Rotational stabilization of pinch instabilities in Taylor-Couette flow

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The axisymmetric linear stability of dissipative Taylor-Couette flow with an azimuthal magnetic field is considered. The magnetic field can be unstable without a rotation. This is the well-known pinch type instability. The stable rotation stabilizes the unstable azimuthal magnetic field. The dissipative flow stability can be classified according to Michael's stability condition for an ideal flow. The dissipative effects stabilize the flow and an ideally unstable flow becomes really unstable only when both the angular velocity and the magnetic field exceed some critical values.

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INTRODUCTION

The Taylor-Couette flow between concentric rotating cylinders is a classical problem of hydrodynamic and hydromagnetic stability [1,2]. According to the Rayleigh criterion, the ideal flow is stable to axisymmetric perturbations whenever the specific angular momentum increases outwards:

$$\frac{d}{dR}(R^2\Omega)^2 > 0, \tag{1}$$

where the cylindrical system of coordinates (R,ϕ,z) is used and Ω is the angular velocity. The viscosity has a stabilizing effect and a dissipative Taylor-Couette flow, which is unstable due to (1), becomes really unstable only if the angular velocity of an inner cylinder (or its Reynolds number) exceeds some critical value.

In the presence of an azimuthal magnetic field, the necessary and sufficient condition for the axisymmetric stability of ideal Taylor-Couette flow is [3]

$$\frac{1}{R^3}\frac{d}{dR}(R^2\Omega)^2 - \frac{R}{\mu_0\rho}\frac{d}{dR}\left(\frac{B_\phi}{R}\right)^2 > 0, \tag{2}$$

where B_ϕ is the azimuthal magnetic field, ρ is the density, and μ_0 is the magnetic constant.

According to (2), the current-free azimuthal magnetic fields ($B_\phi \propto 1/R$) stabilize the flow [4]. This stabilization has been confirmed for dissipative Taylor-Couette flow [5]. Nevertheless, the azimuthal magnetic fields, which stabilize the flow, are restricted by a narrow interval nearby to the current-free field [6].

For static configurations, the stability is defined by the Hartmann number only [7]. Like the stabilization of unstable rotation by the viscosity, the magnetic diffusivity stabilizes the unstable magnetic field and the ideally unstable magnetic field becomes really unstable only if the azimuthal magnetic field of inner cylinder (or Hartmann number) exceeds some critical value [6]. This critical Hartmann number does not depend on the rotation. This fact leads us to the conclusion that the unstable magnetic field, which has a value larger than the critical one, cannot be stabilized by the rotation [6]. In this paper, we will show that this conclusion is true only for a slow rotation. Fast stable rotation stabilizes the unstable magnetic field in accordance with the condition (2).

BASIC EQUATIONS

Consider a viscous electrically conducting incompressible fluid between two rotating infinite cylinders in the presence of an azimuthal magnetic field. The equations governing the problem are

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\,\nabla)\mathbf{U} = -\frac{1}{\rho}\nabla P + \nu\Delta\mathbf{U} + \frac{1}{\mu_0}\mathrm{curl}\mathbf{B}\times\mathbf{B},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{U}\times\mathbf{B}) + \eta\Delta\mathbf{B},$$

$$\mathrm{div}\,\mathbf{U} = \mathrm{div}\,\mathbf{B} = 0, \tag{3}$$

where $\mathbf{U}$ is the velocity, $\mathbf{B}$ is the magnetic field, P is the pressure, ν is the kinematic viscosity, and η is the magnetic diffusivity.

Equations (3) admit the basic solution in the cylindrical system of coordinates (R,ϕ,z):

$$U_R = U_z = B_R = B_z = 0,$$

$$B_\phi = a_B R + \frac{b_B}{R}, \quad U_\phi = R\Omega = a_\Omega R + \frac{b_\Omega}{R}, \tag{4}$$

where a_Ω, b_Ω, a_B, and b_B are constants defined by the boundary conditions

$$a_\Omega = \Omega_{\rm in}\frac{\hat{\mu}_\Omega - \hat{\eta}^2}{1-\hat{\eta}^2}, \quad b_\Omega = \Omega_{\rm in}R_{\rm in}^2\frac{1-\hat{\mu}_\Omega}{1-\hat{\eta}^2},$$

$$a_B = \frac{B_{\rm in}}{R_{\rm in}}\frac{\hat{\eta}(\hat{\mu}_B - \hat{\eta})}{1-\hat{\eta}^2}, \quad b_B = B_{\rm in}R_{\rm in}\frac{1-\hat{\mu}_B\hat{\eta}}{1-\hat{\eta}^2}, \tag{5}$$

where

$$\hat{\eta} = \frac{R_{\rm in}}{R_{\rm out}}, \quad \hat{\mu}_\Omega = \frac{\Omega_{\rm out}}{\Omega_{\rm in}}, \quad \hat{\mu}_B = \frac{B_{\rm out}}{B_{\rm in}}, \tag{6}$$

$R_{\rm in}$ and $R_{\rm out}$ are the radii, $\Omega_{\rm in}$ and $\Omega_{\rm out}$ are the angular velocities, and $B_{\rm in}$ and $B_{\rm out}$ are the azimuthal magnetic fields of the inner and outer cylinders, respectively.

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Note that for viscous flow the magnetic field profile and the angular velocity profile are completely defined by the three parameters only: $\hat{\eta}$, $\hat{\mu}_\Omega$, and $\hat{\mu}_B$. The first magnetic field term in Eqs. (4) corresponds to a constant axial electric current density into the fluid. The second term is current free.

We are interested in the stability of the basic solution (4). The linear stability problem is considered. By developing the disturbances into normal modes, solutions of the linearized equations (3) are considered in the form

$$F = F(R)\exp[i(kz + \omega t)], \tag{7}$$

where F is all of the disturbances.

The dimensionless numbers of the problem are the magnetic Prandtl number Pm, Hartmann number Ha, and Reynolds number Re,

$$\mathrm{Pm} = \frac{\nu}{\eta}, \quad \mathrm{Ha} = \frac{B_{\mathrm{in}} R_0}{\sqrt{\mu_0 \rho \nu \eta}}, \quad \mathrm{Re} = \frac{\Omega_{\mathrm{in}} R_0^2}{\nu}, \tag{8}$$

where $R_0=[R_{\mathrm{in}}(R_{\mathrm{out}}-R_{\mathrm{in}})]^{1/2}$ is the length unit.

A detailed description of the equations and the numerical method used have been given in our earlier paper [6] and will not be reproduced here. Always no-slip boundary conditions for the velocity on the walls are used. The tangential electrical currents and the radial component of the magnetic field vanish on the conducting walls. The magnetic field must match the external magnetic field for the insulating walls [6].

There are some indications that the instability originates as a monotonic instability for the problem in hand [6].[1] To the author's knowledge, the above statement has not been formally proved. Nevertheless, for the sake of simplicity, we take that $\mathcal{R}(\omega)=0$ [where $\mathcal{R}(\omega)$ is the real part of ω] for the marginal stability lines below.

RESULTS

According to the condition (1) or the first term in (2), the rotation is stable if

$$\hat{\mu}_\Omega > \hat{\eta}^2. \tag{9}$$

Due to the condition (2), the magnetic field is stable for static configurations if

$$0 \leqslant \hat{\mu}_B \leqslant \frac{1}{\hat{\eta}}. \tag{10}$$

The rotation is called stable (unstable) if $\hat{\mu}_\Omega > \hat{\eta}^2$ ($\hat{\mu}_\Omega < \hat{\eta}^2$). The azimuthal magnetic field is called stable (unstable) if $\hat{\mu}_B$ lies inside (outside) of the interval Eq. (10).

It has been demonstrated [6] that the dissipative Taylor-Couette flow can be destabilized by the magnetic field with $\hat{\mu}_B$ out of the range (10) only. For the unstable magnetic field and the unstable rotation, the critical Reynolds numbers decrease with increasing Hartmann numbers and become zero for some critical Hartmann number Ha_0. Thus, the magnetic field is unstable without the rotation for $\mathrm{Ha} > \mathrm{Ha}_0$. The critical Hartmann number does not depend on the magnetic Prandtl number and the angular velocity profile ($\hat{\mu}_\Omega$). So the unstable magnetic field cannot be stabilized by the slow rotation.

[1]Note that a monotonic instability was wrongly called "overstability" instead of "exchange of stabilities" in this paper.

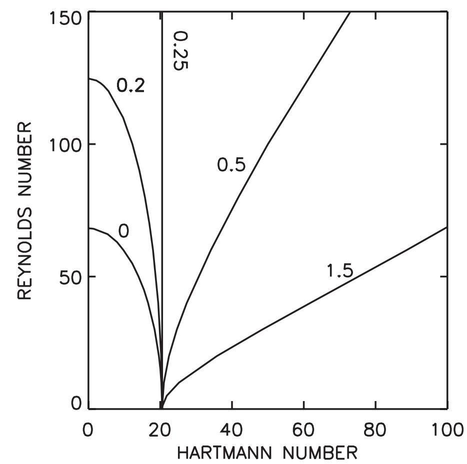

FIG. 1. The marginal stability lines for *insulating* cylinders with $\hat{\eta}=0.5$ and $\hat{\mu}_B=4$. The lines are labeled by the $\hat{\mu}_\Omega$ values. The flow is stable (unstable) to the left (right) of the lines. The line with $\hat{\mu}_\Omega=\hat{\eta}^2$ ($\hat{\mu}_\Omega=0.25$) is the Rayleigh line.

Can this instability be suppressed by the fast stable rotation? The answer can be found in Fig. 1, which presents the marginal stability lines for the insulating cylinders. Figure 1 represents the typical behavior of marginal stability lines for the unstable magnetic field. The lines do not depend on the magnetic Prandtl number. The flow is stable to the left of the lines and unstable to the right. So the flow is stable for small Reynolds numbers to the left of the Rayleigh line ($\hat{\mu}=\hat{\eta}^2$). The situation is opposite to the right of the Rayleigh line, where flow is stable for large Reynolds numbers. The rotation is unstable even without the magnetic field to the left of the Rayleigh line. The rotational instability is increased by the unstable magnetic field (the flow becomes unstable for a smaller Reynolds number for the larger Hartmann number). The rotation is neutrally stable on the Rayleigh line and does not influence the instability of the magnetic field (the stability depends on the Hartmann number only). To the right of the Rayleigh line, the rotation is stable and stabilizes the unstable magnetic field (we need a larger Hartmann number for the larger Reynolds number to destabilize the flow). The flow with the larger $\hat{\mu}_\Omega$ becomes stable for smaller Reynolds numbers at fixed Hartmann number.

For conducting cylinders, the results are similar. The fast stable rotation stabilizes the unstable magnetic field. Nevertheless, the conducting cylinders should be considered more carefully. There is a spurious solution $b_\phi \sim R^{-1}$ for the marginal stability line ($\omega=0$) with $k=0$ (where k is the axial wave number). For this spurious solution, an unstable magnetic field is not stabilized by the stable rotation. Nevertheless, the mode with $k=0$ is always stable (does not cross the marginal stability line) [8]. To overcome this difficulty, calculations have been performed for the slightly unstable lines with $\omega=-10^{-3}i$. These slightly unstable lines depend on the magnetic Prandtl number. The results are demonstrated in Fig. 2.

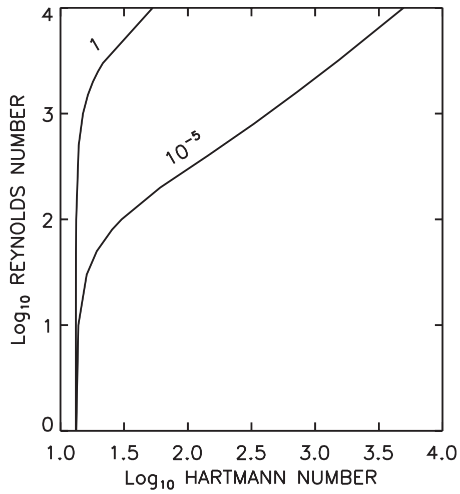

FIG. 2. The slightly unstable lines ($\omega=-10^{-3}i$) for *conducting* cylinders with $\hat{\eta}=0.5$, $\hat{\mu}_\Omega=1.5$, and $\hat{\mu}_B=4$. The lines are labeled by the Pm values.

CONCLUSION

We have demonstrated that the axisymmetric stability properties of dissipative Taylor-Couette flow can be classified exactly in accordance with the ideal stability condition (2). The rotation is called stable (unstable) when it is stable (unstable) without the magnetic field. The magnetic field is called stable (unstable) when it is stable (unstable) without the rotation. For the unstable rotation, the instability is rotational (or centrifugal) by its physical nature and called rotational instability. The instability can be called magnetic (or pinch) for the unstable magnetic field and magnetorotational when both the rotation and the magnetic field are unstable. To distinguish it from the well-known magnetorotational instability in the presence of an axial magnetic field [9], the instability discussed can be called azimuthal magnetorotational instability (AMRI).

According to (2), every stable rotation can be destabilized by an unstable magnetic field with the large enough magnetic field value. Obviously, we can reverse the point of view and say that every unstable magnetic field can be stabilized by a stable rotation. Similarly, every stable magnetic field can be destabilized by an unstable rotation (or unstable rotation can be stabilized by a stable magnetic field). The stability properties of ideal flow do not change when both the rotation and the magnetic field are either stable or unstable.

The dissipation leads to the appearance of two critical numbers Reynolds, Re_0, and Hartmann, Ha_0 (Re_0 is calculated for $\mathrm{Ha}=0$ and Ha_0 is calculated for $\mathrm{Re}=0$). The rotation, which is ideally unstable without the magnetic field, becomes really unstable only if $\mathrm{Re}>\mathrm{Re}_0$. Similarly, the azimuthal magnetic field, which is ideally unstable without the rotation, becomes really unstable only if $\mathrm{Ha}>\mathrm{Ha}_0$.

The stable magnetic field stabilizes the unstable rotation and the critical Reynolds numbers, above which the flow becomes unstable, increase with increasing Hartmann numbers [6]. Similarly, the stable rotation stabilizes the unstable magnetic field and the critical Hartmann numbers, above which there is the instability, increases with the increasing Reynolds number (see the lines to the right of the Rayleigh line on Fig. 1).

The instability is subcritical when both the rotation and the magnetic field are unstable (i.e., there is the instability for $\mathrm{Re}<\mathrm{Re}_0$ and $\mathrm{Ha}<\mathrm{Ha}_0$).

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