Analytical calculation for the percolation crossover in deterministic partially self-avoiding walks in one-dimensional random media

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Consider N points randomly distributed along a line segment of unitary length. A walker explores this disordered medium, moving according to a partially self-avoiding deterministic walk. The walker, with memory μ, leaves from the leftmost point and moves, at each discrete time step, to the nearest point that has not been visited in the preceding μ steps. Using open boundary conditions, we have calculated analytically the probability $P_N(\mu)=(1-2^{-\mu})^{N-\mu-1}$ that all N points are visited, with $N \gg \mu \gg 1$. This approximated expression for $P_N(\mu)$ is reasonable even for small N and μ values, as validated by Monte Carlo simulations. We show the existence of a critical memory $\mu_1=\ln N/\ln 2$. For $\mu<\mu_1-e/(2\ln 2)$, the walker gets trapped in cycles and does not fully explore the system. For $\mu>\mu_1+e/(2\ln 2)$, the walker explores the whole system. Since the intermediate region increases as $\ln N$ and its width is constant, a sharp transition is obtained for one-dimensional large systems. This means that the walker need not have full memory of its trajectory to explore the whole system. Instead, it suffices to have memory of order $\log_2 N$.

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I. INTRODUCTION

While random walks in regular or disordered media have been thoroughly explored [1], deterministic walks in regular [2] and disordered media [3–6] have been much less studied. Here we are concerned with the properties of deterministic walks in random media.

Given N points distributed in a d-dimensional space, a possible question to ask is how efficiently these points can be visited by a walker who follows a simple movement rule. The search for the shortest closed path passing each point once is the well-known traveling salesman problem (TSP), which has been extensively studied. In particular, if the point coordinates are distributed following a uniform deviate, results concerning the statistics of the shortest paths have been obtained analytically [7–9]. To tackle the TSP, one imperatively needs to know the coordinates of all the points in advance. Global system information must be at the walker's disposal.

Nevertheless, other situations may be envisaged. For instance, suppose that only local information about the neighborhood ranking of the current point is at the walker's disposal. In this case, one can think of several deterministic and stochastic strategies to maximize the number of visited points while trying to minimize the traveled distance.

Our aim is to study the way a walker explores the medium following the deterministic rule of going to the nearest point that has not been visited in the previous μ discrete time steps. We call this partially self-avoiding walk the *deterministic tourist walk*. In this dynamics, each trajectory depends on the starting point, presents a transient time, and ends in nontrivial cycles.

The paper presentation is arranged as follows. In Sec. II a brief review of results obtained for the deterministic dynamics proposed is presented. The model is then mapped into two other physical systems. The first one is a particle with constant energy moving in a quenched random rugged landscape. The second one is a walker moving in a random, directed, and weighted graph. In Sec. III, we consider a walker moving according to the deterministic tourist rule in semi-infinite disordered media. First, we calculate exactly the distribution of visited points, which allows us to justify a very good approximation using a simple mean-field argument. Second, we propose an alternative derivation for this distribution using the exploration and return probabilities, which allows application of the tourist walk in finite disordered media. This is done in Sec. IV, where we obtain the percolation probability and show the existence of a crossover in the walker's exploratory behavior at a critical memory $\mu_1=\ln N/\ln 2$ in a narrow memory range of width $\varepsilon=e/\ln 2$. This crossover splits the walker's behavior into essentially two regimes. For $\mu<\mu_1-\varepsilon/2$, the walker gets trapped in cycles, and for $\mu>\mu_1+\varepsilon/2$, the walker visits all the points. The calculated quantities have been validated by Monte Carlo simulations. In this way we show that the walker needs to have only a small memory (of order $\log_2 N$) to explore the whole disordered medium. The percolation in one-dimensional (1D) deterministic tourist walks shares some common features with other systems, as shown in Sec. V.

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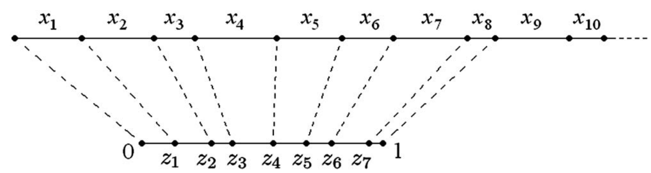

FIG. 1. Scheme showing the equivalence between a finite and a semi-infinite disordered medium. Along the upper line segment the points are generated using the random distances x_k with exponential probability distribution function. In the lower line segment, the number of points (N) is fixed and normalized to its total length, where z_k are the normalized coordinates.

II. PHYSICAL ANALOGIES TO THE DETERMINISTIC TOURIST WALKS

Consider a walker who explores the medium following the deterministic rule of going to the nearest point that has not been visited in the previous μ discrete time steps. Each trajectory produced by this deterministic rule has an initial transient of length t and ends in a cycle of period p. Both transient time and cycle period can be combined in the joint distribution $S_{\mu,d}^{(N)}(t,p)$. The deterministic tourist walk with memory $\mu=0$ is trivial. Every starting point is its own nearest neighbor, so the trajectory contains only one point. The transient and period joint distribution is simply $S_{0,d}^{(N)}(t,p)=\delta_{t,0}\delta_{p,1}$, where $\delta_{i,j}$ is the Kronecker delta function. With memory $\mu=1$, the walker must leave the current point at each time step. The transient and period joint distribution has been obtained analytically for $N\gg 1$ [10]. This memoryless rule ($\mu=1$) does not lead to exploration of the random medium, since, after a very short transient, the tourist gets trapped in pairs of points that are mutually nearest neighbors. Interesting phenomena occur when greater memory values are considered. In this case, the cycle distribution is no longer peaked at $p_{min}=\mu+1$, but presents a whole spectrum of cycles with period $p\geq p_{min}$, with possible power-law decay [11–13]. These cycles have been used as a clusterization method [14] and in image texture analysis [15,16].

It is interesting to point out that, for 1D systems, determinism imposes serious restrictions. For any μ value, cycles of period $2\mu+1\leq p\leq 2\mu+3$ are forbidden. Additionally, for $\mu=2$ all odd periods but $p_{min}=3$ are forbidden. Also, the heavy tail of the period marginal distribution $S_{\mu,1}^{(N)}(p)=\Sigma_t S_{\mu,1}^{(N)}(t,p)$ may lead to often-visited large-period cycles [11]. This allows system exploration even for small memory values ($\mu\ll N$).

In Euclidean space, the partially self-avoiding deterministic walk in one dimension occurs along a line segment, where N points are drawn from a uniform probability density. The first point (site s_1) is the starting point of the walk, the second point (s_2) is x_1 apart from the first point, s_3 is x_2 apart from s_2, and so forth until the last point. The dynamics is that the walker goes to the nearest point that has not been visited in the previous μ steps. It is convenient to consider the scaled distances $x_i'=x_i/L$, where $L=\Sigma_{i=1}^{n}x_i$ is the length of the line segment (see Fig. 1).

Next we show that this partially self-avoiding deterministic walk can be implemented in a regular lattice with N sites, with lattice constant $1/N$, so that it can be viewed as a particle moving in a rugged potential landscape. To be compatible with the deterministic tourist walk, the landscape must be constructed as follows. Initially we associate a potential $V_i'^{(+)}=x_{i+1}'$ and $V_i'^{(-)}=x_{i-1}'$ with the site i. We define the energy of the particle arriving at site i from the left as $E_i^{(+)}(\mu)=\Sigma_{j=i-\mu}^{i}x_j'$ and from the right as $E_i^{(-)}(\mu)=\Sigma_{j=i+1}^{i+\mu+1}x_j'$. The direction flip condition is $V_i'>E_i(\mu)$ (valid for both $\pm$ cases). Since the quenched random energy is not standard for mechanical particles, we write $E_i(\mu)=\bar{E}+(E_i-\bar{E})=\bar{E}+\eta_i$, where $\bar{E}$ can be thought of a constant particle energy and η is the energy fluctuation. Next, we rescale the potential landscape as $V_i=V_i'-\eta_i$, so that the flip condition simply reads $V_i>\bar{E}$. With this analogy we try to mimic the physical picture that, when the particle gets trapped within two potential barriers larger than its energy $\bar{E}$, a cycle is found. Notice that the cycle period depends on the number of bumps between the two bordering larger barriers, and that the landscape depends on the memory μ. This landscape is not fixed, since it depends on the direction of the movement of the particle. Also, the dimensionality d of the system affects the potential heights, through the distances.

In the previous analogy we transformed the random distances to random potentials in a regular lattice. Now we consider a more abstract description of the problem. Notice that, to implement the dynamics of the tourist walk, the walker only has to know the μ next nearest neighbors from his location. We stress that the walker does not have to know in advance the location of the N sites; only local information is required. Consider each point as a node of a graph. Each node is connected to μ neighboring nodes. These connections are directed and represent the movement from one point to one of its μ nearest neighbors. Also these connections are weighted, representing the neighborhood rank. This random directed weighted (disordered) graph has a special property: the number n of outgoing links of each node is fixed ($\delta_{n,\mu}$), but the number m of incoming links is variable and depends on the dimensionality d of the underlying Euclidean metric space used to construct the graph. The quantity m follows a binomial distribution parametrized by d [17]. The dynamics is implemented in this graph with a walker going from one node to the one with the smallest link weight that has not being visited in the previous μ steps.

Our objective is to obtain a typical μ value that allows the tourist to visit all points. The same value of μ makes all the lattice sites accessible to the particle in the random rugged landscape or, equivalently, all the graph nodes to the walker in the network formulation.

III. SEMI-INFINITE DISORDERED MEDIA

A random static semi-infinite medium is constructed of uncountable points that are randomly and uniformly distributed along a semi-infinite line segment with a mean density r. The upper line segment of Fig. 1 represents this medium, where the distances x_k between consecutive points are independent and identically distributed (i.i.d.) variables with exponential probability density function (PDF) $g(x)=re^{-rx}$ for

$x \geq 0$ and $g(x)=0$, otherwise. In the following we analytically obtain the statistics related to the deterministic tourist walk performed on semi-infinite random media.

A. Distribution of the number of visited points

Here we obtain analytically the probability $S^{(\infty)}_{\mu,si}(n)$ for a walker, with memory μ and moving according to the deterministic tourist rule, to visit n points of a semi-infinite medium. The exact result is obtained and this allows us to justify a simple mean-field approach.

1. Exact result

Consider a walker who leaves from the leftmost point s_1, placed at the origin of the upper line segment of Fig. 1. The conditions for the walker to visit $n \geq \mu+1$ distinct points are as follows.

(1) The distances $x_1, x_2, \ldots, x_\mu$ may assume any value in the interval $[0;\infty)$, since the memory μ prohibits the walker from moving backward in the first μ steps, so that the first $\mu+1$ points are indeed visited.

(2) Each of the following distances $x_{\mu+1}, x_{\mu+2}, \ldots, x_{n-1}$ must be smaller than the sum of the μ preceding step distances, until the tourist reaches the point s_n.

(3) The distance x_n must be greater than the sum of the μ preceding ones, to force the walker to move back to the point $s_{n-\mu}$, instead of exploring a new point s_{n+1}.

Once the walker has returned to the point $s_{n-\mu}$, he (she) may revisit the starting point s_1, get trapped in an attractor, or even revisit the point s_n, but he (she) will not be able to overpass the distance barrier x_n between the points s_n and s_{n+1}. Actually, no new points will be visited any longer. Combining these conditions, the probability for the walker to visit n distinct points is

$$S^{(\infty)}_{\mu,si}(n) = \prod_{j=1}^{\mu} \int_0^\infty dx_j r e^{-rx_j} \prod_{j=\mu+1}^{n-1} \int_0^{\sum_{k=j-\mu}^{j-1} x_k} dx_j r e^{-rx_j} \times \int_{\sum_{k=n-\mu}^{n-1} x_k}^{\infty} dx_n r e^{-rx_n}. \qquad (1)$$

The difficulty of obtaining $S^{(\infty)}_{\mu,si}(n)$ is that the n integrals are chained and the integration procedure must start from the rightmost factor. Applying the substitutions $y_j = e^{-rx_j}$, with $1 \leq j \leq n$, one has

$$S^{(\infty)}_{\mu,si}(n) = \prod_{j=1}^{n} \mathcal{I}_j, \qquad (2)$$

where the form of each functional $\mathcal{I}_j$ depends on j:

$$\mathcal{I}_j = \begin{cases} \int_0^1 dy_j & \text{for } 1 \leq j \leq \mu, \\ \int_{\tilde{y}_j}^1 dy_j & \text{for } \mu+1 \leq j \leq n-1, \\ \int_0^{\tilde{y}_j} dy_j & \text{for } j = n, \end{cases} \qquad (3)$$

and each integration limit $\tilde{y}_j = \prod_{k=j-\mu}^{j-1} y_k$ links $\mathcal{I}_j$ to the preceding μ integrals. This means that Eq. (2) must be evaluated from $\mathcal{I}_n$ to $\mathcal{I}_1$. Notice that r has been eliminated, indicating that the number of visited points does not depend on the medium density.

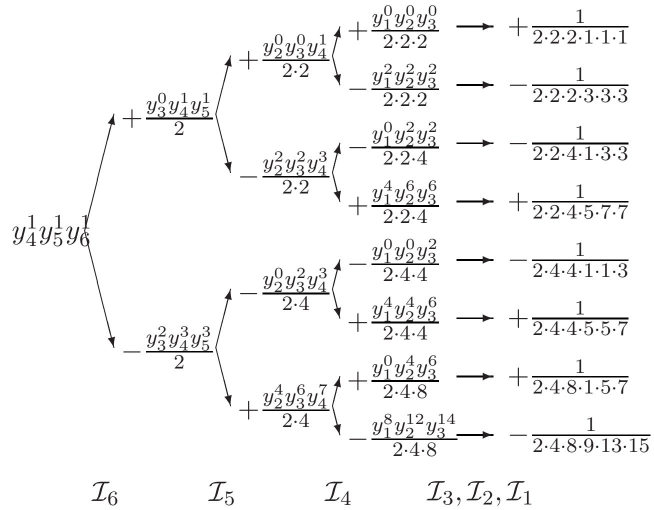

FIG. 2. Calculation scheme for the chained integrals of Eq. (2). Here we have considered the example of $\mu=3$ and $n=7$. We focus on the dynamics of the powers of y along the bifurcation path, which leads to the recursive relation Eq. (5).

We note that this calculation concerns dealing with the powers of y_j. Figure 2 illustrates the calculation of Eq. (2) for the particular case $\mu=3$ and $n=7$. In this scheme, the relevant quantities are the y_j powers in the integrand, since all y's disappear after all integration levels are performed. The integration process consists basically of three steps, where each one of them represents a case of Eq. (3).

(1) The first integral $\mathcal{I}_7$ [third case of Eq. (3)] is trivially evaluated to its upper limit $\tilde{y}_7$, yielding the root node $y_4^1 y_5^1 y_6^1$, with all the integrand variables raised to the first power. These powers are denoted as $a_1, a_2, \ldots, a_\mu$, where, in particular, a_μ is the power of the integrand of the current level.

(2) Each bifurcation level represents an integral from $\mathcal{I}_6$ to $\mathcal{I}_4$ [second case of Eq. (3)]. (a) A unit is added to the power a_μ and it becomes a new factor $a_\mu+1$ at the denominator of the following level [this is just $\int y^a dy = y^{a+1}/(a+1)$]. (b) For each bifurcation, in the upper fractions, the remaining variables keep their powers and the new y is raised to 0. (c) In the lower fraction, we sum $a_\mu+1$ to each power of y and the fraction sign is switched.

(3) The last level represents the integrals from $\mathcal{I}_3$ to $\mathcal{I}_1$ [first case of Eq. (3)], where a unit is added to all powers a_1, $a_2, \ldots, a_\mu$ and they become new factors in the denominator.

Generalizing the reasoning of the scheme of Fig. 2 for arbitrary μ and n, Eq. (2) may be written as the following recursive formula:

$$S^{(\infty)}_{\mu,si}(n) = f_\mu[n, \vec{1}], \quad n = \mu+1, \mu+2, \ldots, \infty, \qquad (4)$$

with

$$f_\mu[j,\vec{a}] = \begin{cases} \dfrac{f_\mu[j-1,\mathrm{shift}(\vec{a})] - f_\mu[j-1,\mathrm{shift}(\vec{a}) + (a_\mu+1)\cdot\vec{1}]}{a_\mu+1} & \text{if } j > \mu+1, \\ 1 \Big/ \prod_{k=1}^{\mu}(a_k+1) & \text{if } j = \mu+1, \end{cases} \tag{5}$$

where $\vec{1}=(1,1,1,\ldots,1)$ and $\vec{a}=(a_1,a_2,a_3,\ldots,a_\mu)$ are μ-dimensional vectors, $\mathrm{shift}(\vec{a})=(0,a_1,a_2,\ldots,a_{\mu-1})$ is the acyclic shifting of coordinates, and j is the integration level, also used as the stop condition. Observe that the initial condition $\vec{1}$ of Eq. (4) and the upper and lower cases of Eq. (5) represent the third, second, and first cases of Eq. (3), respectively.

The minimum allowed cycle period in the deterministic tourist walk is $p_{min}=\mu+1$ [11]. Once the memory μ assures that the walker visits at least $\mu+1$ points, the number of extra visited points $n_e=n-p_{min}$ is the relevant quantity since all the distributions $S^{(\infty)}_{\mu,si}$ start at the same point $n_e=0$, regardless of the μ value.

Although the recursive relation of Eq. (5) is exact, it is not efficient for algebraic treatment. Even for numerical calculation it presents several disadvantages. It is difficult to implement due to its recursive structure and the processing time grows exponentially. Such exponential time dependence limited the plots of Figs. 3 and 4 to $n_e \leq 30$. The continuous lines of Fig. 3 represent Eq. (4) for different values of μ. As one can see from this figure, a remarkable property is

$$S^{(\infty)}_{\mu,si}(n_e=0) = \frac{1}{2^\mu} \tag{6}$$

for all μ. This is exactly the probability to have a null transient and a cycle with minimum period p_{min} in the one-dimensional tourist walk.

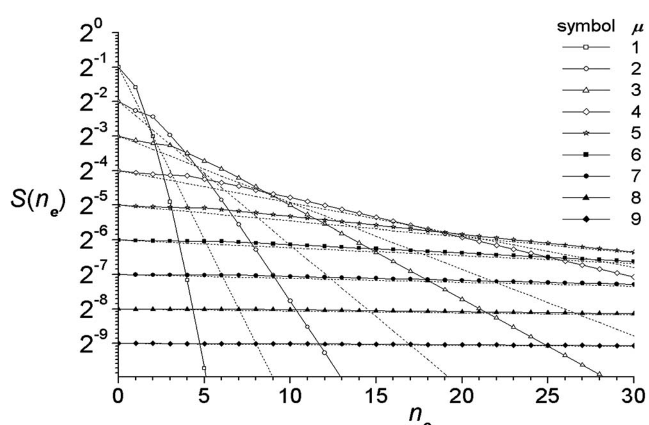

FIG. 3. Distribution of n_e for μ varying from 1 to 9. Continuous lines refer to exact form of Eq. (4) and dotted lines refer to approximate form of Eq. (8).

2. Mean-field approximation

The recursivity of Eq. (5) has been inherited from the chained integrals of Eq. (2). However, for $\mu \gg 1$ a mean-field approximation may be used to untie those integrals. It consists of replacing the products $\tilde{y}_j$ by their mean values.

To fully appreciate this mean-field argument, consider first the distribution of a product of uniform deviates. Let y_1, $y_2,\ldots,y_\mu$ be μ independent random variables uniformly distributed on the interval $(0,1]$. To obtain the PDF $p(\tilde{y})$ of the product $\tilde{y}=\Pi_{k=1}^{\mu} y_k$, let us apply the transformation $\tilde{w}=-\ln \tilde{y} = \Sigma_{k=1}^{\mu} w_k$, where $w_k=-\ln y_k$ with $1\leq k\leq \mu$ are i.i.d. variables with exponential PDF of unitary mean. Thus, the sum $\tilde{w}$ follows a Γ PDF $p(\tilde{w})=\tilde{w}^{\mu-1}e^{-\tilde{w}}/\Gamma(\mu)$. Since $|p(\tilde{y})d\tilde{y}|=|p(\tilde{w})d\tilde{w}|$, one obtains the distribution of $\tilde{y}$: $p(\tilde{y}) = (-\ln \tilde{y})^{\mu-1}/\Gamma(\mu)$, whose mth moment is $\langle \tilde{y}^m\rangle=(m+1)^{-\mu}$.

The above tools can be used due to the fact that all the variables $y_j=e^{-rx_j}$ [applied to Eq. (1)] are i.i.d. for a uniform deviate in the interval $(0,1]$. The first condition $(1\leq j\leq \mu)$ of Eq. (3) states that the variables y_1, $y_2,\ldots,y_\mu$ may freely vary from 0 to 1. Since for $\mu\gg 1$ the product $\tilde{y}_{\mu+1}=\Pi_{k=1}^{\mu} y_k$ has a small variance, it can be approximated by its mean value $\langle \tilde{y}_{\mu+1}\rangle = 2^{-\mu}$.

Concerning the next product $\tilde{y}_{\mu+2}=\Pi_{k=2}^{\mu+1} y_k$, the variables y_2, $y_3,\ldots,y_{\mu+1}$ are not all i.i.d., because $y_{\mu+1}$ has just been constrained to the interval $[2^{-\mu},1]$. However, for $\mu\gg 1$, the interval $[2^{-\mu},1]$ becomes close to $[0,1]$, allowing $\tilde{y}_{\mu+2}$ to be also approximated by the mean value $2^{-\mu}$. This reasoning can be inductively applied for the remaining integration limits $\tilde{y}_j$. Thus, Eq. (3) is approximated to

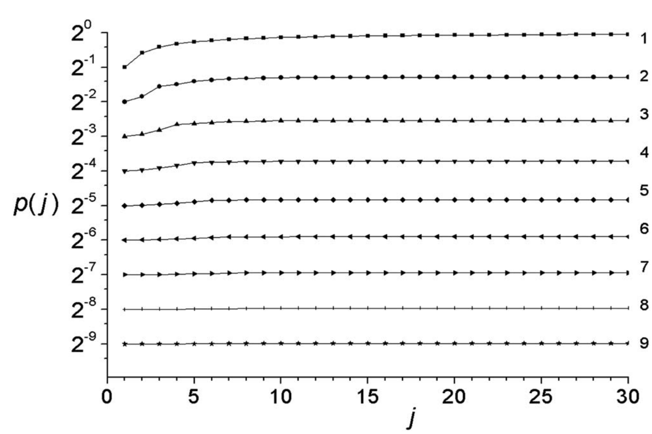

FIG. 4. Return probability given by Eq. (12), with μ varying from 1 to 9.

$$\mathcal{I}_j \approx \begin{cases} \int_0^1 dy_j & \text{for } 1 \le j \le \mu, \\ \int_{2^{-\mu}}^1 dy_j & \text{for } \mu+1 \le j \le n-1, \\ \int_0^{2^{-\mu}} dy_j & \text{for } j = n. \end{cases} \tag{7}$$

Observe that these integrals are no longer chained and that $S_{\mu,si}^{(\infty)}(n)$ is still given by Eq. (2), which leads to

$$S_{\mu,si}^{(\infty)}(n) \approx 2^{-\mu}(1-2^{-\mu})^{n-\mu-1}, \tag{8}$$

with $n=\mu+1,\mu+2,\ldots,\infty$, and yields $E(n)=2^{\mu}+\mu$, which may be interpreted as the *characteristic range* of the walk, and $\mathrm{var}(n)=2^{2\mu}-2^{\mu}$. The dotted lines in Fig. 3 represent this approximation for $1\le\mu\le 9$.

B. Exploration and return probabilities

The purpose of the calculation of the exploration and return probabilities is twofold. It is an alternative argument to obtain Eq. (8), and these probabilities lead to simple arguments to obtain the percolation probability for a finite disordered medium.

1. Upper tail cumulative probability: An exact calculation

A similar argument to that used to obtain Eq. (4) may be used to obtain the upper tail cumulative distribution $\overline{F}_{\mu,si}^{(\infty)}(n)=\Sigma_{k=n}^{\infty}S_{\mu,si}^{(\infty)}(k)$, which gives the probability for the walker to visit at least n points. The only modification is that, once the walker has reached the point s_n, he (she) can move either backward or forward. Therefore, the rightmost integral of Eq. (1) is no longer necessary, so

$$\overline{F}_{\mu,si}^{(\infty)}(n) = \prod_{j=1}^{n-1} \mathcal{I}_j, \tag{9}$$

where each functional $\mathcal{I}_j$ is given by Eq. (3). The root node of Fig. 2 is now set to 1 (or, equivalently, $y_4^0y_5^0y_6^0$), which leads to

$$\overline{F}_{\mu,si}^{(\infty)}(n) = f_\mu[n,\vec{0}], \quad n=\mu+1,\mu+2,\ldots,\infty, \tag{10}$$

where $\vec{0}=(0,0,\ldots,0)$ is the μ-dimensional null vector and f_μ is given by Eq. (5). Observe that $\overline{F}_{\mu,si}^{(\infty)}(n)$ uses the same recursive structure as Eq. (4), but with a different initial condition ($\vec{0}$ instead of $\vec{1}$). If Eq. (7) is used as an approximation to evaluate Eq. (9), one readily has

$$\overline{F}_{\mu,si}^{(\infty)}(n) \approx (1-2^{-\mu})^{n-\mu-1}. \tag{11}$$

The memory μ assures that the walker, leaving from the point s_1, moves forward in the first μ steps. In contrast, the following steps are uncertain, since the walker may either move forward and visit a new point or return and stop the medium exploration. In analogy to the geometric distribution, it is useful to define the exploration probability $q_\mu(j)$ (taken as failure) as the probability for the walker to visit a new point at the jth uncertain step.

Therefore, the return probability $p_\mu(j)$ (taken as success) for the jth uncertain step is equal to the probability for the walker to visit exactly $n=\mu+j$ points conditioned on the fact that he (she) has already visited $n=\mu+j-1$ points. This probability is given by

$$p_\mu(j) = \frac{S_{\mu,si}^{(\infty)}(n=\mu+j)}{\overline{F}_{\mu,si}^{(\infty)}(n=\mu+j)} = \frac{f_\mu[\mu+j,\vec{1}]}{f_\mu[\mu+j,\vec{0}]}, \tag{12}$$

where f_μ is given by Eq. (4).

Figure 4 shows the behavior of $p_\mu(j)$ for the first 30 uncertain steps, with μ varying from 1 to 9. One can observe that for $\mu\gg 1$ the return probability $p_\mu(j)$ along the walk is almost constant and equal its initial value $p_\mu(1)=2^{-\mu}$. In this way, one can verify empirically that for $\mu\gg 1$ the return probabilities can be taken as $p_\mu=2^{-\mu}$ for all steps, and $q_\mu=1-2^{-\mu}$ can be taken for all exploration probabilities.

This empirical approximation for the return probability can be justified analytically using Eqs. (8) and (11) in its definition:

$$p_\mu(j) = \frac{S_{\mu,si}^{(\infty)}(n=\mu+j)}{\overline{F}_{\mu,si}^{(\infty)}(n=\mu+j)} \approx \frac{1}{2^\mu}. \tag{13}$$

For $\mu=1$, n_e is numerically equal to the transient time t (which does not mean that they are the same part of the trajectory; the transient is the beginning of it and n_e counts the final points), and in this case Eqs. (4), (10), and (12) assume the simple exact closed forms $S_{1,si}^{(\infty)}(n_e)=(n_e+1)/(n_e+2)!$, $\overline{F}_{1,si}^{(\infty)}(n_e)=1/(n_e+1)!$, and $p_1(j)=j/(j+1)$, which have been previously found in Ref. [10].

2. An alternative derivation

The approximate expressions for exploration and return probabilities can also be obtained by analytical means through a more direct derivation. Consider again the tourist dynamics with a walker who leaves from the point s_1, placed at the origin of the semi-infinite medium.

The first $\mu+1$ points are indeed visited, because the memory μ prohibits the walker from returning. Thus, the distances $x_1,x_2,\ldots,x_\mu$ may assume any value in the interval $[0,\infty)$.

The exploration probability $q_\mu(1)$ for the first uncertain step can be obtained by imposing that the distance $x_{\mu+1}$ is smaller than the sum $\tilde{x}_1=\Sigma_{k=1}^{\mu}x_k$. Since the variables x_1, $x_2,\ldots,x_\mu$ are i.i.d. with exponential PDF, $\tilde{x}_1$ has a Γ PDF. Hence $q_\mu(1)=[\int_0^\infty d\tilde{x}_1 r^\mu \tilde{x}_1^{\mu-1}e^{-r\tilde{x}_1}/\Gamma(\mu)]\int_0^{y_1}dx_{\mu+1}re^{-rx_{\mu+1}}=1-2^{-\mu}$.

The exploration probability $q_\mu(2)$ for the second uncertain step is not exactly equal to $q_\mu(1)$. Since the distance $x_{\mu+1}$ must vary in the interval $[0,\tilde{x}_1]$, the variables $x_2,x_3,\ldots,x_{\mu+1}$ are not all independent, and consequently $\tilde{x}_2=\Sigma_{k=2}^{\mu+1}x_k$ has not exactly a Γ PDF. However, for $\mu\gg 1$, $x_{\mu+1}$ rarely exceeds $\tilde{x}_1$ [this probability is just $P(x_{\mu+1}>\tilde{x}_1)=1-q_\mu(1)=2^{-\mu}$, meaning that a weak correlation is present for $\mu\gg 1$]. Therefore, one can make an approximation assuming that $\tilde{x}_2$ still follows a Γ PDF and considering $q_\mu(2)\approx q_\mu(1)$. The same arguments can be used for the succeeding steps.

When the point s_n is reached, the walker must turn back, stopping the medium exploration. Once $q_\mu(1)$ is taken for all q_μ, the return probability is $p_\mu=1-q_\mu=2^{-\mu}$ and one has $S^{(\infty)}_{\mu,si}(n_e)=2^{-\mu}(1-2^{-\mu})^{n_e}$, which is the result of Eq. (8).

IV. PERCOLATION PROBABILITY FOR FINITE DISORDERED MEDIA

The finite disordered medium is constructed of N points whose coordinates z_k are randomly generated in the interval $[0,1]$ following a uniform deviate as depicted in Fig. 1.

Numerical simulation results pointed out that the exploration and return probabilities obtained for the semi-infinite medium may also be applied to this finite medium. This is not trivial, since all results for the semi-infinite medium have been obtained assuming that the distances between consecutive points are i.i.d. variables with exponential distribution. Obviously the distances between consecutive points in the finite medium are not i.i.d. variables, nor do they have exponential distribution.

Nevertheless, the equivalence between these two media can be argued as follows. On one hand, the abscissas of the ranked points in the finite medium follow a β PDF [18]. On the other hand, if one restricts the semi-infinite medium length to the first $N+1$ distances and normalizes it to fit in the interval $[0,1]$, then the abscissa of its kth ranked point is $z_k=U_k/(U_k+V_k)$, where $U_k=x_1+x_2+\cdots+x_k$ and $V_k=x_{k+1}+x_{k+2}+\cdots+x_{N+1}$. Figure 1 shows an example for $N=7$ normalization. Since U_k and V_k have Γ PDFs, z_k has a β PDF [18], as in the genuine finite medium. This normalization does not affect the tourist walk, because in this walk only the neighborhood ranking is relevant, not the distances themselves [11,13].

The probability $P_N(\mu)$ for the exploration of the whole N-point medium can be derived by noticing that the walker must move forward $N-(\mu+1)$ uncertain steps and, when the last point s_N is reached, there is no need to impose a return step. Therefore the percolation probability is

$$P_N(\mu)=q_\mu^{N-(\mu+1)}=(1-2^{-\mu})^{N-\mu-1}. \tag{14}$$

It is interesting to note that the percolation probability relates directly to the upper tail cumulative function as shown by Eq. (11). The difference between them is only in the interpretation of the number of visited points N, but this can be justified because of the normalization to the finite medium discussed above.

Figure 5 shows a comparison of the evaluation of Eq. (14) and the results of Monte Carlo simulations. From this figure one clearly sees that the probability of full exploration increases abruptly from almost 0 to almost 1.

From the analogy with a first-order phase transition, we define the crossover point as the maximum of the derivative of $P_N(\mu)$, with respect to μ. This implies that the second derivative vanishes at the maximum $d^2_\mu P_N(\mu)|_{\mu_1^{(c)}}=0$, leading to a transcendental equation, which cannot be solved analytically to obtain $\mu_1^{(c)}$. An estimated value of $\mu_1^{(c)}$ can be calculated considering $N\gg\mu\gg 1$, Eq. (14) may be approximated to $P_N(\mu)=(1-2^{-\mu})^N$ and, at the inflection point, one has

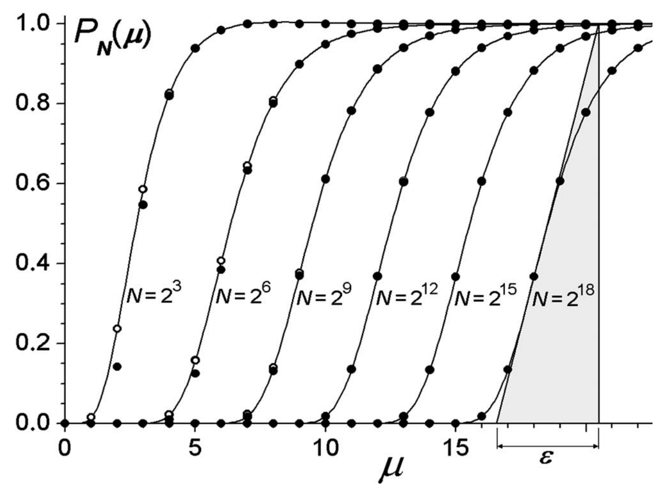

FIG. 5. Percolation probability for some fixed N values. Empty circles are given by Eq. (14) and full ones represent numerical simulations ($M=100\,000$ maps for each N and μ value); error bars are smaller than the symbol size. Continuous lines are plotted only to guide eyes. Analytical results are satisfactory, when compared to numerical simulation, even for small N and μ values. The crossover points μ_1 are given by Eq. (15); they are weakly dependent on N but all of them have the same constant dispersion $\varepsilon\sim 4$ [Eq. (16)].

$$\mu_1=\log_2 N. \tag{15}$$

A simple interpretation can be given to μ_1. It is just the number of necessary bits to represent the system size N. Also μ_1 is typically the size of the extra array needed to find a given element in an ordered list by the QUICKSORT method [[19], p. 333]. To evaluate the width of the crossover region, use the slope of $P_N(\mu)$ at μ_1, which results in $\ln 2/e$, for all N (see Fig. 5). The crossover region has a constant width

$$\varepsilon=\frac{e}{\ln 2}\approx 3.92. \tag{16}$$

On one hand, as N increases, the critical memory slowly increases (as $\log_2 N$), but its deviation is independent of the system size, so that a sharp crossover is found in the thermodynamic limit ($N\gg 1$). We stress that the approximations employed lead to satisfactory results even for small N and μ values.

On the other hand, if one uses the reduced memory $\tilde{\mu}=(\mu-\mu_1)/\mu_1$, the crossover occurs at $\tilde{\mu}_1=0$, but now the crossover width depends on the size of the system as $1/\log_2 N$.

V. CONCLUSION

Our main result is that the walker does not need to have memory of order N to explore the whole medium; a small memory (of order $\ln N$) allows this full exploration. All the exact results calculated here are in accordance with the limiting case $\mu=1$ obtained in Ref. [10]. An interesting exact result that we have obtained in the one-dimensional deterministic tourist walk is that the probability to have a null transient and a minimum cycle is $2^{-\mu}$, where $\mu<\mu_1$ is the memory of the walker.

The crossover found here is similar to the one found in the k-sat (satisfiability) problem [20–22]. In this problem one considers N logical variables and a set of M AND clauses. Each clause is a logical OR of k logical variables, where each variable can be negated. The objective is to obtain a sat solution, i.e., a realization of the N variables that satisfies the M clauses. This problem can be mapped to a Hamiltonian where the N Ising spins correspond to the logical variables; the exchange coupling can take only values ± 1 to represent negation or not of a spin variable. The OR operation is mapped to a product of k spins and the M AND clauses are represented by a summation. The control parameter is $\alpha = M/N$, and a sharp k-dependent crossover at α_k separates the satisfiable and unsatisfiable regimes. For $\alpha < \alpha_k$, sat solutions are obtained with times depending algebraically on system size N, while for $\alpha > \alpha_k$ no sat solutions can be found. The worst case is $\alpha \sim \alpha_k$, where the sat solutions exist, but they are obtained typically with times that increase exponentially with N.

At μ_1 one can think of the appearance of the first walker to cross the disordered medium with the deterministic tourist walk. This crossing typically appears for walker memory of the order of $\log N$ [23]. The disordered Erdös-Rényi random graph and scale-free networks present weights associated with their links. These weights are exponentially distributed, and the mean value (model parameter) represents the disorder strength; for weak disorder, the optimal path connecting any two nodes increases as $\ln N$.

The distance constraints can be generalized to a d-dimensional Euclidean space and possibly this calculation scheme can be employed in this interesting situation.

Finally, the tourist rule can be relaxed to a stochastic walk. In this case, the walker goes to nearer cities with greater probabilities, given by a one-parameter (inverse of the temperature) exponential distribution. This situation has been studied for the nonmemory cases (μ=0 [24] and 1 [25]), and we have detected the existence of a critical temperature separating the localized from the extended regime. It would be interesting to combine in the tourist walks both stochastic movement (driven by a temperature parameter) and memory (μ).

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