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Clustering of matter in waves and currents

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The growth rate of small-scale density inhomogeneities (the entropy production rate) is given by the sum of the Lyapunov exponents in a random flow. We derive an analytic formula for the rate in a flow of weakly interacting waves and show that in most cases it is zero up to the fourth order in the wave amplitude. We then derive an analytic formula for the rate in a flow of waves and currents. Estimates of the rate and the fractal dimension of the density distribution show that the interplay between waves and currents is a realistic mechanism for providing patchiness of the pollutant distribution on the ocean surface.

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Random compressible flows produce a very inhomogeneous distribution of density; see [1–9] for theory and [10–16] for experiments. Here we study the growth of density inhomogeneities at small scales, where the flow can be considered spatially smooth. It can then be characterized by the Lyapunov exponents whose sum is the logarithmic rate of change of an infinitesimal volume element—that is, minus the density rate of change, λ. It is called also the entropy production rate or the clustering rate. Since contracting regions contain more statistical weight than expanding ones, λ is generally positive in a random compressible flow [2–5] (an analog of the second law of thermodynamics). As a result, density grows on most trajectories and, in the limit of infinite time, concentrates on a constantly evolving fractal set characterized by a singular (Sinai-Ruelle-Bowen) measure [4,17–19]. Our goal here is to establish what determines the rate λ in fluid flows with waves, particularly on liquid surfaces. Patchiness in the distribution of litter on the surface of lakes and pools and of oil slicks and seaweeds on the sea surface is well known empirically while there is no general theory that describes it. Dynamic processes like wave breaking and Langmuir circulations produce streaks of flotsam [20]. In random flows, patchiness is expected to be a signature of a fractal measure forming on a surface [11–13,15,16]. Our purpose is to establish the role of low-amplitude waves (ubiquitous on water surfaces) in that process and to estimate how fast this fractal set is formed and what its fractal dimension is.

Surface flows can be compressible, even for incompressible fluids. For example, underwater turbulence produces compressible surface currents that lead to fractal distributions of surface density [11–13,15,16,21]. However, underwater turbulence is relatively rare in natural environment (because of stable stratification) and large-scale currents are usually incompressible. A compressible component of the surface flows is then provided by waves. Linear waves just oscillate; net effects are produced by nonlinearity. Every running plane wave provides for a (Stokes) drift proportional to the square of the wave amplitude. A set of random waves provides for the Lyapunov exponents proportional to the fourth power of the wave amplitudes, yet the sum of the exponents $-\lambda$ is found to be zero for purely longitudinal waves with Gaussian statistics (the nonzero rate appears only in the sixth order in wave amplitudes; i.e., it is so small as to be practically unobservable in most cases) [8]. Here we use the general formula for the entropy production rate [22] and show that the account of wave interactions (making the statistics weakly non-Gaussian) does not bring nonzero λ in the fourth order in wave amplitudes. We then suggest that in many situations (particularly on liquid surfaces) the growth of density inhomogeneities is due to an interplay between waves and currents. For such flows, we calculate λ and the fractal dimension of the resulting measure and consider different limits.

In the velocity field $\boldsymbol{v}(t,\boldsymbol{x})$, the trajectory $\boldsymbol{X}(t,\boldsymbol{x})$ satisfies the equation $\partial_t \boldsymbol{X}=\boldsymbol{v}(t,\boldsymbol{X})$ with the initial condition $\boldsymbol{X}(0,\boldsymbol{x})=\boldsymbol{x}$. The rate of density change along the trajectory averaged over $\boldsymbol{x}$ is given by [22]

$$\lambda = -\lim_{t\to\infty}\langle w(t,\boldsymbol{X})\rangle = \int_0^\infty dt\langle w(0,\boldsymbol{x})w(t,\boldsymbol{X})\rangle, \qquad (1)$$

with $w\equiv\boldsymbol{\nabla}\cdot\boldsymbol{v}$. This is a generalization of the Kawasaki formula [23] (obtained in the context of statistical physics) to time-dependent flows with a steady statistics.

For a general flow, it is impossible to relate the Lagrangian integral (1) to the velocity spectra or correlation functions given usually in the Eulerian frame. However, it is possible for flows where fluid particles shift little during the correlation time. That happens, in particular, for sufficiently wide wave packets of low amplitude. Indeed, for packets with both the wavenumber and the width of order k, we estimate the correlation time of w as $\Omega_{\boldsymbol{k}}^{-1}$ and the correlation length as k^{-1}. For waves with (a small) amplitude v, the fluid particle shift during a period $|\boldsymbol{X}-\boldsymbol{x}|\sim v/\Omega_k$ is much smaller than the wavelength. The small parameter $\epsilon=kv/\Omega_{\boldsymbol{k}}$ allows for an analytical treatment: one expands Eq. (1) near $\boldsymbol{x}$ up to ϵ^4, using $\boldsymbol{X}-\boldsymbol{x}=\int_0^t\boldsymbol{v}(t',\boldsymbol{x})dt'+\cdots$, $w(\boldsymbol{X})=w(\boldsymbol{x})+(\boldsymbol{X}-\boldsymbol{x})\nabla w(\boldsymbol{x})+\cdots$, etc.:

$$\lambda \approx \int_0^\infty dt\Bigg[\langle w(0)w(t)\rangle + \left\langle w(0)\frac{\partial w(t)}{\partial x^\alpha}\int_0^t dt_1 v^\alpha(t_1)\right\rangle + \left\langle w(0)\frac{\partial w(t)}{\partial x^\alpha}\int_0^t dt_1\frac{\partial v^\alpha(t_1)}{\partial x^\beta}\int_0^{t_1}dt_2 v^\beta(t_2)\right\rangle + \frac{1}{2}\left\langle w(0)\frac{\partial w(t)}{\partial x^\beta\partial x^\alpha}\int_0^t dt_1 v^\alpha(t_1)\int_0^t dt_2 v^\beta(t_2)\right\rangle\Bigg]. \qquad (2)$$

All quantities here are taken at the same point in space. We start the consideration of (2) from the simplest case when the flow is solely due to weakly nonlinear waves. The velocity Fourier component is expressed via the polarization vector $\boldsymbol{A}_{\boldsymbol{k}}$ and the normal coordinates $a_{\boldsymbol{k}}$ by $\boldsymbol{v}_{\boldsymbol{k}}=\boldsymbol{A}_{\boldsymbol{k}}(a_{\boldsymbol{k}}-a^*_{-\boldsymbol{k}})$. The normal coordinates satisfy the equation $\partial_t a_{\boldsymbol{k}} = -i\delta\mathcal{H}/\delta a^*_{\boldsymbol{k}}$ where the Hamiltonian $\mathcal{H}$ can be expanded in wave amplitudes as follows [24]: $\mathcal{H}=\int d\boldsymbol{k}\Omega_{\boldsymbol{k}}|a_{\boldsymbol{k}}|^2 + \frac{1}{2}\int d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3(V_{123}a_1a_2^*a_3^*+\text{c.c.})+\cdots$. We do not write explicitly here other (third- and fourth-order) terms since they will not contribute λ up to $\sim\epsilon^4$. We use throughout the shorthand notation $\Omega(\pm\boldsymbol{k}_i)=\Omega_{\pm i}$, $\boldsymbol{A}_{\boldsymbol{k}_i}=\boldsymbol{A}_i$, and for the interaction coefficients $\boldsymbol{V}_{123}=V_{123}\delta(\boldsymbol{k}_1-\boldsymbol{k}_2-\boldsymbol{k}_3)$ and $V_{123}=V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$.

One derives the clustering rate up to ϵ^4 using a standard perturbation theory for weakly interacting waves [24]. The first term in (2) is the time integral (the zero-frequency value) of the second moment. At the order ϵ^2, the second moment in the frequency representation is proportional to the delta function: $\langle a^*(\boldsymbol{k},\omega)a(\boldsymbol{k}',\omega')\rangle=(2\pi)^{d+2}n(\boldsymbol{k})\delta(\omega-\Omega_{\boldsymbol{k}})\delta(\boldsymbol{k}-\boldsymbol{k}')\delta(\omega-\omega')$. A finite width over ω and a finite value at $\omega=0$ appear either due to finite linear attenuation (the case considered in [25]) or due to nonlinearity in the second order of perturbation theory (which gives ϵ^4 and is considered here). The second term in (2) is the triple moment, which appears in the first order of the perturbation theory, and the last two terms contain the fourth moment, which is to be taken at the zeroth order (i.e., as a product of two second moments). Straightforward calculations then give for weakly nonlinear waves the ϵ^4 contribution

$$\lambda = \text{Re}\int\frac{d\boldsymbol{k}_2 d\boldsymbol{k}_3}{(2\pi)^{2d}}\delta(\Omega_2-\Omega_3)n(\boldsymbol{k}_2)n(\boldsymbol{k}_3)\Bigg\{\int\frac{d\boldsymbol{k}_1}{(2\pi)^d}$$
$$\times\Bigg[\left(\frac{\boldsymbol{V}^*_{213}}{\Omega_1}-\frac{\boldsymbol{V}_{3-12}}{\Omega_{-1}}\right)\Bigg((2\pi)^{3d+1}|\boldsymbol{A}_1\cdot\boldsymbol{k}_1|^2\frac{V_{213}}{\Omega_1} \tag{3}$$
$$-\frac{(2\pi)^{2d}}{\Omega_2}(\boldsymbol{A}^*_1\cdot\boldsymbol{k}_1)(\boldsymbol{A}_2\cdot\boldsymbol{k}_2)[\boldsymbol{A}^*_3\cdot(\boldsymbol{k}_2+\boldsymbol{k}_1)]\Bigg)\Bigg] \tag{4}$$
$$+\frac{\pi}{\Omega_2^2}|(\boldsymbol{A}_3\cdot\boldsymbol{k}_3)(\boldsymbol{A}^*_2\cdot\boldsymbol{k}_3)-(\boldsymbol{A}_3\cdot\boldsymbol{k}_2)(\boldsymbol{A}^*_2\cdot\boldsymbol{k}_2)|^2\Bigg\}. \tag{5}$$

The common factor $\delta(\Omega_2-\Omega_3)n(\boldsymbol{k}_2)n(\boldsymbol{k}_3)$ tells us that we have here the contribution of two pairs of waves with the same frequencies. All three terms are generally nonzero (and positive) when the dispersion law is nonmonotonic or nonisotropic so that $\Omega_2=\Omega_3$ does not require $k_2=k_3$. In most interesting cases, however, $\Omega_{\boldsymbol{k}}$ is a monotonous function of the modulus k so that $k_2=k_3$. Let us show first that the wave interaction does not contribute λ in this case. Indeed, the first two terms, (3) and (4), which came out of the first two terms of (2), are proportional to the difference $\boldsymbol{V}^*_{213}-\boldsymbol{V}_{3-12}$ between the amplitude of decay into a wave with $\boldsymbol{k}_1$ and confluence with a wave with $-\boldsymbol{k}_1$. Interaction coefficients for $k_2=k_3$ have rotational symmetry and are thus functions of wavenumbers so that $V_{213}-V_{3-12}=V_{213}-V_{312}=V_{212}-V_{212}=0$.

The last term (5) comes from the last two terms of (2) and does not contain the interaction coefficient V. This term is due to the nonlinear relation between Eulerian and Lagrangian variables rather than due to wave interactions. We can compare (5) with the growth rate of the squared density for noninteracting waves [see (12) in [8]], written there in terms of the energy spectrum, $E^{\alpha\beta}(\boldsymbol{k},\omega)=2\pi A^\alpha_{\boldsymbol{k}}A^{*\beta}_{\boldsymbol{k}}[n(\boldsymbol{k})\delta(\omega-\Omega_{\boldsymbol{k}})+n(-\boldsymbol{k})\delta(\omega+\Omega_{-\boldsymbol{k}})]$. The comparison shows this part of our logarithmic growth rate being exactly half the growth rate for the second moment as it should be for a short-correlated flow [2]. Indeed, the process of the creation of density inhomogeneities is effectively short-correlated since the time it takes ($1/\Omega_k\epsilon^4$ or longer) exceeds the correlation time of velocity divergence in the Lagrangian frame, $1/\Omega_k$. For monotonous $\Omega(k)$, (5) is nonzero only if the polarization vector $\boldsymbol{A}_{\boldsymbol{k}}$ is neither parallel nor perpendicular to $\boldsymbol{k}$—i.e., it contains both longitudinal and transverse components. This is not the case for most waves in continuous media. We thus conclude that for most common situations (in particular, for sound or surface waves) the entropy production rate is zero in the order ϵ^4. Note that for surface waves, the canonical variables are elevation $\eta(\boldsymbol{r},t)$ and the potential $\phi(\boldsymbol{r},z=\eta,t)$ which are related to the surface velocity by a nonlinear relation $\boldsymbol{v}=\boldsymbol{\nabla}\phi(\boldsymbol{r},\eta,t)$. Expanding it in the powers of η, one can show that this extra nonlinearity does not contribute λ in the order ϵ^4 [26]. We find it remarkable that the flow of random longitudinal waves is only weakly compressible (i.e., the senior Lyapunov exponent is much larger than the sum of the exponents).

Therefore, we consider now the clustering rate in the flow of incompressible surface currents $\boldsymbol{u}$ and longitudinal (compressible) waves $\boldsymbol{v}$, the situation most relevant for oceanological applications [20,27]. To derive λ in the lowest (second) order in $\epsilon=kv/\Omega_k$, we neglect the contribution of $\boldsymbol{v}$ into $\boldsymbol{X}$ in Eq. (1) and assume $\partial_t\boldsymbol{X}(t,\boldsymbol{x})\approx\boldsymbol{u}(t,\boldsymbol{X}(t,\boldsymbol{x}))$. In this order, $w=\boldsymbol{\nabla}\cdot\boldsymbol{v}$ is Gaussian and one may integrate by parts:

$$\langle w(0,\boldsymbol{x})w(t,\boldsymbol{X}(t,\boldsymbol{x}))\rangle = \int dt'd\boldsymbol{x}'\Phi(t',\boldsymbol{x}'-\boldsymbol{x})$$
$$\times\langle\delta w(t,\boldsymbol{X}(t,\boldsymbol{x}))/\delta w(t',\boldsymbol{x}')\rangle.$$

Here $\Phi(t'-t,\boldsymbol{x}'-\boldsymbol{x})=\langle w(t,\boldsymbol{x})w(t',\boldsymbol{x}')\rangle$ is the Eulerian correlation function and

$$\lambda\approx\int_0^\infty dt\langle\Phi[t,\boldsymbol{J}(t)]\rangle,\quad \boldsymbol{J}(t)\equiv\boldsymbol{X}(t,\boldsymbol{x})-\boldsymbol{x}. \tag{6}$$

Waves and currents are considered statistically independent in this order. Using the spectrum $k^\alpha k^\beta E^{\alpha\beta}_{\boldsymbol{k}}\equiv k^2E_{\boldsymbol{k}}$, we can express $\Phi(t,\boldsymbol{r})=(2\pi)^{-d}\int k^2E_{\boldsymbol{k}}\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\Omega_{\boldsymbol{k}}t)d\boldsymbol{k}$ and rewrite (6) as a weighted spectral integral:

$$\lambda=(2\pi)^{-d}\int k^2E_{\boldsymbol{k}}\mu(\boldsymbol{k})d\boldsymbol{k}, \tag{7}$$

$$\mu(\boldsymbol{k})=\int_0^\infty\langle\cos[\boldsymbol{k}\cdot\boldsymbol{J}(t)-\Omega_{\boldsymbol{k}}t]\rangle dt. \tag{8}$$

The spectral weight $\mu(k)$ is the Lagrangian correlation time of the k harmonic of w and is expressed via the characteristic function of the particle drift $\boldsymbol{J}(t)$. Without currents, Eqs. (7) and (8) reproduce the first term of (2), since only the zero-frequency wave contributes. Already a steady uniform current $\bar{\boldsymbol{u}}$ contributes the clustering rate in the order ϵ^2 if there are waves in a Cherenkov resonance with the current $\lambda = (2\pi)^{-d}\int k^2 E_{\boldsymbol{k}} \delta(\Omega_{\boldsymbol{k}} - \boldsymbol{k}\cdot\bar{\boldsymbol{u}})d\boldsymbol{k}$. Similar resonance has been noticed before for diffusivity [28]. Let us stress that this result is based on the assumption that waves are independent of currents—in particular, that there is no Doppler shift of the wave frequency. That takes place, for instance, when there is only a surface mean current. If, on the contrary, the current is homogeneous across the depth brought into the motion by a wave (of the order of a wavelength for gravity waves or the whole water depth for inertiogravity waves), then Eq. (8) needs replacing $\Omega_k \to \Omega_k + \boldsymbol{k}\cdot\bar{\boldsymbol{u}}$, and the effect of the mean current is zero due to Galilean invariance.

Consider now the fluctuating part of the current velocity characterized by the rms velocity $u_0^2 \equiv \langle (u-\bar{u})^2\rangle$, the correlation time $\tau \equiv \int \langle u_\alpha(0,\boldsymbol{x}) u_\alpha(t,\boldsymbol{X}(t,\boldsymbol{x}))\rangle dt/u_0^2$, and the correlation scale $\ell = u_0\tau$. Spatial and temporal relationships between waves and currents are described by the two dimensionless parameters $L \equiv k\ell$ and $T \equiv \Omega_{\boldsymbol{k}}\tau$. The characteristic function $\langle \exp[i\boldsymbol{k}\cdot\boldsymbol{J}(t)]\rangle$ depends on the details of the currents statistics but it has universal behavior at both $t \ll \tau$ and $t \gg \tau$ where general calculations are possible. On the plane of L, T we distinguish three regions of different asymptotic behavior.

Consider first the ballistic limit when the integral (8) is determined by the times $t \ll \tau$ when the drift velocity does not change and $\boldsymbol{J}(t) \approx \boldsymbol{u}(0,\boldsymbol{x})t$. Again, only those waves contribute that are in a Cherenkov resonance with the current (whose phase velocity coincides with the local projection of the current velocity): $\mu = \pi\langle \delta(\Omega_{\boldsymbol{k}} - \boldsymbol{k}\cdot\boldsymbol{u})\rangle$. In this limit, the weight μ is determined by the single-time probability distribution of the current velocity which we denote $\mathcal{P}(\boldsymbol{u})$. In particular, for the isotropic Gaussian $\mathcal{P}(u) \propto \exp(-u^2/2u_0^2)$, we get

$$\mu(k) = (\pi d/2)^{1/2}(ku_0)^{-1}\exp[-(\sqrt{d}\Omega_k/\sqrt{2}ku_0)^2]. \qquad (9)$$

The ballistic approximation and Eq. (9) hold when $(k\ell)^2/d$ is much larger than both unity and $\Omega_k\tau$.

The second universal limit is that of a slow clustering which proceeds for a time exceeding the correlation time of currents. At $t \gg \tau$, we use the diffusion approximation $\langle \exp[i\boldsymbol{k}\cdot\boldsymbol{J}(t)]\rangle = \exp[-k^2u_0^2\tau t/d]$ in Eq. (8):

$$\mu(k) = \tau\frac{d(k\ell)^2}{(k\ell)^4 + (d\Omega_k\tau)^2}. \qquad (10)$$

That answer and the diffusive approximation hold when both $(k\ell)^2/d$ and $\Omega_k\tau$ are small. Formulas (7) and (10) can be compared with the expression for the clustering rate for waves with a linear damping, $\lambda \simeq \int k^2 E_k \gamma_k(\Omega_k^2 + \gamma_k^2)^{-1} d\boldsymbol{k}$ [25]. We see that in this limit the diffusive motion of fluid particles due to currents is equivalent in its effect to a damping of waves with $\gamma_k = k^2u_0^2\tau/d$, where $u_0^2\tau/d$ is the eddy diffusivity.

The third asymptotic regime takes place for fast-oscillating waves when $\Omega_k\tau$ exceeds both unity and $(k\ell)^2/d$. An integral of the fast oscillating exponent with a slow function, $\int_0^\infty \cos(\Omega_k t)f(t)dt$, decays as Ω_k^{-2n-2} where $2n+1$ is the lowest order of the nonvanishing derivative of $f(t)$

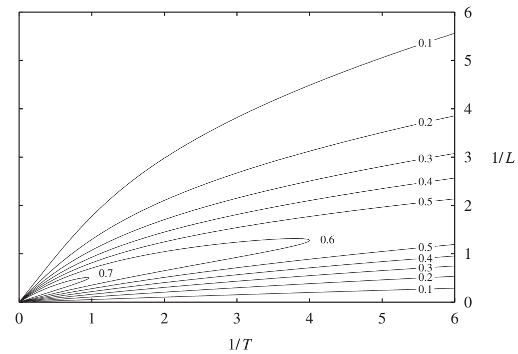

FIG. 1. The isolines of the dimensionless clustering rate $\mu(k)\Omega_{\boldsymbol{k}}$ given by Eq. (11). Here $L = ku_0\tau$, $T = \Omega_{\boldsymbol{k}}\tau$.

$= \langle \exp[i\boldsymbol{k}\cdot\boldsymbol{J}(t)]\rangle$ at $t=0$. When all odd derivatives at zero are zero, the integral decays exponentially. We see that the answer depends on the details of the statistics of currents.

If $\boldsymbol{u}(t,\boldsymbol{X}(t,\boldsymbol{x}))$ is Gaussian and isotropic with $\langle u^\alpha(0,\boldsymbol{x})u^\beta(t,\boldsymbol{X}(t,\boldsymbol{x}))\rangle = (u_0^2/d)\delta^{\alpha\beta}\exp(-|t|/\tau)$, then

$$\mu(k) = \tau\int_0^\infty ds\cos(Ts)\exp[(L^2/d)(1 - s - e^{-s})]. \qquad (11)$$

It gives both limits (9) and (10) and

$$\mu(k) = (ku_0)^2/\tau\Omega_k^4 d, \qquad (12)$$

at large Ω_k since the lowest nonvanishing derivative is $f'''(0)$. Isolines of Eq. (11) are shown in Fig. 1 for arbitrary parameters. Recall that the whole description based on (6) is valid when $v \ll u$.

Note in passing that if one interpolates between the ballistic and diffusive regimes (i.e., between $J^2 \propto t^2$ and $J^2 \propto t$) with the help of the function $\sqrt{1+(t/\tau)^2} - 1$, which is smooth at $t=0$, then the weight factor can be calculated analytically in terms of the Bessel function. That concludes the analysis of the weight $\mu(k)$, and we can now turn to Eq. (7) to get the clustering rate λ.

When the wave spectrum is not very wide (with the width comparable to k), then Eq. (7) gives the estimate

$$\lambda \simeq (kv)^2\mu(k) = \epsilon^2\Omega_k^2\mu(k). \qquad (13)$$

Let us now find out which wavenumbers contribute Eq. (7) when the spectrum is wide—for instance, an isotropic power law $E_{\boldsymbol{k}} \propto k^{b-d}$ between some k_{min} and k_{max}—and when the dispersion relation is $\Omega_{\boldsymbol{k}} = Ck^a$ [24]. Consider first the ballistic regime. For either $a>1$, $b>0$ or $a<1$, $b<0$ the wavenumber $k_* = [bu_0^2/dC^2(a-1)]^{1/(2a-2)}$ determines λ. For either $b \le 0$, $a>1$ or $b<-1$, $a=1$ the clustering rate is determined by k_{min}, while for either $b \ge 0$, $a<1$ or $b \ge -1$, $a=1$ by k_{max}. Let us give physical examples using Kolmogorov spectra of waves. For capillary waves on a deep water, $\Omega_k \propto k^{3/2}$ and $E_k \propto k^{-11/4}$, and λ is determined by k_{min}—i.e., by the longest waves in the wave turbulent spectrum (assuming the ballistic approximation is valid for them). For gravity waves in a deep water, $\Omega_k \propto k^{1/2}$, and for both Kolmogorov solutions, $E_k \propto k^{-20/6}$ and $E_k \propto k^{-7/2}$, the clustering rate is determined by waves around k_*. For the diffusive regime, the clustering rate is determined by k_{max} if $b \geqslant \max[2a-4,0]$ and by k_{min} if $b < \max[2a-4,0]$.

Let us now discuss the results obtained. The estimates (9), (10), and (12) show that the dimensionless ratio $\lambda/(\epsilon^2\Omega_k) \simeq \Omega_k\mu(k)$ has a maximum of order unity either in the ballistic regime where the phase velocity of wave is comparable to the current velocity or in the diffusive regime where the eddy diffusivity $u_0^2\tau$ is comparable to $\Omega_k k^{-2}$ [in the third asymptotic regime $\lambda/(\epsilon^2\Omega_k)$ is always small]. In those cases, $\lambda/\Omega_k \simeq \epsilon^2$; i.e., the degree of clustering during a period is the squared wave nonlinearity (typically ϵ is between 0.1 and 0.01). Such clustering is pretty fast (minutes for meter-sized gravity waves and a week for 50-km-sized inertiogravity waves).

Let us briefly discuss wind-generated gravity waves and surface-layer currents (due to a wind drag). In this case, usually $u_0 \ll \Omega_k/k$ so that the maximal wave-current clustering rate $\lambda \simeq \Omega_k\epsilon^2$ is reached in the diffusive regime (10) when $u_0\ell \simeq \Omega_k/k^2$. Note that turbulent fluctuations of the wind generate surface currents which are generally compressible. Therefore, there is a direct contribution of compressible currents to the clustering rate that can be estimated as $u_0/\ell \simeq \tau^{-1}$. The wave contribution dominates when $\Omega_k\epsilon^2 \simeq u_0 > u_0\ell k^2\epsilon^2/\ell$—i.e., $\epsilon k\ell > 1$.

Clustering leads to a fractal distribution of floaters over the surface. When the compressible component of the velocity is small, the Lyapunov exponents are due to the current flow, $\lambda_1 \sim \lambda_2 \sim \tau^{-1}$. Then, the fractal dimension of the density distribution can be expressed by the Kaplan-Yorke formula $1+\lambda_1/|\lambda_2| = 2-\lambda/|\lambda_2| \approx 2-\lambda\tau$. The fractal part is maximal in the ballistic regime when $\Omega_k \simeq ku_0$; then, $\lambda\tau \simeq \epsilon^2\Omega_k\tau = \epsilon^2 k\ell$ grows with ℓ and reaches order unity when $k\ell \simeq \epsilon^{-2}$. Therefore, the distribution is most fractal when waves are short while currents are long: the current-to-wave ratio of scales, $k\ell$, compensates for a small wave nonlinearity ϵ^2, so that even weak waves with the help of surface currents can produce a very inhomogeneous fractal distribution of matter. To conclude, we have suggested a mechanism for generating patchiness of pollutants on a liquid surface. Our formulas (9), (10), and (12) give realistic timescales so that the interplay between waves and currents can be a source of inhomogeneities in natural environments.

As a final remark, note that apart from fluid mechanics, one can think about the evolution of a dynamical system as a flow in the phase space and treat density as a measure. The Hamiltonian dynamics of a closed system provides for an incompressible flow and a constant (equilibrium) measure. Compressibility corresponds to pumping and damping—i.e., to nonequilibrium. Indeed, the notion of singular (fractal) measures first appeared in nonequilibrium statistical physics [17–19] and then was applied to fluid mechanics [2,3,6,11,12]. Therefore, formulas (6)–(13) also describe the entropy production rate in dynamical systems under the action of perturbations periodic in space and in time.

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