Onset of spatiotemporal chaos in a nonlinear system

J. D. Szezech, Jr., S. R. Lopes, and R. L. Viana
Departamento de Física, Universidade Federal do Paraná, Caixa Postal 19081, 81531-990, Curitiba, Paraná, Brazil
(Received 16 November 2006; revised manuscript received 23 April 2007; published 21 June 2007)

We describe the onset of spatiotemporal chaos in a spatially extended nonlinear dynamical system as a result of the loss of transversal stability of an invariant manifold representing a spatially homogeneous and temporally chaotic state. The onset of spatiotemporal chaos is characterized by the switching between spatially homogeneous and nonhomogeneous states with statistical properties of on-off intermittency.

DOI: 10.1103/PhysRevE.75.067202 PACS number(s): 05.45.Jn, 47.27.Cn, 52.35.Mw

Describing the onset of spatiotemporal chaos in fluids, plasmas, and other spatially extended dynamical systems is an outstanding problem in theoretical physics [1]. An extensively investigated model characterizes the onset of spatiotemporal chaos as a result of a sequence of Hopf bifurcations [2]. While this route has been verified by experiments in fluids, plasmas, and electronic circuits [3], there are situations for which the onset of spatiotemporal chaos is thought to be caused by other mechanisms [4]. On the other hand, purely temporal chaotic behavior in low-dimensional systems is better understood, such that we might ask to what extent it would be possible to use our understanding of low-dimensional chaotic systems to interpret the onset of spatiotemporal chaos in terms of the excitation of a few spatial modes drawing energy from a purely temporal chaotic mode [5].

In this paper we propose that the onset of spatiotemporal chaos in a class of nonlinear systems can be explained from the loss of transversal stability of a spatially homogeneous chaotic state, a mechanism not directly related to the breakup of some high-dimensional torus [6]. In order to support this claim, we present numerical results obtained from a physical model consisting of three waves undergoing nonlinear interactions. Such systems occur in a plethora of problems in fluid dynamics [7], plasma physics [8], and nonlinear optics [9]. Three-wave interactions are mathematically described by a system of coupled-mode nonlinear partial differential equations which exhibit complex behavior [10]. The nonlinear three-wave model describes the exchange of energy among a high-frequency (parent) wave and its sideband (daughters) with quadratic interactions, as well as with a spatial diffusion term. This system is known to present a spatiotemporal chaos for certain parameter intervals [12,14], and we claim that such a scenario can be regarded as stemming from the spatial-mode instability of an invariant manifold for which the dynamics is temporally chaotic but spatially homogeneous. Moreover, this instability causes, just after the transition to spatiotemporal chaos, an intermittent switching between the spatially homogeneous and nonhomogeneous states with the same statistical properties of the so-called on-off intermittency [15].

We used (complex) amplitudes A_α, $\alpha=1,2,3$, to describe monochromatic waves propagating along the x direction, where A_1 stands for the parent wave amplitude, A_2 and A_3 being the corresponding quantities for the faster and slower daughter waves, respectively. Their wave numbers and frequencies must satisfy matching conditions for the triplet, $\mathbf{k}_3=\mathbf{k}_1+\mathbf{k}_2$ and $\Omega_{\mathbf{k}_3}=\Omega_{\mathbf{k}_1}-\Omega_{\mathbf{k}_2}-\delta$, where δ is a small mismatch. Supposing that the nonlinearities are sufficiently weak, such that only quadratic terms in the wave amplitudes need to be considered, the equations of the three-wave model, with energy injection, dissipation, and a diffusive term, read [10,12]

$$\frac{\partial A_1}{\partial t}+v_{g1}\frac{\partial A_1}{\partial x}=A_2A_3+\gamma A_1+D\frac{\partial^2 A_1}{\partial x^2}, \tag{1}$$

$$\frac{\partial A_2}{\partial t}+v_{g2}\frac{\partial A_2}{\partial x}=-A_1A_3^*+\nu A_2, \tag{2}$$

$$\frac{\partial A_3}{\partial t}+v_{g3}\frac{\partial A_3}{\partial x}=i\delta A_3-A_1A_2^*+\nu A_3, \tag{3}$$

where each wave is assumed to have a constant group velocity $v_{g\alpha}=d\Omega_{\mathbf{k}_\alpha}/dk_\alpha$, given by the dispersion relation of the specific wave to be considered, such that $v_{g2}>v_{g1}>v_{g3}$ [10]. The coefficients $\gamma>0$ and $\nu<0$ are introduced phenomenologically to represent energy injection (through wave 1) and dissipation (through waves 2 and 3), respectively [11], and D is a diffusion coefficient that provides a cutoff in the wave growth, being essential to nonlinear saturation [12].

We choose a configuration where a parent wave has a positive linear growth rate and pumps energy to the daughter waves. Let us consider first the absence of the spatial derivatives in Eqs. (1)–(3). The parent wave amplitude, which initially grows linearly, saturates due to the nonlinear terms and imparts its energy to the daughter waves, which have noise-level amplitudes during the linear growth of the parent wave. The transfer of energy causes the daughter-wave amplitudes to increase and decrease rapidly, giving energy back to the parent wave, which grows again, comprising the basic interaction process. Depending on the values of the parameter ν the wave amplitudes vary chaotically with time [13]. We fixed $\gamma=0.01$ and take ν as our tunable parameter, and it suffices to consider diffusion to act only in the parent wave.

Equations (1)–(3) were numerically integrated by a pseudospectral method using a fixed number N of modes in Fourier space (for a one-dimensional box of length L with periodic boundary conditions):

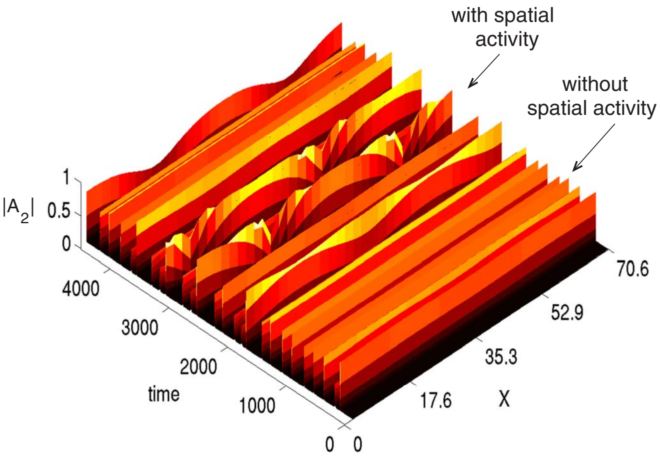

FIG. 1. (Color online) Spatiotemporal evolution of the daughter-wave amplitude for $N=32$ Fourier modes and $\gamma=0.01$, $\nu=-0.5$, $\delta_1=0.1$, $D=1.0$, $v_{g1}=0.0$, $v_{g2}=1.0$, $v_{g3}=-1.0$, and a box length $L=2\pi/\kappa_{1,1}=2\pi/0.089$. The initial conditions are $a_{1,0}(0)=0.45+i0.0$ and $a_{2,\pm1}(0)=0.0+i0.001$, all other modes being set to zero.

$$A_\alpha(x,t)=\sum_{n\in I}|a_{\alpha,n}(t)|e^{i[\kappa_{\alpha,n}x+\phi_{\alpha,n}(t)]}, \tag{4}$$

where $a_{\alpha,n}$ is the time-dependent Fourier coefficient corresponding to the mode number $\kappa_{\alpha,n}=2\pi n/L$, whose evolution is governed by a system of $6N$ coupled ordinary differential equations, and $I=[-(N/2)+1,(N/2)]$. We emphasize that the $\kappa_{\alpha,0}$ mode does not contain the spatial variable and thus reflects only the temporal dynamics of the wave interaction process. This spatially homogeneous dynamics can be regarded as being restricted to an invariant manifold $\mathcal{M}$ embedded in the phase space since, once an initial condition is placed there, the ensuing trajectory remains in $\mathcal{M}$ for all further times. Accordingly, the spatially inhomogeneous modes $\kappa_{\alpha,n}$ are related to directions transversal to $\mathcal{M}$.

A representative example of spatiotemporal chaos in the dynamics generated by the model equations (1)–(3) is depicted in Fig. 1, where the spatiotemporal evolution of the fast daughter-wave modulus ($|A_2|$) is plotted, for nonvanishing initial conditions, only for the modes $\kappa_{2,0}$ and $\kappa_{2,\pm1}$. Except for ν, all the remaining parameters will take on the same numerical values for the forthcoming figures. The spatial profile is initially flat and later alternates with irregular spatial structures. If we think of the mode amplitudes $a_{2,n}$ as coupled oscillators, they have chaotic behavior for this set of parameter values, and we can regard the flat spatial profiles as synchronized chaotic states restricted to $\mathcal{M}$, where $\kappa_{\alpha,0}$ are the only nonvanishing modes. The spatial irregularities we observe for certain time intervals result from the excitation of modes with nonzero $\kappa_{2,n}$. The wave energy, which was initially confined to the $\kappa_{2,0}$ mode, is now imparted to other modes, chiefly the $\kappa_{\alpha,1}$ ones. This nonlinear energy transfer among modes can be observed in Fig. 2, where we plot the time series for the amplitudes of the homogeneous (purely temporal) modes $\kappa_{\alpha,0}$ and the first spatial modes $\kappa_{\alpha,1}$ for the parent and slow daughter waves (respectively, $\alpha=1,3$). Most of the time these spatial-mode amplitudes remain quiescent, meaning that the dynamics is constrained to the homogeneous manifold, except for the spikes corresponding to irregular spatial profiles for which the mode oscillators remain chaotic, although not synchronized. We stress that the number of excited higher transversal modes, $\kappa_{\alpha,n}$ ($n\geq 2$), is bounded near the onset of spatiotemporal chaos.

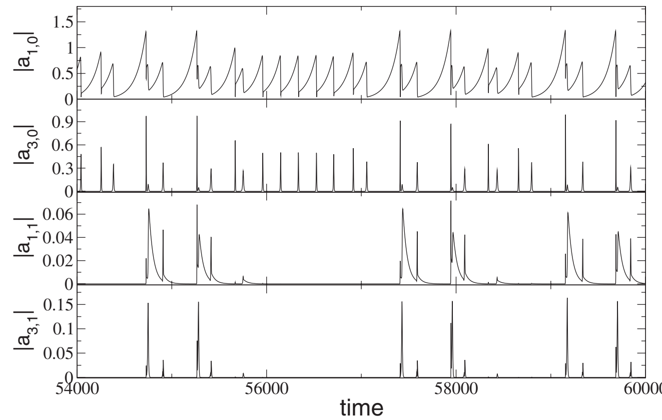

FIG. 2. Time evolution of the $n=0$ and $n=1$ mode amplitudes $a_{\alpha,n}$ for $\nu=-0.37$ of parent ($\alpha=1$) and slow daughter ($\alpha=2$) waves.

The transition from a synchronized chaotic dynamics on the homogeneous manifold $\mathcal{M}$ to a nonsynchronized one, where spatial modes are progressively excited, can be analyzed quantitatively by using the order parameter

$$R_\alpha(t)=\left|\frac{1}{N}\sum_{n\in I}e^{i\phi_{\alpha,n}(t)}\right|^2. \tag{5}$$

Figure 3 depicts the variation of the time-averaged order parameter in terms of the decay rate ν, for both the parent (R_1) and the fast daughter waves (R_2) showing a similar behavior. For $\nu>\nu_{CR}\approx-0.33$ the order parameter reaches its maximum value—namely, $\bar{R}_\alpha=1$—meaning that all transversal Fourier coefficients vanish ($a_{\alpha,n}=0$ for $n\neq 0$) and the chaotic dynamics is constrained to $\mathcal{M}$. On the other hand, if $\nu<\nu_{CR}$, some spatial modes are excited and $\bar{R}_\alpha\lesssim 1$. When the dissipative coefficient ν reaches its critical value ν_{CR} the purely temporal dynamics breaks down and some energy is imparted to spatial modes. In geometrical terms, before the transition ($\nu>\nu_{CR}$) the homogeneous manifold $\mathcal{M}$ is transversely stable, whereas after the transition ($\nu<\nu_{CR}$) some periodic orbits embedded in $\mathcal{M}$ lose transversal stability, causing the occurrence of spatial modes. This is the onset of

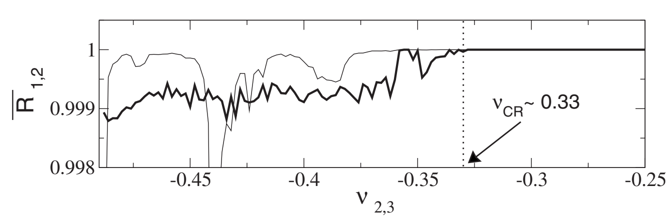

FIG. 3. Time-averaged order parameter for parent wave *versus* decay rate for $T=2\times10^5$, after 10^4 transient iterations.

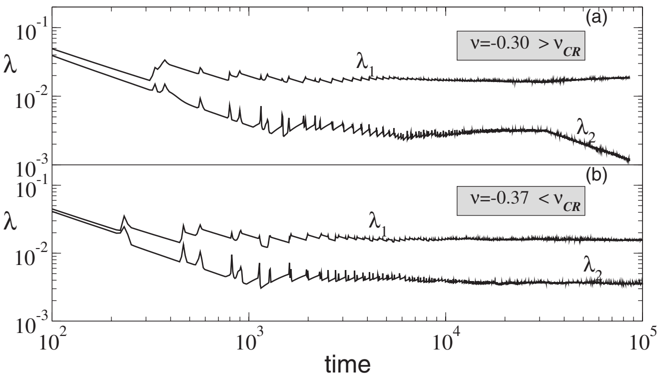

FIG. 4. Time evolution of the two largest Lyapunov exponents of the dynamical system formed by N Fourier modes.

spatiotemporal chaos for the model we study in this work. To support this conclusion we computed the Lyapunov spectrum related to the mode dynamics in Fourier space, and the time evolution of the two largest exponents (out of $6N$ modes considered) is depicted in Fig. 4 [17]. The homogeneous manifold is three dimensional since waves 2 and 3 have the same values for their dissipation parameters. Before the transition [Fig. 4(a)] the largest Lyapunov exponent (λ_1) is positive, indicating that the dynamics on the homogeneous manifold is indeed chaotic, whereas the second exponent (λ_2), which is the largest one along the $N-1$ transversal directions, decays to zero as a power law, such that the $\mathcal{M}$ is transversely stable for $\nu > \nu_{CR}$. After the transition [Fig. 4(b)] the λ_2 exponent is also positive, showing that some orbits on $\mathcal{M}$ have lost transversal stability.

In this context we can explain the intermittent transition to spatial chaos occurring in the vicinity of the critical point ν_{CR} [16]. A trajectory in the (Fourier) phase space which is off but very close to the homogeneous manifold $\mathcal{M}$ will stay in its vicinity as long as it suffers the major influence from the transversely stable periodic orbits embedded in $\mathcal{M}$, the energy being concentrated on the $\kappa_{\alpha,0}$ mode. As the trajectory approaches transversely unstable orbits of $\mathcal{M}$ it experiences excursions far from $\mathcal{M}$, physically imparting some amount of energy to higher $\kappa_{\alpha,n}$ modes and causing the spikes observed in Fig. 2. The trajectory eventually returns to the vicinity of $\mathcal{M}$ due to the influence of those transversely stable orbits, and the interspike intervals found in Fig. 2 may be regarded as laminar phases of duration τ_i interrupted in an intermittent way. There may exist interburst intervals as large as *circa* 30 000 time units, explaining the plateau with almost-constant values of λ_2 before the power-law decay observed in Fig. 4(a). Figure 5 depicts the probability distribution of these interspike intervals, $P(\tau)$, which presents the characteristic scaling of on-off intermittency with noise, where the small interspike intervals are fairly common, their lengths obeying a power-law scaling with universal exponent $-3/2$, whereas the large intervals obey a decreasing exponential scaling characteristic of noise [18]. The latter part is due to the influence of small-amplitude random perturbations on the intermittent scenario, which come from the intrinsic chaotic dynamics in the homogeneous manifold.

The previous analyses were conducted using a modest

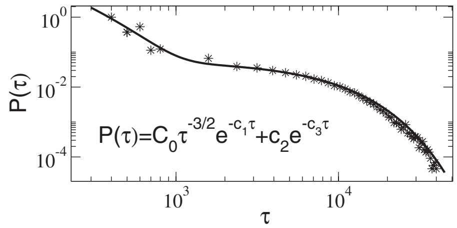

FIG. 5. Probability distribution for the duration of interspike intervals for $\nu = -0.37 < \nu_{CR}$.

number of Fourier modes ($N=32$). Our results were compared with $N=64$ modes with no quantitative differences for the transition to spatiotemporal chaos. In fact, near the onset of spatiotemporal chaos, the wave energy, formerly restricted to the homogeneous manifold, is transferred to the lowest spatial modes first. This scenario, however, is expected to change completely in the fully turbulent case, where the strong interaction there existing among different spatial scales leads to a fast redistribution of the wave energy to the lowest wavelengths, such that the use of a large number N of modes would be necessary.

We can obtain such a fully turbulent scenario for values of ν in the same range we get the onset of spatiotemporal chaos, provided we decrease the magnitude of the diffusion term in Eq. (1). A crude way to explain this observation is to note that, on neglecting the influence of the nonlinear and convective terms, the effective linear growth rate for the parent wave is $\gamma - D\kappa^2$. Hence a weak diffusion reduces the growth of the modes with smaller κ but damps modes with higher κ. On the other hand, letting $D \to 0$ allows a fast distribution of energy among modes. Accordingly, we plot in Fig. 6 the power spectrum of the parent-wave Fourier-mode amplitude $|a_{1,n}|$ for $N=1024$ modes with weak diffusion. Our numerical results are consistent with a power-law scaling $\mathrm{P}(|a_{1,n}|) \sim |a_{1,n}|^{-1.7}$ which compares well with the 5/3 exponent predicted by Kolmogorov theory [1]. This behavior occurs since energy is injected in the waves at the lowest κ modes due to the chaotic dynamics in the homogeneous manifold ($\kappa=0$), and it is transported to smaller length scales or higher κ modes.

In conclusion, we propose an interpretation for the onset of spatiotemporal chaos in a nonlinear system, using as a paradigmatic example the quadratic interaction of three waves and considering both dissipation and diffusion effects.

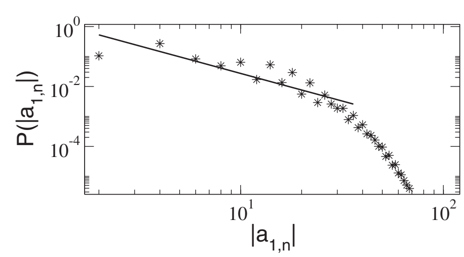

FIG. 6. Power spectrum of the daughter-wave oscillations. We used $N=1024$ modes, $D=10^{-4}$, and considered $T=400$ before numerical mode saturation. The solid line is the Kolmogorov 5/3-scaling law.

We claim that the onset of spatiotemporal chaos in this system occurs via the loss of transversal stability of a spatially homogeneous state for which the temporal dynamics is chaotic. The spatial modes so excited draw energy from the purely temporal chaotic state, eventually leading to fully developed spatiotemporal chaos when the number of excited modes is large enough. The underlying dynamical mechanism of this transition scenario is that some unstable periodic orbit embedded in this homogeneous manifold loses transversal stability, causing the emergence of the spatially heterogeneous modes. This process has been extensively stated in low-dimensional dynamical system, where it is related to fundamental phenomena like riddled basins [19], on-off intermittency [15], and unstable dimension variability. Thanks to the high dimensionality of the dynamics of our spatiotemporal system, it is very difficult to find the particular orbit in the homogeneous manifold $\mathcal{M}$ which loses transversal stability first, although for simpler low-dimensional systems this can be done [19]. In the scenario following this loss of transversal stability only a limited number of spatial modes are excited. Moreover, after the onset of spatiotemporal chaos, we observe an intermittent switching between spatially homogeneous and nonhomogeneous states which possesses a power-law (with the universal exponent $3/2$) distribution of laminar states. As more spatial modes are added we can obtain a fully developed scenario with the Kolmogorov spectral distribution. Our numerical results were drawn from a paradigmatic model of spatiotemporal dynamics, and hence we claim that the same mechanism described here could be applied to other high-dimensional spatially extended dynamical systems [18].

This work was made possible by partial financial support of CNPq, CAPES, and Fundação Araucária.

[1] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).

[2] D. Ruelle and F. Takens, Commun. Math. Phys. **20**, 167 (1971); S. Newhouse, D. Ruelle, and F. Takens, *ibid.* **64**, 35 (1978).

[3] J. P. Gollub and H. L. Swinney, Phys. Rev. Lett. **35**, 927 (1975); T. Klinger *et al.*, *ibid.* **79**, 3913 (1997); A. Cumming and P. S. Linsay, *ibid.* **60**, 2719 (1988).

[4] R. W. Walden, P. Kolodner, A. Passner, and C. M. Surko, Phys. Rev. Lett. **53**, 242 (1984).

[5] The mechanism leading to intermittency in the transition to spatiotemporal chaos is still an open question. See, for instance, B. W. Zeff *et al.*, Nature (London) **421**, 146 (2003); Y. Li and C. Meneveau, Phys. Rev. Lett. **95**, 164502 (2005); E. L. Rempel and Abraham C.-L. Chian, *ibid.* **98**, 014101 (2007).

[6] K. He, Phys. Rev. Lett. **94**, 034101 (2005).

[7] L. Turner, Phys. Rev. E **54**, 5822 (1996); Y. C. Li, Int. J. Bifurcation Chaos Appl. Sci. Eng. **17**, 85 (2007).

[8] A. C.-L. Chian, S. R. Lopes, and M. V. Alves, Astron. Astrophys. **290**, L13 (1994); A. C.-L. Chian and F. B. Rizzato, J. Plasma Phys. **51**, 61 (1994).

[9] A. Rundquist *et al.*, Science **280**, 1412 (1998); G. I. Stegeman and M. Segev, *ibid.* **286**, 1518 (1999); A. Picozzi and M. Haelterman, Phys. Rev. Lett. **86**, 2010 (2001).

[10] D. J. Kaup, A. Reiman, and A. Bers, Rev. Mod. Phys. **51**, 275 (1979).

[11] This sign convention follows from a linear analysis in which the parent wave is supposed to grow exponentially with time, whereas the daughter waves are expected to decay exponentially in each cycle.

[12] C. C. Chow, A. Bers, and A. K. Ram, Phys. Rev. Lett. **68**, 3379 (1992).

[13] A. M. Batista *et al.*, Phys. Plasmas **13**, 042510 (2006).

[14] S. R. Lopes and F. B. Rizzato, Phys. Rev. E **60**, 5375 (1999).

[15] J. F. Heagy, N. Platt, and S. M. Hammel, Phys. Rev. E **49**, 1140 (1994).

[16] K. He and Abraham C.-L. Chian, Phys. Rev. Lett. **91**, 034102 (2003).

[17] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, Physica D **16**, 285 (1985); M. Yamada and K. Ohkitani, Phys. Rev. Lett. **60**, 983 (1988).

[18] E. Covas, P. Ashwin, and R. Tavakol, Phys. Rev. E **56**, 6451 (1997).

[19] Y.-C. Lai, C. Grebogi, J. A. Yorke, and S. C. Venkataramani, Phys. Rev. Lett. **77**, 55 (1996).