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Power-law friction in closely packed granular materials

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In order to understand the nature of friction in dense granular materials, a discrete element simulation on granular layers subjected to isobaric plain shear is performed. It is found that the friction coefficient increases as the power of the shear rate, the exponent of which does not depend on the material constants. Using a nondimensional parameter that is known as the inertial number, the power law can be cast in a generalized form so that the friction coefficients at different confining pressures collapse on the same curve. We show that the volume fraction also obeys a power law.

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Friction is one of the oldest problems in science because it dominates various phenomena in our daily life. In particular, the dynamics of granular flow, which is ubiquitous in earth sciences and engineering, is governed by a law that describes the behavior of the friction coefficient (ratio of the shear stress to the normal stress). Examples are avalanches, landslides, debris flow, silo flow, etc. In addition, the nature of friction on faults, which plays a key role in earthquake mechanics [1,2], is also related to that of granular rock because the fault zone consists of layers of fine rock particles that are ground up by fault motion of the past. To find a suitable law of friction in granular materials under a specific condition is thus an essential problem.

Although the frictional properties of granular materials are so important, our understanding is still limited. In the context of earthquake mechanics, the slip velocity (or shear rate) dependence of the friction coefficient, which is related to the rheology under constant pressure conditions, is a matter of interest [2]. In experiments on thin granular layers that are sheared at relatively low sliding velocities ranging from nm/s to mm/s, the behavior of the friction coefficient can be described by a phenomenological law in which the friction coefficient depends logarithmically on the sliding velocity. This is known as the rate- and state-dependent friction (RSF) law [3]. Note that the RSF law also applies to friction at interfaces between two solids, as well as that in granular layers. Although the RSF law applies well to lower-speed (creeplike) friction, it is violated in high-speed friction. For example, several experiments indicated nonlogarithmic increase of the friction coefficient in granular layers at higher sliding velocities [4–6]. The same tendency was also observed in experiments on friction between two sheets of paper [7,8]. However, at this point, we do not know any friction law that is valid at such higher velocities.

Several recent attempts to understand the nature of friction in granular media under high shear rates are noteworthy here. Jop and co-workers presented a simple friction law that describes flow on inclined planes [9], based on massive simulations and experiments [10]. Although their friction law seems feasible, it involves rather dilute flow and its applicability to denser and slower flow (e.g., quasistatic flow) is not clear. Da Cruz *et al.* [11] performed an extensive simulation that focused on the dense and slow regime and found a friction law that does not contradict that of Jop *et al.* However, because da Cruz *et al.* studied a two-dimensional system, the effect of the dimensionality may be questioned. In particular, in the quasistatic regime, where the nature of interparticle contacts plays an essential role in rheology, the effect of dimensionality should be seriously investigated.

In this Rapid Communication, we perform a three-dimensional simulation in order to understand the nature of friction in a slowly sheared dense granular material. Our particular interest is a dense granular matter under high confining pressure (e.g., tens of megapascals) which roughly corresponds to a typical configuration of faults at a seismic slip. Note that the RSF law is violated in such a situation. We report a law in which the friction coefficient increases as a power of the shear rate.

In the following we describe the computational model of granular layers. The individual constituents are assumed to be spheres, and their diameters range uniformly from $0.7d$ to $1.0d$. The interaction force follows the discrete element method [12]. Consider a grain i of radius R_i located at $\mathbf{r}_i$ with the translational velocity $\mathbf{v}_i$ and the angular velocity $\mathbf{\Omega}_i$. This grain interacts with another grain j whenever it overlaps; i.e., $|\mathbf{r}_{ij}|<R_i+R_j$, where $\mathbf{r}_{ij}=\mathbf{r}_i-\mathbf{r}_j$. The interaction consists of two kinds of forces, which are normal and transverse to $\mathbf{r}_{ij}$, respectively. Introducing the unit normal vector $\mathbf{n}_{ij}=\mathbf{r}_{ij}/|\mathbf{r}_{ij}|$, the normal force acting on i, which is denoted by $\mathbf{F}_{ij}^{(n)}$, is given by $[f(\epsilon_{ij})+\zeta\mathbf{n}_{ij}\cdot\dot{\mathbf{r}}_{ij}]\mathbf{n}_{ij}$, where $\epsilon_{ij}=1-|\mathbf{r}_{ij}|/(R_i+R_j)$. The function $f(\epsilon)$ describes the elastic repulsion between grains. Here we test two models: $f(\epsilon)=k\epsilon$ (the linear force) and $f(\epsilon)=k\epsilon^{3/2}$ (the Hertzian force) [13]. Note that the constant k/d^2 is on the order of the Young's modulus of the grains. In order to define the transverse force, we utilize the relative tangential velocity $\mathbf{v}_{ij}^{(t)}$ defined by $(\dot{\mathbf{r}}_{ij}-\mathbf{n}_{ij}\cdot\dot{\mathbf{r}}_{ij})+(R_i\mathbf{\Omega}_i+R_j\mathbf{\Omega}_j)/(R_i+R_j)\times\mathbf{r}_{ij}$ and introduce the relative tangential displacement vector $\mathbf{\Delta}_{ij}^{(t)}=\int_{\mathrm{roll}}dt\,\mathbf{v}_{ij}^{(t)}$. The subscript in the integral indicates that the integral is performed when the contact is *rolling*; i.e., $|\Delta_{ij}^t|<k_t$ or $\Delta_{ij}^t v_{ij}^t<0$. Then the tangential force acting on the particle i is written as $-\min(\mu\dot{\mathbf{r}}_{ij}/|\dot{\mathbf{r}}_{ij}|,k_t\mathbf{\Delta}_{ij}^{(t)})|\mathbf{F}_{ij}^{(n)}|$. In the case that $\mu=0$, the tangential force vanishes and the rotation of particles does not affect the translational motion. The parameter values adopted in the present simulation are given in Table I.

The configuration of the system mimics a typical experiment on granular layers subjected to simple shear. Note that there is no gravity in the system. The system spans a volume $L_x\times L_y\times L_z$, and is periodic in the x and the y directions. We

TABLE I. The parameters of the discrete element simulation.

Polydispersity	$\zeta\sqrt{d/km}$	$k_t d$	μ	Pd^2/k
30%	1	0.005	0–0.6	$3.8\times10^{-5}-1.1\times10^{-2}$

prepare two systems of different aspect ratios, each of which contains approximately 10 000 particles: $25d\times25d\times8d$ and $15d\times15d\times25d$. As we shall discuss later, the difference of the aspect ratio does not affect the rheology. In the z direction, there exist two rough walls that consist of the same kind of particles as those in the bulk. The particles in the walls are randomly placed on the boundary and their relative positions are fixed. The walls are parallel to each other and displaced antiparallel along the y axis at constant velocities $\pm V/2$, while they are prohibited from moving along the x axis. One of the walls is allowed to move along the z axis so that the pressure is kept constant at P. Using the mass of the wall, M_w, which is defined as the sum of the masses of the constituent particles, the z coordinate of the wall, Z_w, is described by the following equation of motion: $M_w\ddot{Z}_w = F_z - PS$, where $\mathbf{F}$ denotes the sum of the forces between the wall particles and the bulk particles, and S denotes the area of the wall. Then the z component of the velocity of the wall particles is given by $\dot{Z}_w$. Note that the friction coefficient of the system is defined by F_y/PS.

The system reaches a steady state after a certain amount of displacement of the walls. We judge that the system reaches a steady state if each of the following quantities shows no apparent trend and seems to fluctuate around a certain value: the friction coefficient, the z coordinate of the wall (i.e., the density), and the granular temperature. Also snapshots of the velocity profile are observed to ensure the realization of uniform shear flow. We confirm that the transient behaviors of the friction coefficient and of the volume increase are quite similar to those observed in experiments. Here we do not investigate such transients and restrict ourselves to steady-state friction.

Because uniform shear flow is unstable in a certain class of granular systems, we must check the internal velocity profiles at steady states. There is a strict tendency for shear flow to localize near the walls in the case that the confining pressure is small and/or the sliding velocity is large. This kind of spatial inhomogeneity is rather ubiquitous in granular flow, and has been extensively investigated [14,15]. In our simulation, uniform shear flow is realized at lower sliding velocities and higher confining pressures. Here we discuss exclusively the case in which uniform shear is realized. In this case, the shear rate is proportional to the sliding velocity of the walls; i.e., $\gamma = V/L_z$.

We investigate the behaviors of the friction coefficient of the system, $F_y/PS \equiv M$. The control parameters that affect the friction coefficient are the shear rate γ and the pressure P. It is useful to represent the control parameters in terms of nondimensional numbers, because the friction coefficient is a nondimensional number and hence must be a function of nondimensional numbers. Thus γ and P are recast in the following forms: $I=\gamma\sqrt{m/Pd}$ and $\Pi = Pd^2/k$. In particular, the former is referred to as the inertial number [16], which dominates the frictional behavior of granular flows. Hereafter we discuss the nature of friction, taking advantage of these nondimensional numbers.

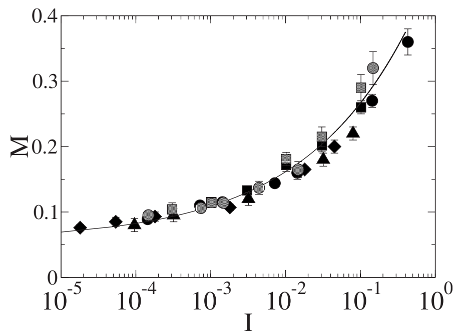

FIG. 1. Friction coefficients M in the models without tangential force, i.e., $\mu=0$. The horizontal axis I denotes the nondimensional shear rates $V/L\sqrt{m/Pd}$. The shape of the symbols and the confining pressure are in one-to-one correspondence: the squares to $\Pi=3.8\times10^{-5}$, the circles to $\Pi=1.9\times10^{-3}$, the triangles to $\Pi=3.9\times10^{-3}$, and the diamonds to $\Pi=1.1\times10^{-2}$. The solid symbols denote the friction coefficient of the Hertzian force model, while the gray symbols denote that of the linear force model. The layer thickness $L_z/d\simeq8$ for the squares and circles, while $L_z/d\simeq26$ for the triangles and diamonds. The solid lines denote Eq. (1) with $\phi=0.3$.

In order to grasp the main point of our result, it is convenient to begin with frictionless particles, i.e., $\mu=0$. We test two models, each of which has different interaction: the Hertzian contact model and the linear force model. As shown in Fig. 1, the friction coefficients of these two models are collapsed on the following master curve:

$$M = M_0 + sI^{\phi}, \tag{1}$$

where M_0 denotes the friction coefficient for $\gamma\to0$. Here $M_0\simeq0.06$. Note that the effect of the inertial number is expressed by a power law, sI^{ϕ}, where $\phi=0.28\pm0.05$. The prefactor s is approximately 0.4 for the both models [17].

In order to check the universality of Eq. (1), we wish to confirm the independence of our results from the details of the model. First we discuss the effect of the tangential force between particles. In Fig. 2 the friction coefficients of the models in which $\mu=0.2$ and 0.6 are shown. It is noteworthy that they range from 0.3 to 0.4, which are not significantly different from those obtained in an experiment on spherical glass beads [18]. More importantly, the friction coefficients again obey Eq. (1) with $\phi\simeq0.3$, regardless of the value of μ and the force model used (the linear or the Hertzian). Indeed the friction coefficients of both models are almost the same. We also remark that the factor s does not depend on μ; $s=0.33\pm0.03$ for $\mu=0$, 0.2, and 0.6.

On the other hand, M_0 depends on μ. In the linear force model, $M_0\simeq0.06$ for $\mu=0$, while $M_0\simeq0.26$ for $\mu=0.2$, and $M_0\simeq0.4$ for $\mu=0.6$. A similar dependence was also observed in Refs. [11,19]. Although M_0 looks like the static friction coefficient, note that it is defined in the $\gamma\to0$ limit and is different from the static friction coefficient above which a static system begins to flow. In order to distinguish the two

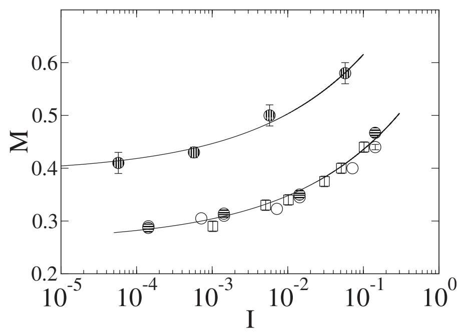

FIG. 2. Friction coefficients in models with tangential force (μ=0.2,0.6). The shape of the symbols and the confining pressure are in one-to-one correspondence as in Fig. 1. The blank symbols denote the friction coefficient of the linear force model with μ=0.2, while the symbols with vertically striped pattern denote that of μ=0.6. The circles of horizontally striped pattern denote the Hertzian force model with μ=0.2. The lines denote Eq. (1) with ϕ=0.3.

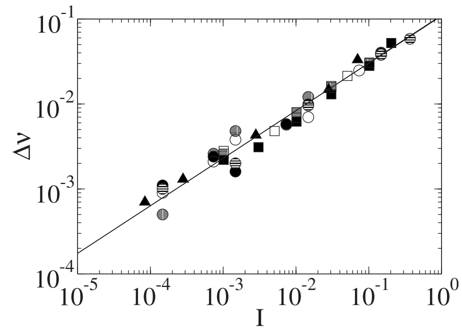

FIG. 3. Dilatation law. Decrease of the volume fraction $\Delta\nu=\nu_0-\nu$ is plotted as a function of the inertial number. Note that ν_0 is the volume fraction in the $\gamma\to 0$ limit, which is estimated by extrapolation. The symbol legends are the same as those in Figs. 1 and 2. The line denotes Eq. (2).

concepts, M_0 is referred to as the dynamic yield strength. The difference is important when we consider the stability of slip, as will be discussed in the last paragraph of this Rapid Communication.

It is important to notice that the data at different confining pressures collapse on the same curve by virtue of the inertial number. This suggests that the friction coefficient of a dense granular material does not depend on Π, as long as it is small (in the present simulation $3.8\times 10^{-5}\leqslant\Pi\leqslant 1.1\times 10^{-2}$). Indeed, da Cruz *et al.* [11] found that the friction coefficient is independent of Π (κ^{-1} in their notation) up to $\Pi\leqslant 2.5\times 10^{-2}$ in a two-dimensional system. Therefore the independence of Π is very likely within the accuracy of these simulations. While one can still expect that dependence may appear for larger Π, a case like $\Pi\sim 0.1$ is meaningless as a model of a granular material. In order to see this, recall that Π roughly corresponds to the average overlap length of contacts divided by the particle diameter; namely, the average strain of individual particles.

Then we discuss the effect of inelasticity that is modeled by the viscous coefficient ζ. The corresponding nondimensional number $\tilde{\zeta}$ is defined by $\zeta\sqrt{d/km}$. We find that decrease of $\tilde{\zeta}$ reduces the friction coefficient in the region where $I\gtrsim 0.01$, while the frictional strength is independent of $\tilde{\zeta}$ for the smaller-I region. This behavior is consistent with those obtained in Refs. [11,19]. Nevertheless, it can be still described by Eq. (1) with s being a smaller value. For example, the friction coefficient of a system in which $\tilde{\zeta}$=0.05 is described by $s\simeq 0.27$ with almost the same values of M_0 and δ. However, the functional form of $s(\tilde{\zeta})$ is not clear at this point.

From the discussions so far, we can conclude that the details of the present model do not affect the validity of Eq. (1), which is the main result of this study. Importantly, the exponent ϕ seems to be universal; it is approximately 0.3 regardless of the details of the model and the control parameters. The velocity-strengthening nature of this friction law does not contradict experiments [4–8]. In addition, it illustrates the universality of the power law in rheological properties of random media [20], including foams [21] and human neutrophils [22]. In the following we discuss four important points that are peripherally related to the main result.

First, we discuss the dependence of the volume fraction on the inertial number. Surprisingly, decrease of the volume fraction caused by shear flow is also described by a power law.

$$\nu_0-\nu=s_2 I^{\delta}, \tag{2}$$

where ν_0 is the volume fraction in the $\gamma\to 0$ limit. Note that the constants s_2 and δ do not depend on the details of the model. Figure 3 shows that all of the data obtained in our model collapse on Eq. (2) with $s_2\simeq 0.11$ and δ=0.56±0.02. This dilatation law also illustrates the ubiquity of the power law in granular materials.

The next point we wish to discuss is the relation between the present result and power-law rheology in systems at constant volume. In particular, Xu and O'Hern [23] found a power-law relation between the shear stress and the shear rate in a two-dimensional granular material consisting of frictionless particles. They estimated the exponent to be 0.65. However, in the constant volume condition, the pressure also depends on the shear rate, so that the behavior of the friction coefficient is generally different from that of the shear stress. Therefore, power-law friction in systems under constant volume conditions are not directly related to the present result. See Ref. [24] for more detailed discussions on this subject.

The third point we wish to discuss is the effect of dimensionality. In contrast to the present study, da Cruz *et al.* [11] obtained a linear friction law in a two-dimensional system. The difference may be attributed to the dimensionalities of the systems, which affect the nature of contacts between particles. In particular, the angular distribution of the tangential force is strongly anisotropic in two-dimensional systems, while such anisotropy is not observed in our three-dimensional system. Accordingly, in the case of frictionless particles, their system exhibited a friction law that is quite similar to ours.

As the fourth point of interest, we discuss the relevance of our result to earthquake mechanics by comparing it to a friction law recently proposed by Jop *et al.* [9], which seems to be validated in experiments on inclined plane flow. We stress that such a flow is characterized by relatively large inertial number (typically $I \gtrsim 10^{-1}$), while our simulation involves much smaller I values ($I \gtrsim 10^{-4}$) as shown in Figs. 1 and 2. In short, Eq. (1) involves a region of much smaller I than Jop *et al.* have investigated.

Such small inertial numbers correspond to a typical configuration of seismic motion of faults. For example, in the case that $d=1$ mm, $V=1$ m/s, $L_z=4$ cm, and $P=100$ MPa, the corresponding inertial number is 10^{-4}. However, one may wonder whether the friction law Eq. (1) can lead to stick-slip motion of faults because the friction law found here is velocity strengthening. Recall that we discuss exclusively stationary-state dynamic friction. Taking static friction into account, unstable slip is inevitable because static friction is always stronger than dynamic friction, which is mainly due to dilatation. Therefore power-law friction in stationary states does not contradict the unstable slip on faults.

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