p renormalized equations in the hierarchy (10), with the second equation solved to order $p-1$, the third to order $p-2,\ldots$, the pth to first order.

Turning to the remnant, P^{N} will contain, order by order, corrections in the form $\hat{P}_l^{\mathrm{R}}(l)-P_l^{\mathrm{R}}(l)$ and $[P_1(m)-P_1(m|l)]-[P_1(m)-P_1(m|l)]_{\mathrm{sec}}$, which do not lead to divergence in Eqs. (16) and (17) and their counterparts for Eq. (10).

Pursuing the analogy with quantum field theory, we see that $\epsilon=0$ plays a role analogous to the upper critical dimension, which suggests to us to calculate the renormalized probability using an ϵ-expansion approach. We choose to keep in $[\cdots]_{\mathrm{sec}}$ only the scaling part, and, substituting Eq. (14) into (16), we obtain

$$\Delta_1 P_n^{\mathrm{R}}(m)=-\frac{1}{2}q^2AP_1(\infty)\sum_{l=1}^{n-1}P_l^{\mathrm{R}}(l)(m-l)^{-1+\epsilon}. \quad (18)$$

For $n\to\infty$, we approximate the sum by an integral; defining $f(z)=P_k^{\mathrm{R}}(k)$ with $k=n(1-z)$, we write

$$\sum_{l=1}^{n-1}P_l^{\mathrm{R}}(l)(m-l)^{-1+\epsilon}\simeq n^{\epsilon}\int_{z_{\min}}^{z_{\max}}dzf(z)[m/n-1+z]^{-1+\epsilon},$$

where $z_{\min}=1/n$, $z_{\max}=1-1/n$, and the factor n^{ϵ} comes from $n^{-1+\epsilon}/\Delta z$ with $\Delta z=1/n$ the discrete increment in the integral. The interesting case is $m=n$. Integrating by parts and expanding in ϵ,

$$n^{\epsilon}\int_{z_{\min}}^{z_{\max}}dzf(z)z^{-1+\epsilon}=\epsilon^{-1}(n^{\epsilon}-1)P_n^{\mathrm{R}}(n)+O(\epsilon).$$

Substituting into Eq. (18) and then into Eq. (16), we obtain the result

$$P_n^{\mathrm{R}}(n)=\left[1+\frac{Aq^2P_1(\infty)}{2\epsilon}(n^{\epsilon}-1)\right]^{-1}P_1(n), \quad (19)$$

and, sending $n\to\infty$,

$$P_n^{\mathrm{R}}(n)=\bar{P}n^{-\epsilon}+O(\epsilon^2), \quad (20)$$

where $\bar{P}=2\epsilon/(Aq^2)$. Notice again the analogy with quantum field theory, with $\bar{P}$ behaving similar to the fixed point value of a renormalized coupling constant. Substituting into Eq. (6), we obtain the stretched exponential scaling for the return time PDF,

$$P(R_q>n)\propto\exp(-\bar{P}n^{1-\epsilon}). \quad (21)$$

The expectation that the result in [8] extends to the return time statistics is therefore confirmed, and we have gained knowledge of the prefactor in the exponent. Conversely, in the range $\epsilon<0$, no divergence would have arisen in Eqs. (14) and (15), so that no renormalization would have been necessary. Hence, the bare perturbation theory would have been appropriate, with the ground state $P_n^{(0)}(n)\to P_1(\infty)$ leading to exponential scaling for the return time distribution.

We can use Eqs. (6) and (19) to study the approach to the asymptotic regime (21). In this transient regime, it is appropriate to use the expression for $P_1(n)$ that is obtained from Eqs. (3) and (12) rather than from the asymptotic formula (13). We see in Fig. 2 that the analytical approximation does a good job even up to $\epsilon=0.5$, which is a consequence of the fact that the first correction to $\Delta_1 P_n^{\mathrm{R}}(n)$ arises only at $O(\epsilon)$.

Is this stretched exponential scaling? Actually it is possible to fit $P(R_q)$ with stretched exponentials, but the fit does not match Eq. (21). Using Eq. (3), in the two cases $\epsilon=0.5$ and $\epsilon=0.2$, we evaluate $A\simeq 0.5$ and $A\simeq 0.4$; for $q=3$, this would correspond to $\bar{P}\simeq 0.2$ in both cases, which goes from a factor of 2 to two orders of magnitude away from the fits in Fig. 2.

What is happening is that Eq. (19) reaches its asymptotic limit (20) only for $n\gtrsim[\bar{P}/P_1(\infty)]^{1/\epsilon}$, which grows very rapidly with q and $1/\epsilon$. To have an idea, for $\epsilon=0.2$ and $q=3$, we

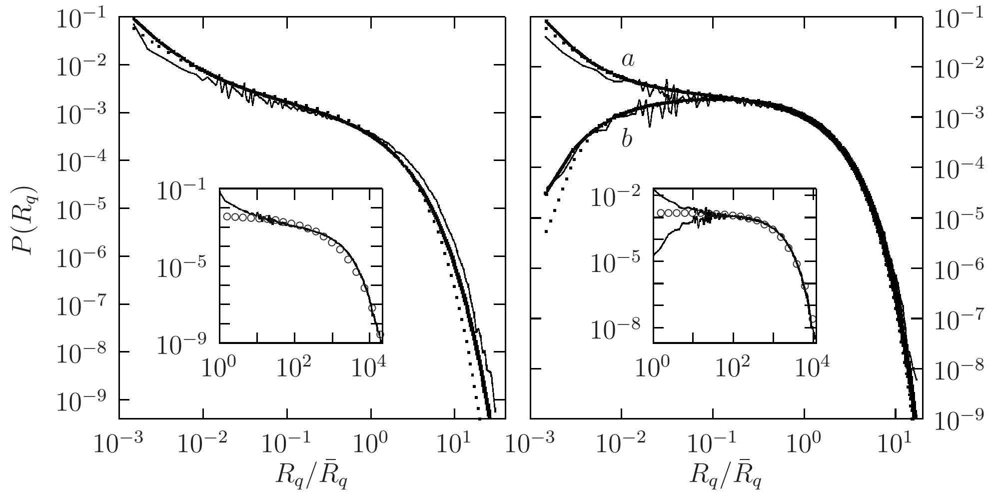

FIG. 2. Return time probability for $\epsilon=0.5$ (left) and $\epsilon=0.2$ (right, curve a). Curve b on the right refers to exit from the initial condition $y(0)=-q$, again for $\epsilon=0.2$. In all cases $\Delta=2$ and $q=3$, corresponding to $\bar{R}_q \simeq [P_1(\infty)]^{-1} \simeq 677$. Thin lines: numerical integration of $y(t)=\Sigma_n x_n(t)$, using Eq. (3) with $N=24$. Dotted lines: theory using Eqs. (6) and (19). Heavy lines: theory using Eqs. (26) and (27). Inset: same results without rescaling, superimposed with the stretched exponential fits (circles): $0.004\exp(-0.1R_q^{0.5})$ and $0.0016\exp(-0.007R_q^{0.8})$.

would have $P_1(\infty)\simeq 0.0015$ and $[\bar{P}/P_1(\infty)]^{1/\epsilon}\sim 10^{10}$. Clearly such long return times would occur only with vanishingly small probability.

Some idea of the higher orders in the perturbative expansion could be obtained by studying Eq. (17) for $p=2$. We focus on the case $m=n$ and start by analyzing whether the second line in Eq. (17), $\Delta_1[P_l(n)-P_l(n|l)]$, gives a secular contribution. From Eq. (16), we thus have to evaluate

$$\Delta_1 P_l(n) = \sum_{j=1}^{l-1} \hat{P}_j^{\mathrm{R}}(j)[P_1(n) - P_1(n|j)]_{\mathrm{sec}},$$

$$\Delta_1 P_l(n|l) = \sum_{j=1}^{l-1} \hat{P}_j^{\mathrm{R}}(j|l)[P_1(n|l) - P_1(n|lj)]_{\mathrm{sec}}. \qquad (22)$$

From Eq. (14) and the argument leading to Eq. (15), we see that $[P_1(n)-P_1(n|j)]_{\mathrm{sec}}\sim|n-j|^{-1+\epsilon}$ and $[P_1(n|l)-P_1(n|lj)]_{\mathrm{sec}}\sim|l-j|^{-1+\epsilon}$. We know from Eq. (20) that $P_j^{\mathrm{R}}(j)\sim j^{-\epsilon}$; we still need $P_j^{\mathrm{R}}(j|l)$. We can repeat the steps from Eqs. (18) and (19), substituting $\Sigma_{j=1}^{l-1}P_j^{\mathrm{R}}(j|l)(l-1)^{-1+\epsilon}$ in the sum of Eq. (18) and the final result is again

$$P_j^{\mathrm{R}}(j|l) \sim j^{-\epsilon}.$$

Indicating $j=n(1-z)$ and $f(z)=P_j^{\mathrm{R}}(j),P_j^{\mathrm{R}}(l|j)$, the leading order behavior for $l/n\to 0$ of the sums in Eq. (22) is therefore

$$l^{\epsilon}\int_0^1 f(z)[n/l-1+z]^{-1+\epsilon} \simeq ln^{-1+\epsilon}.$$

We see that no divergences are present at small l in the sum

$$\sum_{l=1}^{n-1} P_l^{\mathrm{R}}(l)\Delta_1[P_l(n) - P_l(n|l)], \qquad (23)$$

so that $[P_l(n)-P_l(n|l)]_{\mathrm{sec}}=0$ and no renormalizations are necessary to this order [see Eq. (17)].

We consider now the remnant, which receives contributions at both orders $p=1$ and $p=2$. The only contribution surviving for $n\to\infty$ turns out to be that at order $p=2$, produced by the sum in Eq. (23). The sum is dominated by $l\to n$; hence $\Delta_1 P_l(n)\sim P_n^{\mathrm{R}}(n)-P_1(n)\sim -P_1(n)$, $\Delta_1 P_{n-1}(n|n-1)\sim -P_1(n|n-1)$, and, for sufficiently large Δ, $P_1(n|n-1)\gg P_1(n)$. For $n\to\infty$, $P_1(n|n-1)=P_1(1)$ and we estimate

$$P_n^{\mathrm{N}}(n) \sim P_n^{\mathrm{R}}(n)P_1(1). \qquad (24)$$

Thus, the validity of the renormalized expansion rests on the smallness of the exit probability after one step, $\lambda=P_1(1)$, which behaves similar to an expansion parameter for the theory beside ϵ. In order for the theory to work, it is then necessary that the sampling constant Δ be sufficiently large. However, this appears to be a rather weak constrain, as, already for $\Delta=1$, $q=3$ and $\epsilon=0.5$, $\lambda\simeq 0.15$.

Substituting into Eq. (17) for $p=3$, we see that $P_n^{\mathrm{N}}(n)$ contributes to $\Delta_2[P_l(n)-P_l(n|l)]_{\mathrm{sec}}$ and to the renormalization of $P_n(n)$, while terms such as $P_l(lji)$ contribute to $P_l^{\mathrm{N}}(n|l)$ with the same mechanism that leads to Eq. (24). This suggests that the higher-order renormalizations to $P_n(n)$ are $O(\lambda^n)$ corrections to prefactors in Eqs. (19)–(21), while the exponent in Eq. (20) should remain invariant. This exponent depends in fact only on the part of $P_l(n|\mathbf{k})-P_l(n|\mathbf{k})|_{t_0\to-\infty}$ with the slowest decay, which is $\propto|t_0|^{-1+\epsilon}$ for all p [see the discussion leading to Eq. (15)].

V. PDF STRUCTURE

The key element of the analysis carried out so far is that the conditioned return probability $P_n^{\mathrm{R}}(n)\simeq\bar{P}n^{-\epsilon}$ [see Eq. (20)] goes to zero for $n\to\infty$. Recalling the discussion at the end of Sec. II, it is this behavior, in contrast to that produced by bare perturbation theory $P_n(n)\to P_1(\infty)>0$, that leads to the anomalous scaling of the return time distribution. This means that the PDF $\rho_n^{\mathrm{R}}(n)$ determining $P_n^{\mathrm{R}}(n)$ through the relation $P_n^{\mathrm{R}}(n)=\int_q^{\infty}\rho_n^{\mathrm{R}}(n)dy_n$ must have vanishing tails at $y_n>q$. The fact that for $n\to\infty$, $\rho_n^{\mathrm{R}}(n)\neq\rho(y_n)$ is the signature of the long memory of the process.

An equation for the PDF $\rho_n(n)$ could be derived repeating the steps leading to Eq. (9),

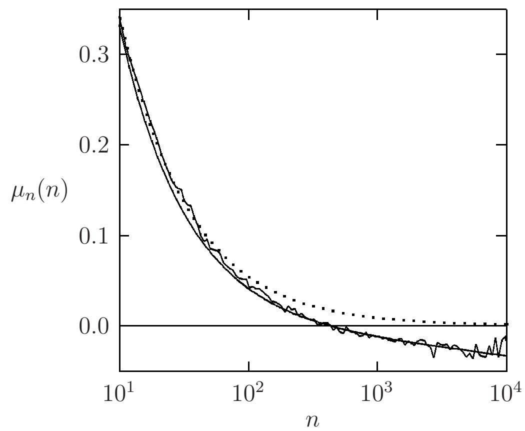

FIG. 3. Scaling for the conditional mean $\mu_n(n)$ for $q=3$, $\epsilon=0.2$, and $\Delta=2$. Dotted line: "bare result" $\mu_1(n)$; heavy line: renormalized result $\mu_n^{\mathrm{R}}(n)$ using Eq. (26); thin line: numerical simulation using the same parameters of Fig. 2.

$$\rho_n(m) = \rho_1(m) + \sum_{l=1}^{n-1} \hat{P}_l(l)[\rho_l(m) - \rho_l(m|l)],$$

with obvious definitions for the various PDFs appearing in the formula. Taking moments, we obtain equations for the conditional mean

$$\mu_n(m) = \mu_1(m) + \sum_{l=1}^{n-1} \hat{P}_l(l)[\mu_l(m) - \mu_l(m|l)], \tag{25}$$

and similarly for the higher moments of $\rho_n(n)$. The perturbative analysis of Eq. (25) is identical to that of Eq. (9). Using Eqs. (12) and (13), we see that the same pattern of divergences is produced,

$$[\mu_1(n) - \mu_1(n|l)]_{\mathrm{sec}} = -qA|n-l|^{-1+\epsilon},$$

and this confirms the role of the conditional mean $\mu_1(n)=\langle y_n|y_0>q\rangle$ in the renormalization procedure. Generalizing Eq. (25) to the higher moments $M_{n,p}(m)=\int\rho_n(m)y_m^p dy_m$, in fact, it is possible to see that $M_{1,p}(n)-M_{1,p}(n|l)\sim|n-l|^{-p(1-\epsilon)}$ and for $\epsilon<0.5$, the higher moment equations do not have divergent behaviors.

Using Eq. (25), it is possible to renormalize Eq. (24) using the same procedure leading from Eqs. (18)–(20). This leads to the result

$$\mu_n^{\mathrm{R}}(n) = \mu_1(n) - (q/\epsilon)(n^\epsilon - 1)P_n^{\mathrm{R}}(n) \to -2/q, \tag{26}$$

which is confirmed in Fig. 3 [as in Fig. 2 with $P_1(n)$, the expression from Eq. (12) is adopted here for $\mu_1(n)$]. Notice the constant value for $n\to\infty$ of $\mu_n^{\mathrm{R}}(n)$, much larger than the value $-qP(y>q)$ that would be obtained subtracting from the equilibrium PDF the values of y above threshold.

Clearly, knowledge of the conditional mean $\mu_n^{\mathrm{R}}(n)$ is not sufficient by itself to guarantee vanishing PDF tails at $y>q$. However, using such a simple approximation for $\rho_n(n)$ as

$$\rho_n(n) \simeq (2\pi)^{-1/2}\exp\{-[y_n - \mu_n^{\mathrm{R}}(n)]^2\}, \tag{27}$$

with $\mu_n^{\mathrm{R}}(n)$ given by Eq. (26), and substituting into $P_n(n)=\int_q^\infty \rho_n(n)dy_n$ and in Eq. (6), produces results in Fig. 2 in some case better than from Eq. (19). One reason for this is the better scaling properties of $\mu_1(n)-\mu_1(n|l)\simeq-\mu_1(n-l)$ as compared with $P_1(n)-P_1(n|l)$ (see Fig. 1).

VI. THE NON-GAUSSIAN CASE

We have seen that the scaling of the return time distribution depends on the tails of the conditioned PDF $\rho_n(n)$. These in turn are determined by the behavior of a bulk quantity such as the conditional mean $\mu_1(n)=\langle y_n|y_0>q\rangle$. It seems therefore that it is not the extreme statistics of the process, rather, its correlation structure that determines the return time distribution. It is natural at this point to question whether Gaussian statistics is strictly necessary for (quasi)stretched exponential scaling.

We examine the conditions under which $\rho_1(\infty)-\rho_1(l)$ and therefore also the difference $P_1(\infty)-P_1(l)$ in Eq. (14), scale similar to $\mu_1(l)\sim l^{-1+\epsilon}$. We provide a sufficient condition for this scaling in the form of a requirement of weak correlation between scales in the process.

Let us write $y_0=x_0+z$ and $y_l=x_l+z$, with z the result on y of some low-pass filtering at scale l in the region in exam. From Eqs. (1) and (2), we have, for large l: $\langle z^2\rangle\sim l^{-1+\epsilon}$. We can obtain $\rho_1(l)$ and $\rho_1(\infty)$ from $\rho(y)$ and $\rho(y,y';l)$, which are respectively the equilibrium PDF for y and the joint PDF that $y_l=y$ and $y_0=y'$. Indicating by $\rho_<$ and $\rho_>$ the PDFs for the low-pass filtered signal z and for $x=y-z$, we can write

$$\rho(y,y',l) = \int dz \rho_<(z)\rho_>(x,x';l|z),$$

$$\rho(y') = \int dz \rho_<(z)\rho_>(x'|z). \tag{28}$$

We can introduce a function $g(y,y',z,l)$ parametrizing the correlation between the small scale components $x_l=y_l-z$ and $x_0=y_0-z$:

$$\rho_>(x,x';l|z) = [1 + g(y,y',z,l)]\rho_>(x|z)\rho_>(x'|z).$$

If correlation between scales are weak and l is large, we can Taylor expand $\rho_>(x|z)$ and $\rho(x'|z)$ around $z=0$ and consider $g(y,y',z,l)$ a small quantity. With these substitutions, Eq. (28) becomes

$$\rho(y,y';l) \simeq \rho_>(y|0)\rho_>(y'|0)[1 + \langle g|y,y';l\rangle + A\langle z^2\rangle],$$

$$\rho(y') \simeq \rho_>(y'|0)[1 + B\langle z^2\rangle],$$

where $A=A(y,y')$ and $B=B(y')$. From here, we obtain finally

$$\rho(y|y';l) \simeq \rho(y)[1 + \langle g|y,y';l\rangle + (A-B)\langle z^2\rangle]. \tag{29}$$

The two contributions to $\rho_1(l)-\rho_1(\infty)\simeq\rho(y|y';l)-\rho(y)$ are deeply different in nature. The term $(A-B)\langle z^2\rangle$ is the direct

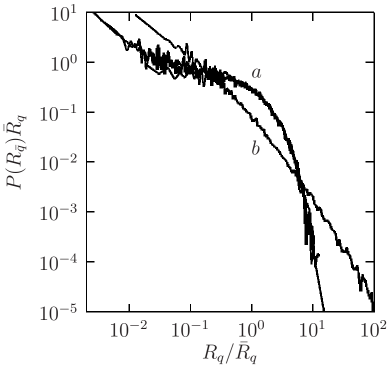

FIG. 4. Return time probability for different non-Gaussian processes. In all cases, $\epsilon=0.2$ and $\Delta=1$. Case a: three nonmultifractal signals with different types of non-Gaussianity; case b: multifractal signal.

additive contribution from the long time-scale fluctuations, while $\langle g|y,y';l\rangle$ probes long time correlations of the small scale fluctuations. The last contribution and the direct coupling between fluctuations at different scales [15] are typically associated with a multifractal structure of the signal [16]. Dominance of the additive contribution $\langle z^2\rangle \sim l^{-1+\epsilon}$ indicates therefore absence of multifractal properties in the signals. The difference $P_1(\infty)-P_1(l)$ in Eq. (14) will behave in this case as if y were Gaussian, and $P(R_q)$ should become a stretched exponential in the large R_q limit.

The fact that fractal objects are not processes with a stretched exponential distribution of return times is not a surprise. This is easy to see in the case of a middle-third Cantor set, in which the return times can be identified with the "holes" in the measure: at the nth generation there are 2^{n-1} holes of length $R_q=3^{-n}$ and this gives the power-law distribution $P(R_q)\propto R_q^{-D}$ with $D=\ln 2/\ln 3$ the fractal dimension of the set. Actually, there have been some recent attempts to characterize multifractal sets by a return time spectrum, beside the more standard singularity and dimension spectra [17,18].

The prediction that the return time statistics is dominated by the correlation structure of the process is confirmed by numerical simulation of non-Gaussian power-law correlated processes, as illustrated in Fig. 4. The four curves in the figure are all characterized by the same power-law correlation with $\epsilon=0.2$, but are generated with different forms of intermittency in the subprocesses $x_n(t)$ in $y(t)=\Sigma_n x_n(t)$. The intermittency is generated letting the noise amplitude in the Langevin equation governing each process fluctuate on the time scale of the process,

$$\dot{x}_n(t) = -\tau_n x_n(t)^{-1} + b_n(t)\xi(t), \quad \langle \xi(t)\xi(0)\rangle = \delta(t),$$

with $\langle b_n^2\rangle \simeq 2\sigma_{x_n}^2/\tau_n$. In the multifractal case, the noise amplitude fluctuations are generated by a multiplicative process of the kind utilized in [16]. In the other cases, the noise amplitude fluctuations are independent. In the first case, the fluctuations are tuned to produce the same kurtosis $\langle y^4\rangle \simeq 12.5$ as in the fractal case. In the second case, the intermittency grows with scale [which would produce nontrivial scaling of the higher diffusion exponents for $r(t)=\int_0^t y(\tau)d\tau$]. In the third case, the signal is Gaussian. As expected, the return time distributions of the nonmultifractal signals collapse on one another if one rescales with $\bar{R}_q$, while the multifractal one leads to a distribution that is closer to a power law.

VII. PERMANENCE TIME DISTRIBUTION

Let us conclude the analysis turning to the permanence times and verifying that their distribution is characterized by stretched exponential scaling, as an extension of the theory in [8] would suggest. The analysis is limited to the Gaussian case. The analog of Eq. (5) in the case of the permanence times S_q reads

$$P(S_q > t) = P^{-1}(y > q)P(y(\tau) > q, \tau \in [0,t]),$$

where the factor $P^{-1}(y>q)$ gives the condition that the stochastic variable is initially above threshold. Let us isolate in $y(\tau)$ its average in $[0,t]$: $y(\tau)=z+x(\tau)$, where $z=t^{-1}\int_0^t y(\tau)d\tau$. We then obtain

$$P(S_q > t) = P^{-1}(y > q) \times \int_q^\infty dz \rho_<(z) P(x(\tau) > q - z, \tau \in [0,t]|z). \tag{30}$$

The PDF for z, for large t, is obtained eliminating frequencies $|\omega|>t^{-1}$ in the power spectrum for y; from Eq. (2): $\langle z^2\rangle \sim \int_{|\omega t|<1} C_\omega d\omega \sim t^{-1+\epsilon}$. In the same limit, the condition on z in $P(x(\tau)>q-z, \tau\in[0,t]|z)$ can be disregarded for Gaussian statistics.

To evaluate Eq. (30), we consider the simpler problem of discrete sampling in time: $t\to t_n=n\Delta$. This allows us to write

$$1 > P(x(\tau) > q - z, \tau \in [0,t]) > [P(x > q - z)]^{t/\Delta}.$$

Substituting into Eq. (30), the first inequality allows us to write, to leading order in q and t,

$$P(S_q > t) < \exp(-Kq^2 t^{1-\epsilon}), \tag{31}$$

where we have estimated $\rho_<(z)\sim \exp(-Kz^2 t^{1-\epsilon})$ and then, for large q, $P(z>q)\sim \exp(-Kq^2 t^{1-\epsilon})$.

Passing to the second inequality, making the substitution in Eq. (30): $P(x(\tau)>q-z, \tau\in[0,t])\to [P(x>q-z)]^{t/\Delta}$, the integrand in that formula will take the form

$$\exp\{-Kz^2 t^{1-\epsilon} + (t/\Delta)\ln[1 - P(x < q - z)]\}.$$

For $t/\Delta\to\infty$, the contribution to the integral will be from values of z for which $\ln[1-P(x<q-z)$ is small, i.e., $z-q$ is large; we can approximate

$$\ln[1 - P(x < q - z)] \sim -\exp(-K(q - z)^2 t^{1-\epsilon}).$$

The integrand in Eq. (30) will then take the form

$$\exp\{-Kz^2 t^{1-\epsilon} - K' t \exp(-K(q - z)^2 t^{1-\epsilon})\},$$

which is peaked, for $t\to\infty$, at $z\simeq q+\sqrt{\epsilon \ln t}$. Estimating the integral in Eq. (30) by the steepest descent gives the result

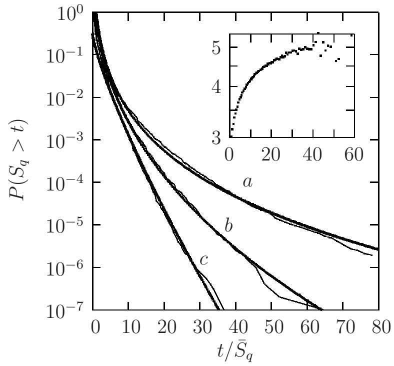

FIG. 5. Permanence time distribution from numerical simulation of a power-law correlated time series (thin lines) and stretched exponential fit (heavy lines) for $q=3$ and three different values of ϵ: (curves a) $\epsilon=0.8$; (curves b) $\epsilon=0.4$; (curves c) $\epsilon=0.2$. Inset: scaling of $\langle y_{\mathrm{MAX}}|S_q\rangle$ vs S_q for $\epsilon=0.4$.

$\exp(-K\epsilon t^{1-\epsilon}\ln t)$. Combining with Eq. (31), we obtain the bound, valid to leading order in q and t,

$$\exp(-K\epsilon t^{1-\epsilon}\ln t) < P(S_q > t) < \exp(-Kq^2 t^{1-\epsilon}), \quad (32)$$

which is similar in form to the one in [8]. Contrary to the case of the return times, the value of the sampling constant Δ is not crucial to the theory. This is confirmed by the fact that, in the present limit, all dependence on Δ, accounted for by K', disappears. We compare in Fig. 5 with the result of numerical simulation using Eq. (3) and see that stretched exponential scaling is compatible with the permanence time distributions in the range considered.

Notice that replacing the upper bound in Eq. (32) with equality would imply

$$P(S_q > t) \sim P(z > q) \sim P(y > qt^{1-\epsilon}),$$

in other words, the probability of a single peak of height $qS_q^{1-\epsilon}$ would be the same as that of a permanence S_q. This is not surprising: given an initial condition $y(0) \gg q \gg 1$, from Eqs. (11) and (13), $\rho(y(t)|y(0))$ would be narrowly peaked around $y(0)\tilde{A}t^{-1+\epsilon}$ and the time it takes to $y(t)$ to go below q would be $t \sim (\tilde{A}y(0)/q)^{1/(1-\epsilon)}$. Substituting $y(0)=qS_q^{1-\epsilon}$, we obtain precisely $t \sim S_q$. As confirmed also in Fig. 5, one expects, therefore, that longer permanences above threshold, be associated with higher peaks.

VIII. CONCLUSION

The analysis carried out in this paper confirms the observation in [4–6] (among others) that long correlations in stochastic processes lead to return time distributions with scaling close to stretched exponential. Similar properties, albeit with different mechanisms, are confirmed for the permanence time distribution [see Eq. (32)].

These results are consistent with the extension to thresholds different from zero, of the bounds derived in [8] on no zero-crossing probabilities. The same analysis also suggests that what can be observed in experimental time series is only a very slow transient regime. The stretched exponential scaling predicted in [8] is achieved only as an asymptotic limit, requiring exceedingly long return times and vanishing probabilities. However, as the theory is based on an ϵ expansion, it is not ruled out that the asymptotic limit may occur earlier in the range $\epsilon \to 1$, where new physics may become important.

From the practical point of view, it is probably irrelevant whether a return time distribution that can be fitted by a stretched exponential is really a stretched exponential. More important, the present theory provides approximate expressions for the return time distribution, valid for any value of the argument and working well up to $\epsilon \simeq 0.5$, i.e., the middle of the range considered [see Eqs. (6) and (20) or (26) and (27)]. The theory is limited to discrete sampling and is actually in the form of a double expansion in ϵ and the parameter $\lambda = P(y_1 > q | y_0 > q)$, which is heavily dependent on the sampling constant Δ [see Eq. (24)]. However, for large enough q, in the range of ϵ in which the theory works, considering Δ in the scaling range for the correlation $C(t)$ appears to be sufficient.

Another important fact is that, although the approximate expressions in the present paper are derived in the Gaussian case, they continue to be valid for non-Gaussian processes, provided one rescales the relevant quantities by the mean return time $\bar{R}_q$: $R_q \to R_q/\bar{R}_q$ and $P(R_q) \to P(R_q)\bar{R}_q$. Basically, only multifractal processes are excluded. (The importance in this context of rescaling by $\bar{R}_q$ was pointed out in [4]).

The mechanism for the nonexponential scaling of the return time distribution appears to be that trajectories originating from an above-threshold event are distributed around a mean $\langle y(t)|y(0)>q\rangle$ that decays very slowly with time. When imposing that the trajectories remain below threshold up to the return time R_q, loosely speaking, this slow decay produces correlations between the conditions of no exit at different times that cannot be treated as independent. The important point is that, as long as no slower scaling quantities are characterizing the process (due, e.g., to multifractality), this mechanism will be insensitive to whether or not the process is Gaussian [see Eq. (29) and following discussion]. This insensitivity on the tail structure of the statistics was recently observed in [19].

From the conceptual point of view, it is interesting that the cumulative effect of the correlation between the below-threshold conditions becomes manifest through secular behaviors in the equations for the evolution of the trajectory distributions. It is also interesting that the most natural way to treat these behaviors is renormalization, with a strategy similar to [14]. In particular, the stretched exponential scaling limit appears to be associated with the fixed point of the renormalized theory. The range $\epsilon < 0$, corresponding to a process with finite correlation time and exponential scaling for the return distribution, conversely, is associated with a trivial theory that does not require renormalization.

It is to be noted that equations similar to the ones considered in the present paper arise in the context of avalanche models [20], where they have been treated as well within an ϵ-expansion approach.

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