Noise sensitivity of phase-synchronization time in stochastic resonance: Theory and experiment

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Recent numerical and heuristic arguments have revealed that the average phase-synchronization time between the input and the output associated with stochastic resonance is highly sensitive to noise variation. In particular there is evidence that this average time exhibits a cusplike behavior as the noise strength varies through the optimal value. Here we present an explicit formula for the average phase-synchronization time in terms of the phase diffusion coefficient and the average frequency difference between the input and the output signals. We also provide experimental evidence for the cusplike behavior by using a bistable microelectronic-circuit system.

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I. INTRODUCTION

One of the most remarkable phenomena in nonlinear and statistical physics is stochastic resonance (SR) [1,2], through which noise plays a beneficial role in enhancing the system's response to weak signals. Traditional measures for characterizing SR include the signal-to-noise ratio [3], the correlation between the input and the output signals [4], the residence time distribution [5], spectral amplification [6], and quantities from the information theory [7,8]. Recent years have seen efforts [9,10] in relating SR with another important phenomenon in nonlinear dynamics: phase synchronization (PS) [11]. This is motivated by the consideration that an SR system typically requires input and output signals and, as such, a question is whether there can be synchrony between the signals. Due to nonlinearity and noise, the output signals usually differ from the input ones in their detailed evolution with time. As a result, complete synchronization between these signals, defined by their approaching each other in the asymptotic time limit, can be ruled out. Instead, a weaker type of synchronization, phase synchronization as signified by a *tendency* for the input and the output signals to follow each other, can occur. In particular, let $x(t)$ and $y(t)$ represent the input and the output signal, respectively, and assume they are oscillatory so that the corresponding phase variables $\phi_x(t)$ and $\phi_y(t)$ can be defined, where one cycle of oscillation in $x(t)$ $[y(t)]$ generates a 2π phase increase in $\phi_x(t)$ $[\phi_y(t)]$. Now consider a time interval $[t_1, t_2]$ that contains many cycles of oscillation. There is phase synchronization if the phase variables satisfy $|\phi_y(t) - \phi_x(t)| < 2\pi$ for all t in this interval. Since noise is present, the interval $\Delta t = t_2 - t_1$ cannot be arbitrarily long, and one can consider an ensemble of the identical SR system and define the *average phase-synchronization time* as the ensemble average: $\tau \equiv \langle \Delta t \rangle$. This average time depends on the noise amplitude ε and a natural question is how.

Insights to this question have been obtained recently in Ref. [10] where a heuristic theory based on analyzing the transition probabilities for a mechanical particle in a stochastic double-well potential system is derived. The result and numerical computations point to an extremely fast rising and falling, *cusplike* behavior in τ as the noise amplitude ε is varied through the optimal value $\bar{\varepsilon}$ for which traditional measures such as the signal-to-noise ratio reach maximum. Since the approximate theory in Ref. [10] does not apply to the noise regime in the close vicinity of $\bar{\varepsilon}$, it is interesting to assess the behavior of the function $\tau(\varepsilon)$ for ε near $\bar{\varepsilon}$. In this aspect, the recent analysis of the phase diffusion process by Casado-Pascual *et al.* [12] indicates that the effective diffusion coefficient D_{eff} varies smoothly through $\bar{\varepsilon}$. Since the average phase-synchronization time τ can be related to D_{eff}, the dependence of τ on ε should also be smooth about $\bar{\varepsilon}$. One aim of this paper is to provide an explicit formula to show that, despite the anticipated smooth behavior, τ typically varies drastically (e.g., over several orders of magnitude) as ε changes through $\bar{\varepsilon}$. This is to be contrasted to the behavior of, say, the signal-to-noise ratio, where typically it hardly varies about $\bar{\varepsilon}$. Another purpose of this paper is to provide the experimental evidence for the cusplike behavior in τ. We shall use the Schmitt-trigger circuit [13] and demonstrate the power of understanding SR using PS and in particular, the quantity τ.

A potential use of our result lies in signal processing. A known example is to develop a device to assess the working environment based on the principle of SR [14]. In particular, for various types of measuring devices in a noisy environment, it is desirable to have the signal spectral peak as pronounced as possible with respect to the broad, noisy background. The principle of SR can naturally be used to detect the optimal noise level (or the optimal working condition). Andó and Graziani recognized that the SNR is in general not suitable for this purpose, as it does not allow for online tuning of the noise variance because of its insensitivity to noise variation about the optimal level. In a series of papers [14], they developed mathematical models utilizing feed-forward estimation theory and tested experimental devices based on the Schmitt trigger to overcome the difficulty. Having a measure that is highly sensitive to noise variation is desirable.

Our result may also be useful for understanding the working of biological systems. There has been extensive experimental evidence that biological systems use the mechanism of SR for various tasks [15]. Since timing is commonly used (for instance, by neural systems) for information processing, it is possible that biological systems may take the advantage of the average synchronization time to tune to the optimal noise level for SR. It may be of interest to carry out biological experiments to test this conjecture. In this regard, we expect our experimental methodology to be useful.

II. THEORY

We consider heavily damped motion of a classical particle in the double-well potential given by $V(x)=x^4/4-x^2/2$, subject to periodic forcing and noise. The Langevin equation is

$$\dot{x}(t) = -\frac{dV(x)}{dx} + F(t) + \xi(t), \tag{1}$$

where $F(t)$ is a rectangular periodic input signal of period T_0 and $\xi(t)$ represents Gaussian white noise that satisfies $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(t')\rangle=2\varepsilon\delta(t-t')$. Mathematically, the forcing function can be written as $F(t)=(-1)^{n(t)}A$, where $n(t)=\lfloor 2t/T_0\rfloor$ and $\lfloor\cdot\rfloor$ denotes the floor function. That is, $F(t)=A(-A)$ if $t\in[nT_0/2,(n+1)T_0/2]$ for n even (odd). The frequency of the input signal is $\Omega_{\rm in}=2\pi/T_0$. Letting $U(x,t)=V(x)-F(t)x$ be the effective potential, we rewrite Eq. (1) as

$$\dot{x}(t) = -\frac{dU(x,t)}{dx} + \xi(t). \tag{2}$$

Since the effective potential is time dependent and the driving is periodic, the shape of the potential changes periodically with time. For $A<A_{\rm th}=\sqrt{4/27}$, the potential $U(x,t)$ has a "double-well" shape in the sense that it has two minima, located at $x_1(t)<0$ and $x_2(t)>0$, respectively, and a maximum at $x_m(t)$ for all t. By the symmetry of the effective potential, these three positions satisfy the following conditions: $x_m(t)=(-1)^{n(t)}x_m(0)$, $x_1(t)+x_2(t)+x_m(t)=0$, and $x_j(t)=(-1)^j\frac{\Delta x(0)}{2}-(-1)^{n(t)}\frac{x_m(0)}{2}$, where $\Delta x(0)=x_2(0)-x_1(0)$.

The average phase-synchronization time τ can be calculated if the average frequency of the output signal $\Omega_{\rm out}$ and the effective diffusion coefficient $D_{\rm eff}$ are known. The diffusion coefficient is defined by

$$D_{\rm eff} = \frac{1}{2}\frac{d}{dt}[\langle\Psi^2\rangle - \langle\Psi\rangle^2], \tag{3}$$

where $\Psi(t)$ is the phase difference between the input and the output signal and thus $\langle\Psi(t)\rangle\equiv(\Omega_{\rm out}-\Omega_{\rm in})t$. Formulas for $\Omega_{\rm out}$ and $D_{\rm eff}$ have recently been obtained by Casado-Pascual *et al.* [12]. Here we shall use their results to obtain an explicit formula for τ.

When the noise strength ε is sufficiently small, intrawell relaxation time scale of the particle is small compared with both the interwell transition time scale and the driving period T_0. The interwell relaxation time is typically independent of the intrawell transition time. In this adiabatic regime, the Langevin dynamics can be approximated by the following pair of rate equations:

$$\dot{P}(1,t) = -\gamma_1(t)P(1,t) + \gamma_2(t)P(2,t),$$

$$\dot{P}(2,t) = -\gamma_2(t)P(2,t) + \gamma_1(t)P(1,t), \tag{4}$$

where $P(1,t)$ and $P(2,t)$ are the probabilities that the particle is in the left and in the right well, respectively, and $\gamma_j(t)$ is the Kramers rate [16] of escape from the well j at time t ($j=1,2$):

$$\gamma_j(t) = \frac{w_j(t)w_m(t)}{2\pi}\exp\left\{-\frac{U[x_m(t),t]-U[x_j(t),t]}{\epsilon}\right\}. \tag{5}$$

In Eq. (5), the quantities $w_j(t)$ and $w_m(t)$ are given by

$$\omega_j(t) = \sqrt{d^2U[x_j(t),t]/dx^2} = \sqrt{3[x_j(t)]^2-1},$$

$$\omega_m(t) = \sqrt{d^2U[x_m(t),t]/dx^2} = \sqrt{1-3[x_m(t)]^2}.$$

Equation (4) describes the evolutions of the probabilities between two consecutive changes in the shape of the potential. The Kramers rate formula Eq. (5) can be simplified by using the symmetry of $U(x,t)$ [12]. One obtains

$$\gamma_j(t) = \frac{\Gamma}{2}[1-(-1)^{n(t)+j}\Delta P_{\rm eq}(0)], \tag{6}$$

where $\Gamma=\gamma_1(0)+\gamma_2(0)=\gamma_1(t)+\gamma_2(t)$, $\Delta P_{\rm eq}(0)=P_{\rm eq}(2,0)-P_{\rm eq}(1,0)$, and $P_{\rm eq}(j,0)=[\delta_{j,1}\gamma_2(0)+\delta_{j,2}\gamma_1(0)]/\Gamma$ is the equilibrium probability of state j at $t=0$.

The average frequency of the output signal is

$$\Omega_{\rm out} = \frac{\pi}{T_0}\int_0^{T_0}[\gamma_1(t)P(1,t)+\gamma_2(t)P(2,t)]dt. \tag{7}$$

An extensive derivation by Casado-Pascual *et al.* [12] yields the following formula for $\Omega_{\rm out}$:

$$\Omega_{\rm out} = \frac{\pi\Gamma}{2}\left[1-[\Delta P_{\rm eq}(0)]^2\left(1-\frac{4\tanh(\Gamma T_0/4)}{\Gamma T_0}\right)\right]. \tag{8}$$

They have also obtained the following formula for the diffusion coefficient:

$$D_{\rm eff} = \pi\Omega_{\rm out} - \frac{2\pi^2}{T_0}[\Delta P_{\rm eq}(0)]^4\left[\tanh\left(\frac{\Gamma T_0}{4}\right)\right]^3 - \frac{\pi^2}{2T_0}[\Delta P_{\rm eq}(0)]^2\{1-[\Delta P_{\rm eq}(0)]^2\}\left(12\tanh\left(\frac{\Gamma T_0}{4}\right) - \Gamma T_0\left\{1+2\left[\mathrm{sech}\left(\frac{\Gamma T_0}{4}\right)\right]^2\right\}\right). \tag{9}$$

To obtain the average phase-synchronization time τ, we can use the definition of the diffusion coefficient Eq. (3) to write [17]

$$\langle\Psi^2\rangle \sim \langle\dot{\Psi}\rangle^2t^2 + 2D_{\rm eff}t. \tag{10}$$

Since τ is the average time required for a 2π change in Ψ to occur, we have $\langle\Psi^2(n\tau)\rangle=(2n\pi)^2$, leading to $\sqrt{\langle\Psi^2(t)\rangle}|_{t=\tau}=2\pi$. Using this result and $\langle\dot{\Psi}\rangle=\Omega_{\rm out}-\Omega_{\rm in}$ yield the following formula for τ:

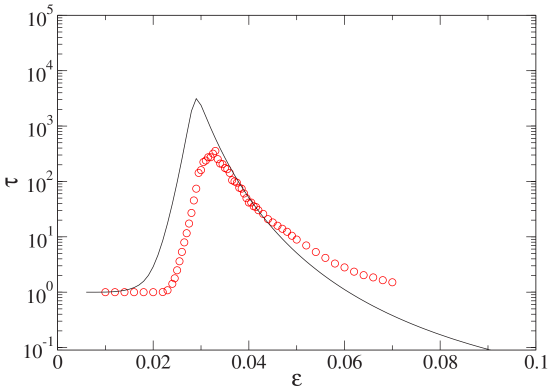

FIG. 1. (Color online) The average phase-synchronization time τ for $\Omega_{in}=0.002$ and $A=0.18$. The solid curve is from the theoretical formula Eq. (11) and the circles are from direct numerical computation of Eq. (2). A proportional constant is used in Eq. (11) to compare the theory with the numerics. The unit of τ is the period of the input signal T_0.

$$\tau \sim \frac{D_{\mathrm{eff}}}{\langle\dot{\Psi}\rangle^2}\left[\sqrt{1+\left(\frac{2\pi\langle\dot{\Psi}\rangle}{D_{\mathrm{eff}}}\right)^2}-1\right]. \tag{11}$$

Note that this theoretical prediction does not give the absolute value of the average phase-synchronization time. Thus a proportional constant must be chosen to compare the theory with numerics. In general, as the noise strength approaches the optimal value $\bar{\varepsilon}$, a maximal degree of phase synchronization between the input and the output can occur. Thus the average frequency difference $\langle\dot{\Psi}\rangle$ approaches zero. At the same time, the diffusion process slows down as it takes longer for a 2π change in Ψ to occur. As a result, the diffusion coefficient also decreases.

Heuristically, we have $\dot{\Psi}\sim 1/t$ where t is the time required for Ψ to reach a fixed value (say 2π). From Eq. (3) we have a scaling relation $D_{\mathrm{eff}}\sim\dot{\Psi}^2 t$, leading to $D_{\mathrm{eff}}\sim 1/t$. Thus we expect that, about the optimal noise intensity, $\dot{\Psi}$ and D_{eff} decrease in a similar manner as ε approaches $\bar{\varepsilon}$ and hence the term inside the square brackets in Eq. (11) is approximately a constant with respect to the variation in ε. We have

$$\tau \sim 1/\langle\dot{\Psi}\rangle \to \infty \text{ as } \langle\dot{\Psi}\rangle \to 0. \tag{12}$$

Since noise is present, the average frequency difference $\dot{\Psi}$ can be small but not infinitesimal. This means that, for $\varepsilon \to \bar{\varepsilon}$, τ can become large but it does not diverge. Thus the behavior of τ vs ε should be smooth and it does not exhibit a cusp in the mathematical sense of derivative discontinuity. Nonetheless, it is possible that τ can increase drastically for $\varepsilon\to\bar{\varepsilon}$, herewith the term *cusplike* behavior [10], as shown in Fig. 1 for $\Omega_{in}=0.002$ and $A=0.18$, where the solid curve is from the theoretical formula Eq. (11) and the circles are from direct numerical computation of Eq. (2). The theoretical and numerical results are consistent. We wish to emphasize that our derivation is not rigorous but only heuristic. Its validity needs to be checked numerically and experimentally.

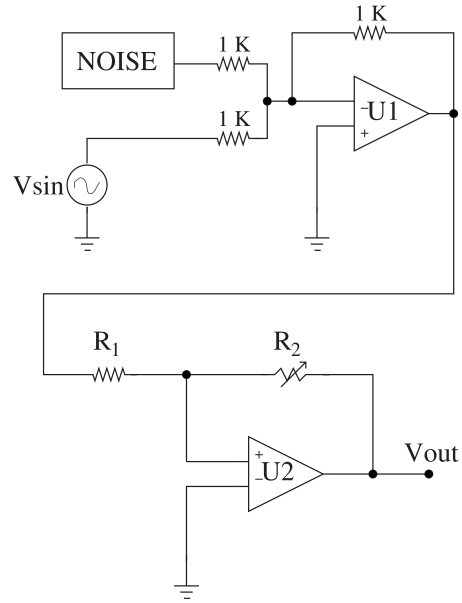

FIG. 2. Schmidtt-trigger circuit system.

III. EXPERIMENT

A class of experimental systems for testing phenomena associated with SR is nonlinear electronic circuits, which nowadays are usually implemented by using metal-oxide-semiconductor field-effect transistor based microelectronic circuits [18]. As we have seen, bistability is convenient for generating SR. Here we use a class of microelectronic circuits known as the bistable multivibrators (e.g., Schmitt triggers) [18] to construct our experimental SR system.

The Schmitt-trigger circuit system with operational amplifiers (op amps) $U1$ and $U2$ and resistors R_1 and R_2, which we have constructed for our experimental study, is shown in Fig. 2. In our experiments, we choose $R_1=1$ KΩ and $R_2=11.7$ KΩ, and use TL082 op amps with saturation voltage 10 V. The threshold voltages are approximately 860 mV. The average phase synchronization time between the input v_{sin} and the output v_{out} are calculated by using long voltage signals (200 s, typically containing between 100 and 25 000 2π phase slips) and by using the Hilbert-transform method for a systematic set of noise levels. The noise signal is from a wide-band analog random voltage generator (SRS-DS345, 30 MHz). The noise amplitude is defined to be the rms value of the random voltage signal. The whole experiment is repeated 30 times to reduce the statistical fluctuations in the measured average phase-synchronization time.

When noise is off, the average phase-synchronization time τ is approximately zero because the output is simply zero and does not follow the input, but τ increases with the noise amplitude. When the noise amplitude is close to the optimal value for SR, which is determined to be about 0.8 V, the circuit is triggered at a rate close to the frequency of the input signal, which is a 1.5-KHz sinusoidal signal of amplitude 110 mV, giving rise to a large value in τ. The result of τ vs the noise amplitude is shown in Fig. 3. We observe a significant increase in τ (of nearly five orders of magnitude!) as the optimal noise amplitude is approached, providing ex-

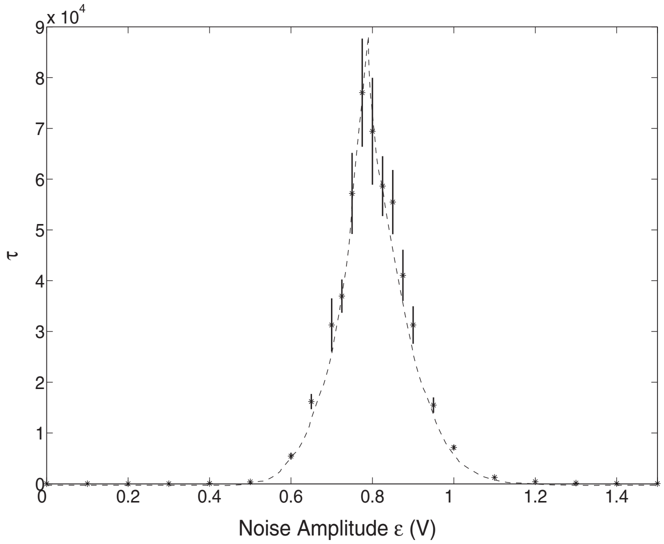

FIG. 3. Experimental result of the average phase synchronization time τ (in the unit of the number of cycles of the input signal) vs the noise intensity ε (V), where the input $V_{\sin}$ is a 1.5-KHz sinusoidal signal of amplitude 110 mV. We observe a significant increase in τ as the optimal noise level for SR is approached (a cusplike behavior).

perimental evidence for the predicted cusplike behavior in the average phase-synchronization time associated with SR.

IV. DISCUSSION

The contributions of this paper are twofold. First, an explicit formula has been worked out for a paradigmatic SR system, the double-well potential system under simple periodic driving, which relates the average phase-synchronization time to the phase-diffusion coefficient and the average frequency difference between the input and the output signals. The formula indicates that, although the time varies smoothly, it can increase drastically as the optimal noise level is approached. The formula is thus consistent with the recent conjecture that the average synchronization time shows a cusplike behavior. Second, we have presented an experimental study of the cusplike behavior via a bistable Schmitt-trigger system implemented using microelectronic circuits, and obtained direct experimental evidence for the cusplike behavior. Given a nonlinear system, the average phase-synchronization time can thus be highly effective for tuning noise to its optimal level to achieve SR.

A number of open issues remains to be explored. For instance, precise frequency tuning may be required in applications of nonlinear systems. The presence of a proper amount of noise can again be advantageous for this purpose (the so-called *bona fide* resonance [19]). Does the average phase-synchronization time also exhibit a cusplike functional relationship with the frequency of the driving so as to serve as an effective measure for precise frequency tuning? So far there has been ample evidence [15] that many biological systems may actually use SR for various purposes. A general question is how do biological systems tune to optimal noise level? Is it possible that the average phase-synchronization time is used?

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