Scaling properties of randomly folded plastic sheets

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We study the scaling properties of randomly folded aluminum sheets of different thicknesses h and widths L. We found that the fractal dimension $D=2.30\pm0.01$ and the force scaling exponent $\delta=0.21\pm0.02$ are independent of the sheet thickness and close to those obtained in numerical simulations with a coarse-grained model of triangulated self-avoiding surfaces with bending and stretching rigidity. So our findings suggest that finite bending rigidity and self-avoidance play the predominant roles in the scaling behavior of randomly folded plastic sheets.

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I. INTRODUCTION

Folded configurations of thin materials are very common. Examples range from folded proteins and polymerized membranes to folded engineering materials and geological formations [1]. Accordingly, the understanding of folding geometry remains an active area of research, both theoretically and experimentally [2–7]. It was noted that, despite the complicated appearance of folded configurations, the folding phenomena in themselves are very robust. Specifically, it was found that a set of balls folded from thin sheets of different sizes $L\times L$ obeys the fractal scaling law

$$M=\rho_a L^2 \propto R^D, \tag{1}$$

where M is the sheet mass, $\rho_a=\rho_m h$ is the sheet areal density, h is the sheet thickness ($h\ll L$), ρ_m is the material mass density, R is the ensemble-averaged diameter of balls folded from sheets of the same size, and $2<D<3$ is the material-dependent fractal dimension of the set of the folded balls [8,9,6]. For phantom sheets (membranes) without bending rigidity, the Flory mean-field approximation predicts that the mean diameter of the folded state behaves as $R\propto\sqrt{\log L}$, whereas the folded state of self-avoiding sheets is expected to obey the scaling behavior (1) with the universal fractal dimension $D=2.5$ (see [10]). On the other hand, it was shown that, for thin sheets with finite bending rigidity, the scaling properties of the folded state are determined by the balance of the bending and stretching energy stored in the folded creases (see [11]).

Numerical simulations with a coarse-grained model of triangulated surfaces with bending and stretching elasticity suggest the universal values of D for the phantom ($D=2.67$) and self-avoiding ($D=2.3<2.5$) sheets of fixed thickness [5]. In addition, numerical simulations performed in [5] suggest that the diameter of folded state scales with the sheet size L, thickness $h\ll L$, and confinement force F as

$$\frac{R}{h}\propto\left(\frac{L}{h}\right)^{2/D}\left(\frac{F}{Yh}\right)^{-\delta}, \tag{2}$$

and so

$$\frac{R}{h}\propto\left(\frac{L}{h}\right)^{2/D}h^{\delta}, \tag{3}$$

when $F=\text{const}$; here Y is the two-dimensional Young modulus and δ is the force scaling exponent, which is found to be different for the phantom ($\delta=3/8$) and the self-avoiding ($\delta\approx1/4$) sheets (see Ref. [5]). We note that the scaling behavior (3) differs from the scaling relation

$$\frac{R}{h}\propto\left(\frac{L}{h}\right)^{2/D}, \tag{4}$$

which was proposed by Bevilacqua [12] for balls folded from plastic sheets of different thickness ($h\ll L$). However, as far as we know, the scaling relations (2) and (4) were never tested experimentally. Accordingly, in this work we studied the scaling properties of sets of balls folded from aluminum sheets of different sizes L and thicknesses h.

II. EXPERIMENTS AND DISCUSSION

In this work, we used square aluminum sheets of thicknesses $h=0.02$, 0.06, 0.12, 0.24, and 0.32 mm with mass density $\rho_m=2.4\pm0.1$ g/mm^3 and stretching yield stress $\sigma_y=15\pm5$ MPa [the bending yield stress is $(L/h)^2$ times lower]. The sheet edge length was varied from $L_0=60$ mm to $L_M=600$ mm with the relation $L=\lambda L_0$ for scaling factors $\lambda=1$, 2, 3, 4, 5, 6, 7, 8, 9, 10. Hence, for the sheets used in this work the ratio of bending to stretching rigidity varies in the range from 2×10^{-9} to 5×10^{-5}. Initially, all sheets were crumpled by hand and then confined in approximately spherical forms (see Fig. 1) by applying the same force $F=60$ N along 15 directions taken at random [13]. In this way 1500 balls were folded.

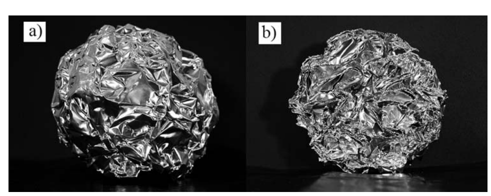

FIG. 1. Images of (a) balls folded from an aluminum sheet of thickness $h=0.06$ mm and edge size $L=60$ cm and (b) the cut through this ball.

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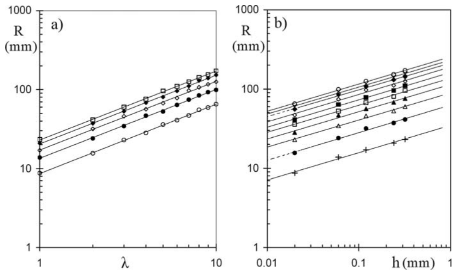

FIG. 2. Log-log plots of (a) R versus λ for different h (from bottom to top, h=0.02, 0.06, 0.12, 0.21, and 0.28 mm); (b) R versus h for different λ. Straight lines are best fits [the slopes of the fitting lines are (a) $2/D$=0.87±0.01 and (b) ϕ=0.35±0.01].

The mean diameter $\overline{R_j(L,h)}=(1/n)\Sigma_i^n R_i$ of each ball was determined from measurements of R_i along $n=15$ directions of confinement. Further, we calculated the ensemble-averaged diameters $R(L,h)=\langle\overline{R_j(L,h)}\rangle$, where the angular brackets denote the average over $N=30$ balls of the same size (L) and thickness (h). We found that, for almost all balls, the distribution of R_i can be best fitted by an inverse Gaussian distribution [14], while the mean diameters $\overline{R_j}$ conform to a normal distribution (see also [7]). We also noted that, in contrast to the case of randomly folded paper balls (see [7]), the diameter of a randomly folded aluminum sheet does not change after the folding force is withdrawn.

A. Fractal dimension and density scaling

The fractal dimension of folded sheets of the same thickness $D(h)$ and the thickness exponent $\phi(L)$ were determined from the scaling relation

$$R \propto L^{2/D}h^{\phi}. \tag{5}$$

We found that the fractal dimension of randomly folded balls does not depend on the sheet thickness [15] and equals D =2.30±0.01 [see Fig. 2(a)]. In addition, the thickness scaling exponent is found to be independent of the sheet size and equals ϕ=0.35±0.01 [see Fig. 2(b)].

We note that the value D=2.30±0.01 is less than D =2.5±0.2 reported in [16] for randomly folded aluminum foils. This difference may be attributed to the fact that in the work [16] the folding force was not controlled. On the other hand, our finding is close to the value D=2.3 found in numerical simulations of self-avoiding elastic sheets with bending and stretching rigidity [5]; nevertheless the folding deformations of aluminum sheets are predominantly plastic. It should be pointed out that, while the fractal dimension is found to be independent of the sheet thickness, and so it is also independent of the bending rigidity $k\propto Yh^2$, the finite bending rigidity of aluminum sheets plays an important role, such that the self-avoiding sheets with and without bending rigidity belong to different universality classes, characterized by D=2.3 and 2.5, respectively.

Furthermore, one can see that the data in Fig. 2 are consistent with the scaling relation (3) with the force exponent

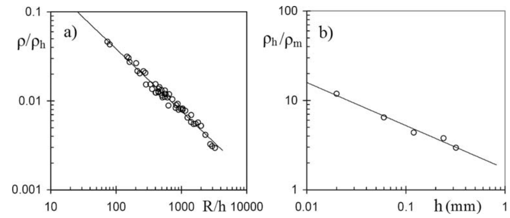

FIG. 3. (a) Data collapse for ρ/ρ_h versus R/h (the slope of the fitting line is $3-D$=0.7009, R^2=0.98); and (b) log-log plot of ρ_h/ρ_m versus h (straight line is given by $y=1.728x^{-0.4816}$, R^2=0.98).

$$\delta = 1-\phi-2/D = 0.21 \pm 0.02, \tag{6}$$

close to the force exponent δ=0.22±0.04 [17] estimated by numerical simulations with the coarse-grained model [5], and so the scaling relation (4) is not valid. This may be attributed to the dependence of the characteristic length of folding creases, Δ, on the sheet thickness (see [7]), which was not taken into account in [12]. Indeed, in Refs. [7,11] it was shown that $\Delta\propto R$, and so one expects that the characteristic length of folding creases scales as $\Delta\propto L^{2/D}F^{-\delta}h^{\phi}$ [18].

Accordingly, the data collapse is shown in Fig. 3(a) in the coordinates ρ/ρ_h versus R/h, where the ball mass density ρ scales as

$$\rho = \rho_h(F)\left(\frac{R}{h}\right)^{D-3} \quad for\ R \gg h, \tag{7}$$

with the normalized factor depending on the sheet thickness and the folding force [see Fig. 3(b)] as

$$\rho_h(F) = \rho_m\left(\frac{F}{Yh}\right)^{\delta D}, \tag{8}$$

and so, from the data in Fig. 3 follows that Y=78 N/mm. Notice that the mass densities of folded balls do not exceed 15% of the sheet density ρ_m, while the normalized factors are found to be $\rho_h > \rho_m$ for all h.

B. Energy scaling

The asymptotic analysis of the Von Karman equations for a thin plate [19] predicts that the total elastic energy accumulated in the folded creases of randomly folded phantom sheet scales as [20]

$$E_f \propto YL^3\left(\frac{L}{h}\right)^{-1}\left(\frac{h}{R}\right)^{5/3}. \tag{9}$$

This scaling was further confirmed in [21] by using analytic methods and lattice simulations. The authors of [20,21] have also argued that the self-avoidance does not affect the scaling behavior of the total elastic energy stored in a randomly crumpled sheet.

On the other hand, the scaling behavior of the folding energy can be determined by the integration of the folding force with respect to the ball diameter, using Eq. (2). In this way, the total folding energy is expected to scale as

$$E_f \propto YL^3\left(\frac{L}{h}\right)^{2/D\delta-3}\left(\frac{h}{R}\right)^{1/\delta-1}, \qquad (10)$$

and so the scaling behavior of the folding energy is expected to be different for the phantom and the self-avoiding sheets, because of the difference in the scaling exponents. Specifically, for randomly folded phantom sheets $D=1/\delta=8/3$ [5], and hence the scaling relation (10) coincides with (9), whereas the self-avoidance leads to drastic deviation from the scaling behavior (9).

Indeed, taking into account that for randomly folded aluminum sheets we found $D=2.3$ and $\delta=0.21$, from (9) it follows that the scaling of the total folding energy,

$$E_d \propto YL^3\left(\frac{L}{h}\right)^{1.07}\left(\frac{h}{R}\right)^{3.72}, \qquad (11)$$

is indistinguishable (within the uncertainty of numerical simulations performed in [5]) from the folding energy scaling expected for randomly folded self-avoiding elastic sheets ($D=2.3$ and $\delta=0.22\pm0.04$ [17]). Hence, the sheet plasticity affects only slightly or not at all affects the scaling properties of the randomly folded state. Notice that, in an elastic sheet, the folding energy is stored in the folding creases, whereas it is dissipated during the irreversible crumpling of plastic materials [22].

III. CONCLUSIONS

In summary, we found that the density of folded balls scales as $\rho=\rho_h(R/h)^{D-3}$, where $\rho_h=\rho_m(F/Yh)^{\delta D}$ and $R\propto F^{-\delta}$. Hence, sets of balls folded from plastic sheets of the same thickness exhibit scale invariance with a mass fractal dimension that is found to be universal $D=2.3\pm0.01$. In addition, the sets of balls folded from sheets of the same size but of different thicknesses also display scale-invariant behavior $M\propto R^{1/\phi}$, but with different fractal dimension $D_h=1/\phi=1/(1-\delta-2/D)=2.86\pm0.08$.

Moreover, our findings suggest that the values of scaling exponents for randomly folded plastic sheets are determined by the effect of sheet self-avoidance, whereas randomly folded elastoplastic paper sheets are characterized by a material-dependent fractal dimension governed by the strain relaxation after withdrawing the folded force (see Ref. [7]). In addition, the numerical coincidence between the values of the scaling exponents determined for plastic aluminum sheets and those expected for self-avoiding elastic sheets suggests that the sheet plasticity only slightly affects, or does not affect at all, the scaling behavior of the folded state.

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[15] Notice that in experiments with different kinds of paper Gomes [8] found that the fractal dimension of randomly folded balls depends on the surface density of paper as $D=4\rho_a^{-0.11}$, whereas the data reported in [7] suggest that D increases with increase of paper bending rigidity $k\propto Yh^2$.

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