Finite-size scaling in anisotropic systems

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We present analytical results for the finite-size scaling in d-dimensional $O(N)$ systems with strong anisotropy where the critical exponents (e.g., $\nu_{\parallel}$ and $\nu_{\perp}$) depend on the direction. Prominent examples are systems with long-range interactions, decaying with the interparticle distance r as $r^{-d-\sigma}$ with different exponents σ in corresponding spatial directions, systems with space-"time" anisotropy near a quantum critical point, and systems with Lifshitz points. The anisotropic properties involve also the geometry of the systems. We consider $O(N)$ systems in the $N\to\infty$ limit, confined to a d-dimensional layer with geometry $L^m\times\infty^n; m+n=d$ and periodic boundary conditions across the finite m dimensions. The arising difficulties are avoided using a technique of calculations based on the analytical properties of the generalized Mittag-Leffler functions.

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I. INTRODUCTION

Anisotropic systems are omnipresent in soft matter and solid state physics. Prominent examples are liquid crystals, dipolar-coupled uniaxial ferromagnets, systems with Lifshitz points, systems with space -"time" anisotropy near a quantum critical point, systems with long-range interactions decaying with the interparticle distance r as $r^{-d-\sigma}$ with different exponents σ in corresponding spatial directions, and some dynamical systems [1–3]. Specific problems arise in the consideration of critical phenomena in such systems. In any of the cases the fundamental idea of scaling must be modified in an appropriate way. In terms of the correlation length one can distinguish two types of anisotropy: weak and strong (see, e.g., [4]). In weakly anisotropic systems, the correlation length has spatially dependent amplitude. In the strongly anisotropic systems, in addition, the critical exponents (e.g., $\nu_{\parallel}$ and $\nu_{\perp}$) depend on the direction. A more general definition based on an anisotropic scale covariance of the n-point correlators and different exhaustive examples can be seen also in Ref. [3]. In [3] a general approach to scale invariance in infinite volume systems with strong anisotropy has been developed.

The object of the present paper is scaling in finite-size systems. Our consideration is in the framework of the $O(N)$-vector model in the $N\to\infty$ limit where the model is exactly solvable. This model (equivalent to the spherical model) has been a classical example in finite-size scaling investigations starting from the seminal works [5,6].

In contrast to the theory of finite-size scaling in isotropic systems (see, e.g., [7,8]) and weakly anisotropic systems (see, e.g., [9] and references therein), the theory of finite-size scaling in strongly anisotropic systems (see [10–17]) is still a field where the lack of results obtained in the framework of simplified and analytically tractable models is noticeable. There exist by now quite a few examples [13,14,17] where the predictions of an anisotropic finite-size scaling hypothesis have been reproduced analytically. In [13,14] anisotropy appears near a quantum critical point as a result of mapping of a "time" dependent problem (in d dimensions) to a "static" problem (in $d+1$ dimensions). In [17] it is due to the spatial direction dependence of the interactions.

Recently [18] a recipe for studying finite-size effects based on some useful properties of the generalized Mittag-Leffler functions has been suggested. It allows one to consider isotropic and some strongly anisotropic systems (including long-range quantum systems) on an equal footing. The interest in Mittag-Leffler functions has grown up because of their applications in some finite-size scaling problems (see, e.g., [7,8,18–20]). The present study (see also [21]) is an illustration of the rare possibility to handle the final expressions of the scaling equations for strongly anisotropic systems analytically. This readily generalizes some isotropic finite-size scaling results also to the anisotropic case.

II. THE MODEL

We restrict our attention to the N-vector spin models defined on a lattice. The Hamiltonian of the model reads

$$H=-N\sum_{x,y}J(x-y)\vec{\sigma_x}\cdot\vec{\sigma_y}, \qquad (2.1)$$

where $\vec{\sigma_x}$ is a classical N-component unit vector defined at site x of the lattice and the spin-spin coupling decays with different power laws in different lattice directions. We assume a d-dimensional system with *mixed* geometry (finite and infinite dimensions) under periodic boundary conditions in the finite dimensions. The interaction between spins enters the expressions of the theory only through its Fourier transform. We will consider the following anisotropic small-$\mathbf{q}$ expansion of the Fourier transform of the spin-spin coupling:

$$J(\mathbf{q})\simeq J(0)+a_{\perp}|\mathbf{q}_{\perp}|^{2\rho}+a_{\parallel}|\mathbf{q}_{\parallel}|^{2\sigma}, \qquad (2.2)$$

where the first m directions (called "transverse" and denoted by $\perp$) are kept finite and the remaining n directions (called "parallel" and denoted by the subscript $\parallel$) are extended to infinity, with $m+n=d$, $a_{\perp}$ and $a_{\parallel}$ are metric factors, and $\rho,\sigma>0$. Further we will use the symbol $L^m(\rho)\times\infty^n(\sigma)$ in order to present the system under consideration. In the finite

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directions the corresponding summations are over the vector $\mathbf{q}_\perp=\{q_{\perp 1},\ldots,q_{\perp m}\}$ that takes values in Λ^m defined by $q_{\perp\nu}=2\pi n_\nu/(aN_0)$ and $-(N_0-1)/2\le n_\nu\le(N_0-1)/2, \nu=1,\ldots,m$. In the infinite directions the sums are substituted with normalized integrals over the corresponding part ofthe first Brillouin zone $[-\frac{\pi}{a},\frac{\pi}{a}]^n$. For our further purposes let us recall that a finite linear dimension $L=N_0a$ in the continuous limit means that the lattice spacing $a\to 0$ and simultaneously $N_0\to\infty$. In our analysis we assume $a_\perp=a_\parallel=-1/2$.

Recently such type of system, focusing on the shape dependence of the finite-size scaling limit, is considered in Ref. [17] (with $0<\rho$, $\sigma<1$). In the large-N limit, the theory is solved in terms of the gap equation for the parameter λ_V related with the finite-volume correlation length of the system. The bulk system is characterized by a vanishing λ_∞, so that the appropriately scaled inverse critical temperature

$$\beta_c=\frac{1}{(2\pi)^d}\int_{[-\pi/a]^d}^{[\pi/a]^d}\frac{d\mathbf{q}}{|\mathbf{q}_\perp|^{2\rho}+|\mathbf{q}_\parallel|^{2\sigma}} \tag{2.3}$$

is finite whenever the effective dimensionality $D=m/\rho+n/\sigma$ is greater than 2. The corresponding critical exponents $\nu_\parallel$ and $\nu_\perp$ associated with the behavior of the infinite system are

$$\nu_\parallel=\frac{1}{\sigma(D-2)},\quad \nu_\perp=\frac{1}{\rho(D-2)}. \tag{2.4}$$

For more details see Ref. [17]. Let us note that the case $\rho=\sigma\neq 1$ will be referred to as "weak" anisotropy [see Eq. (2.2)].

III. THE GAP EQUATION FOR THE REFERENCE SYSTEM

For the system with mixed geometry $L^m(\rho)\times\infty^n(\sigma)$ the gap equation has the form

$$\beta=\frac{1}{(2\pi)^n}\frac{1}{L^m}\int_{[-\pi/a]^n}^{[\pi/a]^n}\sum_{\mathbf{q}_\perp\in\Lambda^m}\frac{d^n\mathbf{q}_\parallel}{|\mathbf{q}_\perp|^{2\rho}+|\mathbf{q}_\parallel|^{2\sigma}+\lambda_V}. \tag{3.1}$$

Our analysis will be limited to systems with an effective dimension D below the upper critical dimension $D_u=4$ and above the lower critical one, $D_l=2$, i.e., for a real dimension d:

$$2\rho+n\left(1-\frac{\rho}{\sigma}\right)<d<4\rho+n\left(1-\frac{\rho}{\sigma}\right),\quad d=m+n. \tag{3.2}$$

From the physical point of view, *the infinite n-dimensional system*, which has a finite size L in the remaining m dimensions, can be found in three qualitatively different situations depending on the value of $\frac{n}{\sigma}$: (i) If $2<\frac{n}{\sigma}$, then the system is above its lower critical dimension $d_l=2\sigma$ and, therefore, it exhibits a true critical behavior. A crossover from n-dimensional to d-dimensional critical behavior takes place when $L\to\infty$. (ii) In the borderline case of $n=2\sigma$, the system is at its lower critical dimension and may have only a zero-temperature critical point. (iii) When $\frac{n}{\sigma}<2$, the system is below its lower critical dimension and a (d-dimensional) critical behavior appears only in the thermodynamic limit $L\to\infty$.

We assume that there is no phase transition for finite L, and in the present study henceforth $n<2\sigma$. For $n<2\sigma$ and $\lambda_V\to 0$, due to the convergence of the integral in Eq. (3.1) over $\mathbf{q}_\parallel$, one can extend the integration over all R^n in consistence with the underlying continuum field theory.

Further, the corresponding n-dimensional integral can be presented as

$$\frac{1}{(2\pi)^n}\frac{S_n}{L^m}\int_0^\infty\sum_{\mathbf{q}_\perp\in\Lambda^m}\frac{p^{n-1}dp}{|\mathbf{q}_\perp|^{2\rho}+p^{2\sigma}+\lambda_V}, \tag{3.3}$$

where $S_n=2(\pi)^{n/2}/\Gamma(n/2)$ is the surface of the n-dimensional unit sphere. With the help of the identity:

$$\int_0^\infty\frac{p^{\alpha-1}dp}{t+p^\eta+|\mathbf{q}_\perp|^\tau}=\frac{\Gamma\left(1-\frac{\alpha}{\eta}\right)\Gamma\left(\frac{\alpha}{\eta}\right)}{\eta}\frac{1}{(t+|\mathbf{q}_\perp|^\tau)^{1-\alpha/\eta}}, \tag{3.4}$$

$$\eta>\alpha>0,$$

if we choose $t=\lambda_V, \alpha=n$, $\tau=2\rho$, and $\eta=2\sigma$, for Eq. (3.3) we end up with the result

$$\frac{A_{n,\sigma}}{L^m}\sum_{\mathbf{q}_\perp\in\Lambda^m}\frac{1}{(\lambda_V+|\mathbf{q}_\perp|^{2\rho})^{1-n/2\sigma}},\quad 2\sigma>n, \tag{3.5}$$

where

$$A_{n,\sigma}=\frac{S_n}{(2\pi)^n}\frac{\Gamma\left(1-\frac{n}{2\sigma}\right)\Gamma\left(\frac{n}{2\sigma}\right)}{2\sigma}. \tag{3.6}$$

If we introduce the notation

$$K:=K(\sigma,n,m)\equiv A_{n,\sigma}^{-1}\beta, \tag{3.7}$$

the gap equation (3.1) may be presented in the equivalent form

$$K=\frac{1}{L^m}\sum_{\mathbf{q}_\perp\in\Lambda^m}\frac{1}{(\lambda_V+|\mathbf{q}_\perp|^{2\rho})^{1-n/2\sigma}},\quad 2\sigma>n. \tag{3.8}$$

Several comments are in order.

First, one can relate Eq. (3.8) with a fictitious *fully finite* m-dimensional reference system in which the memory of the n extended to infinity dimensions and the memory of the anisotropy of the system is retained only in the parameter

$$\gamma:=1-\frac{n}{2\sigma},\quad 0<\gamma<1. \tag{3.9}$$

So, instead of four, only three model parameters m,ρ, and γ are relevant in order to describe the model. The conditions (3.2) take the form

$$\frac{m}{2\rho}-1<\gamma<\frac{m}{2\rho}. \tag{3.10}$$

Second, we emphasize that the presence of anisotropy $\rho \neq \sigma$ does not lead to visible complications in Eq. (3.8) in comparison with the "weakly anisotropic" case $\rho=\sigma$. Since the parameters n and σ enter only in combination $\frac{n}{\sigma}$ in terms of the reference system some specific *crossover rules* take place. For example, symbolically we can write

$$L^m(\rho) \times \infty^n(\sigma) \Leftrightarrow L^m(\rho) \times \infty^{\tilde{n}}(\rho), \quad \tilde{n} := n\frac{\rho}{\sigma}, \quad (3.11)$$

i.e., the finite-size behavior of the strongly anisotropic system ($\rho \neq \sigma$) with m finite and n infinite dimensions is equivalent to a "weakly anisotropic" system ($\rho=\sigma$) with m finite and $\tilde{n}$ infinite dimensions, and vice versa. Likewise, we can write

$$L^m(\rho) \times \infty^n(\sigma) \Leftrightarrow L^m(\rho) \times \infty^m(\tilde{\sigma}), \quad \tilde{\sigma} := \sigma\frac{m}{n}. \quad (3.12)$$

An interesting consequence of this crossover rule is the property reduction of dimension: in the case $n > m$ the crossover counterpart has smaller dimension $2m$. Indeed, Eqs. (3.11) and (3.12) are true under the conditions (3.9) and (3.10) for the model parameters n, m, ρ, and σ.

Third, in the particular case $\rho=1$, $\gamma=1/2$ (i.e., $\tilde{n}=1$) and $1 < m < 3$, from the crossover rule (3.11) follows that the gap equation for λ_V, apart from a trivial rescaling of the temperature and the number of infinite dimensions, is independent of the anisotropy. As a result, in this case it is possible to apply the usual theory for isotropic systems. For example, the value of the universal scaling amplitude $A(m,\rho,\gamma)=\xi_{\perp,L}/L$ may be taken directly from the study of the crossover counterpart of the model (2.2), i.e., the quantum spherical spin model [7,22]. The result is $A(2,1,1/2)=1.5119\cdots$ [22].

IV. FINITE-SIZE SCALING FORM OF THE GAP EQUATION

We will follow the approach of Ref. [18] in order to obtain the finite-size scaling form of Eq. (3.8). Due to some significant properties of the generalized Mittag-Leffler functions used in this approach, one could follow a line of consideration quite close to the usual isotropic case. The corresponding mathematical calculations are presented in Appendix A.

The generalized Mittag-Leffler functions are defined by the power series [23] (see also [18,24,25])

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \alpha,\beta,\gamma \in \mathbb{C}, \quad Re(\alpha) > 0, \quad (4.1)$$

where

$$(\gamma)_0 = 1, (\gamma)_k = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+k-1) = \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)}, \quad (4.2)$$

$$k = 1,2, \ldots .$$

Some necessary properties of these functions are systemized in Appendix B.

Let us first introduce the scaling variables:

$$x = L^{2\rho(m/2\rho-\gamma)}(K - K^c_\infty), \quad y = \lambda_V L^{2\rho} \equiv (L/\xi_{\perp,L})^{2\rho}, \quad (4.3)$$

where K^c_∞ is the inverse critical temperature [see Eq. (A19)] of the "reference" bulk system (3.8). Then for the gap equation, the following scaling form is obtained (see Appendix A):

$$x \approx -a(m;\rho,\gamma)y^{(m/2\rho-\gamma)} + F^{\gamma}_{m,2\rho}(y) + \frac{1}{y^{\gamma}}, \quad (4.4)$$

where

$$a(m;\rho,\gamma) = -\frac{1}{(4\pi)^{m/2}\rho} \frac{\Gamma\left(\frac{m}{2\rho}\right)\Gamma\left(\gamma - \frac{m}{2\rho}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma(\gamma)}. \quad (4.5)$$

In Eq. (4.4),

$$F^{\gamma}_{m,2\rho}(y) = \frac{1}{(2\pi)^{2\gamma\rho}} \int_0^{\infty} dz z^{\gamma\rho-1} E^{\gamma}_{\rho,\gamma\rho}\left(-\frac{z^{\rho}}{(2\pi)^{2\rho}} y\right) \times \left[A^m(z) - 1 - \left(\frac{\pi}{z}\right)^{m/2}\right] \quad (4.6)$$

is the so-called universal scaling function [18], and

$$A(z) \equiv \sum_{n=-\infty}^{+\infty} e^{-zn^2}. \quad (4.7)$$

For $\gamma=1$, Eq. (4.6) reduces to the scaling function introduced earlier by a number of authors in the finite-size scaling theory (see [7,8] and references therein).

Compare with Eq. (3.8), there are complications and simplifications in Eq. (4.4). On one hand, the simplicity of the previous expression is lacking, so that a transparent physical interpretation is hampered. On the other hand the full set of the model parameters ($m, \rho, \gamma = 1 - \frac{n}{2\sigma}$, and L) enters in though complicated but well-studied special functions, so that a subsequent analytical consideration of the finite-size scaling properties is possible.

Our model study confirms the phenomenological assumption [15] that the finite-size scaling behavior in systems with mixed geometry $L^m(\rho) \times \infty^n(\sigma)$ is governed by the "perpendicular" correlation length $\xi_{\perp,L}$ only.

It is well-known that the concept of the standard finite-size scaling ceases to be valid when the dimension d of the considered finite-size system is smaller than the lower and bigger than the upper critical dimensions, respectively. For isotropic systems these are $d^{is}_l = 2\rho$ and $d^{is}_{up} = 4\rho$ (recall that in our case $\rho=1$ corresponds to a short-range interaction). If an anisotropic factor appears in the problem, such as the directionally dependent correlation length exponents in conjunction with more and more elongated geometries of the finite

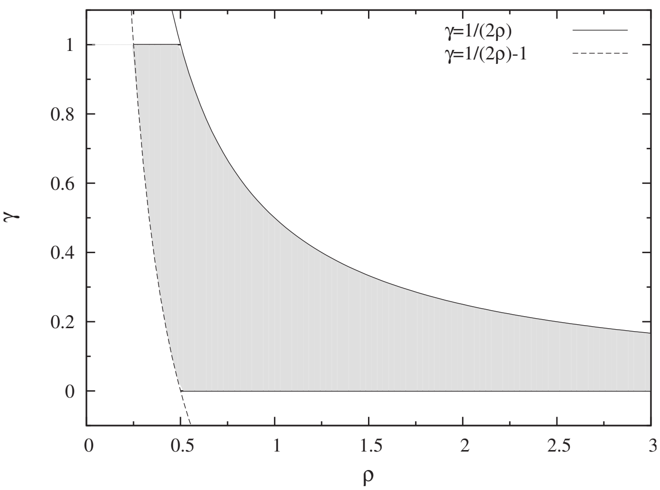

FIG. 1. Curves $\gamma=1/(2\rho)$ and $\gamma=1/(2\rho)-1$, in the ρ-γ plane, resulting from the upper and lower critical dimensions [left-hand side and right-hand side of inequalities (3.2)], respectively. In the shaded region, bounded by portions of these curves together with the conditions $0<\gamma<1$ the considered finite-size scaling behavior occurs.

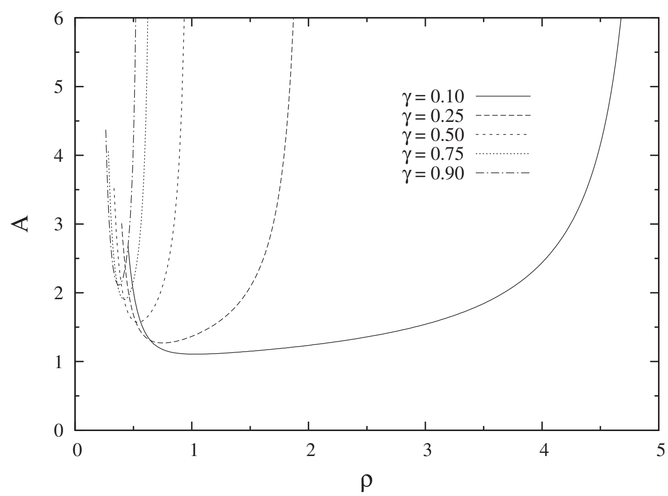

FIG. 2. The universal scaling amplitude $A=A(1,\rho,\gamma)$, for some values of γ, as a function of ρ. We recall that ρ and γ must belong to the domain presented in Fig. 1.

system, what happens is that d_l^{is} and d_{up}^{is} are shifted, e.g., $d_l^{anis}=d_l^{is}-n\Delta$ and $d_{up}^{anis}=d_{up}^{is}-n\Delta$, where $\Delta:=\rho/\sigma-1$ is the so-called anisotropy exponent [cf. Eq. (3.2)]. As a result, depending on the value of $n\Delta$ that may be either positive or negative, the standard finite-size scaling [e.g., Eq. (4.4)] may be relevant for different integer values of the dimensionality d. For example, it may be the case of low-dimensional systems (1d and 2d) excluding the 3d case, etc. At this level, the anisotropy of the system appears only in the generalized form of the scaling equation, Eq. (4.4), through parameter $\gamma\neq 1$.

V. FINITE-SIZE CORRECTIONS

Given the gap equation in scaling form, we are now in a position to explore the various finite-size corrections. In this section, for the sake of simplicity, we will consider the important particular case of slab geometry, $m=1$. Fixing m, we have two independent model parameters ρ and γ subject to condition (3.10) jointed with Eq. (3.9). The visualization of the corresponding restrictions on ρ and γ is shown in Fig. 1.

Here, we look at different regimes: the finite-size scaling regime defined by the condition $y\sim 1$, crossover to the thermodynamic critical behavior $y\gg 1$, and the regime $y\ll 1$.

A. $y\sim 1$

We see that the finite-size scaling regime is characterized by $\lambda_V\to 0$ and $L^{2\rho}\to\infty$, so that $y:=\lambda_V L^{2\rho}=O(1)$. In order to consider this case we will use a new representation for $F_{1,2\rho}^{\gamma}(y)$ (see Appendix C),

$$F_{1,2\rho}^{\gamma}(y)=F_{1,2\rho}^{\gamma}(0)+a(1;\rho,\gamma)y^{1/2\rho-\gamma}+2\sum_{l=1}^{\infty}\frac{(4\pi^2l^2)^{\rho\gamma}-[y+(4\pi^2l^2)^{\rho}]^{\gamma}}{(4\pi^2l^2)^{\rho\gamma}[y+(4\pi^2l^2)^{\rho}]^{\gamma}},\quad 1>2\gamma\rho, \tag{5.1}$$

and rewrite Eq. (4.4) in a form suitable for obtaining the shift of the bulk critical temperature. Substituting Eq. (5.1) in Eq. (4.4) $(m=1)$ we obtain the gap equation in the form

$$x\simeq F_{1,2\rho}^{\gamma}(0)+2\sum_{l=1}^{\infty}\frac{(4\pi^2l^2)^{\rho\gamma}-[y+(4\pi^2l^2)^{\rho}]^{\gamma}}{(4\pi^2l^2)^{\rho\gamma}[y+(4\pi^2l^2)^{\rho}]^{\gamma}}+\frac{1}{y^{\gamma}}. \tag{5.2}$$

Therefore when $K\to K_\infty^c$, simultaneously with $L\to\infty$, in the way prescribed by the equation

$$K=K_\infty^c+\frac{x}{L^{2\rho(1/2\rho-\gamma)}} \tag{5.3}$$

with $x=O(1)$, the leading-order asymptotic form of λ_V is given by

$$\lambda_V\simeq\frac{y(x)}{L^{2\rho}}, \tag{5.4}$$

where $y(x)$ is the positive solution of Eq. (5.2). Hence at the critical point $x=0$ $(K=K_\infty^c)$, we obtain

$$\xi_{\perp,L}=A(1,\rho,\gamma)L, \tag{5.5}$$

where $A(1,\rho,\gamma)=1/[y(0)]^{1/2\rho}$ is a universal amplitude. In systems with mixed geometry the existence of the universal amplitude of the (finite-size) correlation length $\xi_{\perp,L}$ on the level of the phenomenological scaling has been suggested in [15]. Here this qualitative statement is made quantitative being a model confirmation of the generalized to anisotropic scaling in [15], the Privman-Fisher hypothesis [cf. with Eq. (23) in [15]]. The outcome of the numerical analysis of the behavior of the universal amplitude $A(1,\rho,\gamma)$ is shown in Fig. 2.

The results show that the universal amplitude has a minimum at a fixed γ, being a rapidly increasing function of ρ closer to the corresponding upper and lower borders of validity of Eq. (4.4), defined by $\gamma_>(\rho)=\min\{1,1/2\rho\}$ and $\gamma_<(\rho)=\max\{0,(1/2\rho)-1\}$, respectively. As long as γ goes smaller the minimum becomes deeper and more flat. Since the anisotropy is captured in a number of infinite dimensions, systems with different dimensions and different σ and ρ, but with the same ρ and γ, have identical universal finite-size scaling amplitude. As one can see the behavior of amplitude in the "less anisotropic" case $\gamma=1/2$ does not have special properties, increasing rapidly in the short-range limit (with $\rho\to 1, \rho\neq 1$).

B. $y \gg 1$

In this regime the correlation length is much smaller that the characteristic system size L but we assume that it is still much larger than the lattice spacing a. In this case one expects a crossover to the bulk critical behavior. When $y\to\infty$, we may approximate the sum in Eq. (5.2) by an integral:

$$\sum_{l=1}^{\infty}\frac{(4\pi^2l^2)^{\rho\gamma}-[y+(4\pi^2l^2)^{\rho}]^{\gamma}}{(4\pi^2l^2)^{\rho\gamma}[y+(4\pi^2l^2)^{\rho}]^{\gamma}} \simeq \frac{1}{(2\pi z)^{2\gamma\rho}}\int_0^{\infty}dx\frac{\left(\frac{x}{z}\right)^{2\gamma\rho}-\left(1+\left(\frac{x}{z}\right)^{2\rho}\right)^{\gamma}}{\left(\frac{x}{z}\right)^{2\gamma\rho+1/2}\left(1+\left(\frac{x}{z}\right)^{2\rho}\right)^{\gamma}}, \quad (5.6)$$

$$z:=y^{1/2\rho}/(2\pi).$$

With the use of Eqs. (A21) and (5.6) one finds the asymptotic solution of Eq. (5.2), which to the leading order in $x\gg 1$ is

$$x\approx -a(1;\rho,\gamma)y^{(1/2\rho)-1}, \quad (5.7)$$

i.e., it recovers exactly the familiar *bulk high-temperature* result.

The finite-size correction to the bulk critical behavior can be extracted from the asymptotic form of the functions $F^{\gamma}_{d,\sigma}(y)$ at large argument $y\gg 1$ (see [18,21]). The result is

$$F^{\gamma}_{1,2\rho}(y)\simeq -y^{-\gamma}+[2\gamma(2\pi)^{2\rho}\zeta(-2\rho)]y^{-(1+\gamma)}, \quad (5.8)$$

where $\zeta(\rho)$ is Riemann's zeta function.

Using Eq. (5.8) for the gap equation (3.1) we obtain

$$x\approx -a(1;\rho,\gamma)y^{(1/2\rho)-1}+[2\gamma(2\pi)^{2\rho}\zeta(-2\rho)]y^{-(1+\gamma)}, \quad y\gg 1. \quad (5.9)$$

As one can see the finite-size effects governed by the second term on the right-hand side of Eq. (5.8) vary as an algebraic power of the variable y. Since $\zeta(-2\rho)=0$ for $\rho=k$, k is a natural number, there are not power-law dependent finite size corrections if $\rho=k$. The case $0<\rho<1$ corresponds to the long-range interaction. For $\rho=1$, corresponding to a short-range interaction, the result for the universal finite-size scaling function is

$$F^{\gamma}_{1,2}(y)\simeq -y^{-\gamma}+\left[\frac{1}{2^{\gamma}\Gamma(\gamma)}\right]y^{-\gamma/2}e^{-\sqrt{y}}, \quad (5.10)$$

which leads to an exponential fall of the finite-size corrections in Eq. (5.9) rather than to a power-law one. As long as $\rho>1$, and if ρ is not an integer, the power-law corrections take place in the case of so-called subleading long-range interaction [26] but with strong anisotropy.

C. $y \ll 1$

In this regime the correlation length is much larger that the characteristic system size L. As a result finite-size effects will be very important.

Whenever Eq. (5.2) has a solution $y\ll 1$, use can be made of the asymptotic expansion

$$\sum_{l=1}^{\infty}\frac{(4\pi^2l^2)^{\rho\gamma}-[y+(4\pi^2l^2)^{\rho}]^{\gamma}}{(4\pi^2l^2)^{\rho\gamma}[y+(4\pi^2l^2)^{\rho}]^{\gamma}} = -\frac{\gamma}{(2\pi)^{2\gamma\rho+2\rho}}\zeta(2\gamma\rho+2\rho)y+O(y^2). \quad (5.11)$$

In obtaining the first term on the right-hand side of Eq. (5.11) the fulfillment of the condition $2\gamma\rho+2\rho>1$ is used. Taking into account Eq. (5.11), in the limit $|x-F^{\gamma}_{1,2\rho}(0)|\gg 1$ one finds

$$y\simeq \frac{1}{|x-F^{\gamma}_{1,2\rho}(0)|^{1/\gamma}}. \quad (5.12)$$

Let us recall that when the number of infinite dimensions is less than the lower critical dimension, the singularities of the bulk thermodynamic functions are rounded and no phase transition occurs in the finite-size system. Nevertheless, one can define a pseudocritical temperature, corresponding to the position of the smeared singularities of the finite-size thermodynamic functions, and study its shift with respect to the bulk value of the critical temperature. In the case under consideration the first term on the right-hand side of Eq. (5.2) is identified with the shift of the finite-size pseudocritical temperature. Actually, for the sake of convenience, here we study the quantity K. The corresponding result for the pseudocritical K^c_L is

$$K^c_L-K^c_{\infty}=L^{-\lambda}F^{\gamma}_{1,2\rho}(0), \quad (5.13)$$

i.e., the critical shift exponent is $\lambda=1/\nu_{\perp}$ in accordance with standard finite-size scaling conjecture, see [7]. The coefficient $F^{\gamma}_{d-n,\sigma}(0)$ can be evaluated analytically as well as numerically for different values of the free parameters d, ρ, and γ using the method developed in Ref. [27].

It is interesting to see the behavior of the universal critical amplitude $A=A(1,\rho,\gamma)$ calculated at the pseudocritical temperature K^c_L. The outcome of the numerical analysis is shown in Fig. 3. There are several points worth noting from Fig. 3. There is a region where the universal scaling amplitudes with very close $\rho(\approx 0.5)$ and quite different γ's have approximately equal values A. One can easily see two different behaviors of A as a function of ρ. On the right-hand side of $\rho=1$, the amplitude at a fixed ρ is an increasing function of γ, being at fixed γ a slowly varying function of ρ, close to a constant. On the left-hand side of $\rho=1$ ($\rho<1$ corresponds to the long-range interaction) the amplitude A exhibits a strong ρ-dependence, and it becomes weaker when increasing γ. Since ρ must belong to the domain presented in Fig. 1, the curves end sharply at the points on the border. For $\gamma\to 1$ the

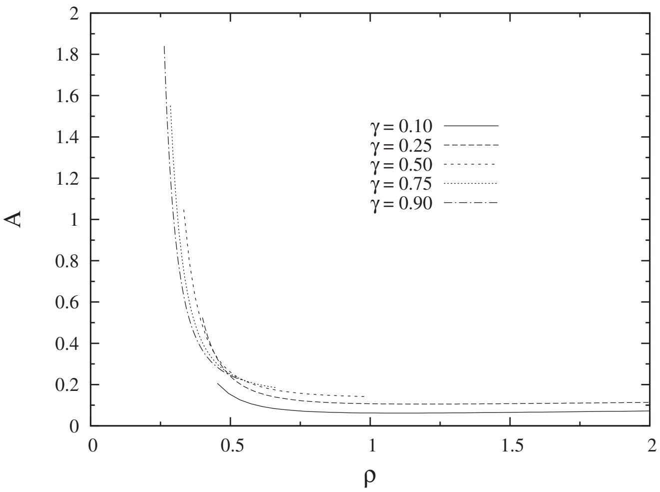

FIG. 3. The universal scaling amplitude $A=A(1,\rho,\gamma)$ obtained from Eq. (5.2), at the pseudocritical point $x=F^{\gamma}_{1,2\rho}(0)$, for different values of γ and as a function of ρ.

initial points are not very resolvable on the scale of the presented graphs, but see the case $\gamma=0.10$ and the end points for decreasing γ. Here it is seen the weaker influence of the change of ρ on the variation of A in comparison to the previous case presented in Fig. 2. This can be explained by the influence of the shift [see Eq. (5.13)] of the critical temperature near end points which in the case of Fig. 3 has been excluded.

VI. CONCLUSIONS

The principal goal of the present study is twofold. First, we extend the scope of systems to which the ideas of Refs. [5,6] (see also [7]) apply; i.e., we study systems which heretofore have been beyond the reach of *analytical* methods. Second, we in a unified way expound the finite-size scaling philosophy. In particular, we demonstrate that finite-size scaling in its standard form takes place for a certain class of systems regardless of the nature of their anisotropic properties.

The statement, that finite-size scaling in systems with mixed geometry $L^m(\rho)\times\infty^n(\sigma)$ is the naturally expected one, is contained in [17] where different shape dependent scaling limits have been studied. In the present study we are much more interested in obtaining an *explicit form* of the scaling equation analytically tractable in different regimes. To this aim an appropriate technique of calculation is developed. We show *how* the mathematical difficulties that arise in the considered anisotropic model with mixed geometry can be avoided.

First, using the identity (3.4) the problem is effectively reduced to the corresponding one related to a fully finite reference system, Eq. (3.8). One advantage of this presentation is evident, i.e., the number of the infinite dimensions n is scaled by σ and the behavior of the system is classified by (m,ρ,γ). This allows one to formulate crossover rules like Eqs. (3.11) and (3.12). In the particular case $\gamma=1/2$ the problem of studying finite-size behavior is mapped onto finite-size behavior of a system with geometry $L^m(\rho)\times\infty^1(1)$ studied earlier in the context of quantum critical phenomena [7,14,22].

A further step is the recognition that with the help of the identity (A2) the appearance of $\gamma\neq 1$ in the summand of the gap equation (3.8) is not an obstacle for our treatment. Knowledge of the properties of the generalized Mittag-Leffler function allows one to carry out all calculations analytically.

We show that though the system is strongly anisotropic, the corresponding gap equation, Eq. (4.4), for the intrinsic scaling variable $y=\lambda_V L^{2\rho}$ has a form very similar to the isotropic case with cubic geometry $L^m(\rho)$ [28]. We stress that the finite-size L is scaled by the perpendicular correlation $\xi_\perp$ only. This verified the Privman-Fisher hypothesis for strongly anisotropic systems formulated in [15]. Let us emphasize that herein an important assumption is that the n "parallel" dimensions are infinite, otherwise we would have to deal with the two distinct correlation lengths $\xi_\perp$ and $\xi_\parallel$. An additional assumption is the hyperscaling to be held; i.e., the dimension d of the system to be between the lower $d_l^{anis}=d_l^{is}-n\Delta$ and upper $d_{up}^{anis}=d_{up}^{is}-n\Delta$ critical dimensions. The shift of isotropic critical dimensions becomes larger with increasing the product $n\Delta$.

The case of slab geometry $m=1$ is examined in detail, both analytically and numerically. We conclude that the finite-size contributions to the thermodynamic behavior decay algebraically as a function of L only if $0<\rho\neq k$, where k is a natural number. In the case $\rho=1$, the finite-size contributions decay exponentially as a function of L. The phenomenon that the so-called subleading terms (in our terminology the term with $\rho>1$) lead to dominant finite-size contributions, being unimportant in the bulk limit, was first discussed in Ref. [26]. This characteristic feature of the long-range interactions is revealed also in our consideration.

It seems interesting to check whether the here studied finite-size scaling behavior is just a special feature of the $O(\infty)$ systems or instead indicates a general property of strongly anisotropic models with finite N on these special finite-size geometries. Indeed, there are two possibilities for further investigation of this issue: large-N expansion and numerical simulations. Unfortunately, not much effort has been put into the investigation of corrections of order $1/N$ even for the bulk case. For the particular case of the m-axial Lifshitz points one can see Ref. [29], where obstacles arise already at the bulk level. The lack of results for finite-size systems (Ref. [17] is restricted to the limit $N\to\infty$) is due to the additional complications arising from the combination of the effects of the anisotropy of shape and the anisotropic critical behavior such systems exhibit. On the numerical simulations side, the progress has been hampered by the same reasons. Let us note that it is far from being trivial to extract information about correlation lengths from Monte Carlo (MC) simulation data even in the case of isotropic $O(N)$ systems [30], where a special path of data analysis was used. In order to give a clear picture of systematic dependencies on the spin dimensionality N the results obtained in [30] were juxtaposed with MC simulations directly in the spherical model trying to match finite-N results with analytical calculations. Apart from the subtle problem of the equivalence on a finite lattice of the $N\to\infty$ limit of the $O(N)$ model and the spherical model (see again [30]), the present model (where exact results for comparison are available) can play a useful role in examining similar problems that would arise in MC simulations of strongly anisotropic systems.

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APPENDIX A: DERIVATION OF FINITE-SIZE SCALING FORM OF THE GAP EQUATION

The normalized m dimensional sum in Eq. (3.8),

$$W_{m,2\rho}^{1-n/2\sigma}(\lambda_V,L):=\frac{1}{L^m}\sum_{\mathbf{q}_\perp\in\Lambda^m}\frac{1}{(\lambda_V+|\mathbf{q}_\perp|^{2\rho})^{1-n/2\sigma}},\quad 2\sigma>n, \tag{A1}$$

can be evaluated with the help of the identity [18]

$$\frac{1}{(\lambda_V+y^\alpha)^\gamma}=\int_0^\infty dte^{-yt}t^{\alpha\gamma-1}E_{\alpha,\alpha\gamma}^\gamma(-\lambda_V t^\alpha), \tag{A2}$$

in terms of the generalized Mittag-Leffler function $E_{\alpha,\gamma\alpha}^\gamma(z)$ (see Appendix B). If one chooses $\alpha=\rho$, $\gamma=1-\frac{n}{2\sigma}$, and $y=|\mathbf{q}_\perp|^2$ the needed result is

$$W_{m,2\rho}^\gamma(\lambda_V,L)=\int_0^\infty dxx^{\gamma\rho-1}E_{\rho,\gamma\rho}^\gamma(-\lambda_V x^\rho)\times\left[\frac{1}{L}\sum_{q\in\Lambda^1}\exp(-q^2x)\right]^m,\quad \gamma>0. \tag{A3}$$

Now let us define

$$Q_{N_0}(x):=\frac{1}{L}\sum_{q\in\Lambda^1}\exp(-q^2x)=\frac{1}{aN_0}\sum_{l=-N_0/2}^{N_0/2-1}\exp\left(-\frac{4\pi^2l^2x}{a^2N_0^2}\right) \tag{A4}$$

and by using the approximating formula (5.5) of Ref. [26], we obtain the expression

$$Q_{N_0}(x)\cong\frac{1}{\sqrt{4\pi x}}\left[\operatorname{erf}\left(\frac{\pi x^{1/2}}{a}\right)\right]-\frac{2\pi^2x}{3a}\exp\left[-\left(\frac{\pi}{a}\right)^2x\right]+\frac{1}{\sqrt{\pi x}}\left\{\sum_{l=1}^\infty\exp\left[-\frac{(laN_0)^2}{4x}\right]\right\}, \tag{A5}$$

valid in the large N_0 asymptotic regime. The first and the second terms in the above equation are size independent and are precisely the infinite volume limit of $Q_{N_0}(x)$. The remainder of the calculations involves the insertion of Eq. (A5) into Eq. (A3) and as a result we get Eq. (3.8) in finite-size scaling form.

In order to illustrate the derivation Eq. (4.4), first we will consider in more detail the case $m=1$. We can represent the right-hand side of Eq. (A3) as a sum of three terms.

The first one is given by

$$\frac{1}{2\pi}\int_{-\pi/a}^{\pi/a}dk\int_0^\infty dxx^{\gamma\rho-1}E_{\rho,\gamma\rho}^\gamma(-\lambda_V x^\rho)\exp(-xk^2)=\frac{1}{2\pi}\int_{-\pi/a}^{\pi/a}dk\frac{1}{(\lambda_V+k^{2\rho})^{1-n/2\sigma}}, \tag{A6}$$

where the definition of the erf function

$$\frac{\operatorname{erf}(\Lambda\sqrt{x})}{\sqrt{4\pi x}}=\frac{1}{2\pi}\int_{-\Lambda}^{\Lambda}\exp(-xk^2)dk \tag{A7}$$

and the identity (A2) have been used.

The second term is

$$-\frac{2\pi^2}{3a}\int_0^\infty dxx^{\gamma\rho}E_{\rho,\gamma\rho}^\gamma(-\lambda_V x^\rho)\exp\left[-\left(\frac{\pi}{a}\right)^2x\right]=-\frac{2\pi\gamma\rho}{3}\frac{\left(\frac{\pi}{a}\right)^{2\rho-1}}{\left[\lambda_V+\left(\frac{\pi}{a}\right)^{2\rho}\right]^{\gamma+1}}. \tag{A8}$$

The third one equals

$$\int_0^\infty dxx^{\gamma\rho-1}E_{\rho,\gamma\rho}^\gamma(-\lambda_V x^\rho)\frac{1}{\sqrt{\pi x}}\left\{\sum_{j=1}^\infty\exp[-(jN_0a)^2/4x]\right\}. \tag{A9}$$

The first term is exactly the bulk limit $W_{1,2\rho}^\gamma(\lambda_V,\infty)$. The second one in the considered regime $\lambda_V\to0$ and $a\to0$ is of order $O(a^{1+2\gamma\rho})$ and can be omitted.

It is convenient to write the third term, Eq. (A9), in terms of the function (a particular case of the Jacobi Θ_3 function)

$$A(x)\equiv\sum_{n=-\infty}^{+\infty}e^{-xn^2} \tag{A10}$$

and the function [18]

$$F_{m,2\rho}^\gamma(y)=\frac{1}{(2\pi)^{2\gamma\rho}}\int_0^\infty dxx^{\gamma\rho-1}E_{\rho,\gamma\rho}^\gamma\left(-\frac{x^\rho}{(2\pi)^{2\rho}}y\right)\times\left[A^m(x)-1-\left(\frac{\pi}{x}\right)^{m/2}\right]. \tag{A11}$$

This can be done with the help of the Poisson transformation formula

$$A(x)=\sqrt{\frac{\pi}{x}}A\left(\frac{\pi^2}{x}\right) \tag{A12}$$

and the identity (B2).

After some algebra the result for the third term is

$$L^{2\gamma\rho-1}\left[F^{\gamma}_{1,2\rho}(\lambda_V L^{2\rho})+\frac{1}{(\lambda_V L^{2\rho})^{\gamma}}\right]. \tag{A13}$$

Collecting the above results for Eq. (A3), if $m=1$, we obtain

$$W^{\gamma}_{1,2\rho}(\lambda_V,L)=\frac{1}{2\pi}\int_{-\pi/a}^{\pi/a}dk\frac{1}{(\lambda_V+k^{2\rho})^{1-n/2\sigma}}-\frac{2\pi\gamma\rho}{3}\frac{\left(\frac{\pi}{a}\right)^{2\rho-1}}{\left[\lambda_V+\left(\frac{\pi}{a}\right)^{2\rho}\right]^{\gamma+1}}+L^{2\gamma\rho-1}\left[F^{\gamma}_{1,2\rho}(\lambda_V L^{2\rho})+\frac{1}{(\lambda_V L^{2\rho})^{\gamma}}\right], \tag{A14}$$

$$\gamma\equiv 1-\frac{n}{2\sigma}>0.$$

In the last term of Eq. (A14) apart from the factor $L^{2\gamma\rho-1}$ the intrinsic scaling combination

$$y=\lambda_V L^{2\rho}=(L/\xi_{\perp,L})^{2\rho} \tag{A15}$$

emerges, where $\xi_{\perp,L}$ is the finite-size transverse correlation length (see [17]). The limitation $m=1$ is not principal. If $m>1$, in view of Eq. (A5), the product $[Q_{N_0}(x)]^m$ in Eq. (A3) contains sums of terms of the form

$$\left\{\frac{1}{\sqrt{4\pi x}}\right\}^{m}\left\{\left[\mathrm{erf}\left(\frac{\pi x^{1/2}}{a}\right)\right]\right\}^{m'}\left\{\exp\left[-\sum_{i=1}^{m-m'}(l_i a N_0)^2/4x\right]\right\}, \tag{A16}$$

with $1\leq m'\leq m-1$ and $l_i\neq 0, i=1,\ldots,m-m'$. In such terms the error function $\mathrm{erf}\left(\frac{\pi x^{1/2}}{a}\right)$ can be replaced by unity, since the exponential function on the right-hand side of Eq. (A16) cuts off the contribution from values of $x^{1/2}\ll aN_0$. Note that all the other terms that contain as a multiplier $\frac{2\pi^2 x}{3a}\exp\left[-\left(\frac{\pi}{a}\right)^2 x\right]$ can be estimated. They are of order $O(a)$ and must be omitted in the considered continuum limit. As a result instead of Eq. (A14) we get

$$W^{\gamma}_{m,2\rho}(\lambda_V,L)\simeq\frac{1}{(2\pi)^m}\int_{[-\pi/a]^m}^{[\pi/a]^m}\frac{d^m\mathbf{k}}{(\lambda_V+|\mathbf{k}|^{2\rho})^{1-n/2\sigma}}+L^{2\gamma\rho-m}\left[F^{\gamma}_{m,2\rho}(\lambda_V L^{2\rho})+\frac{1}{(\lambda_V L^{2\rho})^{\gamma}}\right], \tag{A17}$$

$$\gamma\equiv 1-\frac{n}{2\sigma}>0.$$

Now, Eq. (3.8) can be rewritten as

$$K-K^c_{\infty}=W^{\gamma}_{m,2\rho}(\lambda_V,L)-W^{\gamma}_{m,2\rho}(0,\infty), \tag{A18}$$

where

$$K^c_{\infty}:=K^c_{\infty}(\sigma,\rho,n,m)\equiv W^{\gamma}_{1,2\rho}(0,\infty)=\frac{1}{(2\pi)^m}\int_{[-\pi/a]^m}^{[\pi/a]^m}d^m\mathbf{k}\frac{1}{(|\mathbf{k}|^{2\rho})^{\gamma}}. \tag{A19}$$

The first term on the right-hand side of Eq. (A17) can be presented in the form

$$W^{\gamma}_{m,2\rho}(\lambda_V,\infty)=\frac{1}{(2\pi)^m}\int_{[-\pi/a]^m}^{[\pi/a]^m}d^m\mathbf{k}\frac{1}{(\lambda_V+|\mathbf{k}|^{2\rho})^{\gamma}}\simeq K^c_{\infty}+\frac{S_m}{2(2\pi)^m}\lambda_V^{m/2\rho-\gamma}\int_0^{\infty}dx\frac{x^{\gamma\rho}-(1+x^{\rho})^{\gamma}}{x^{\gamma\rho+1-m/2}(1+x^{\rho})^{\gamma}}, \tag{A20}$$

valid for $\xi_{\perp,L}\gg a$. The integral over x converges, provided $m>2\gamma\rho>m-2\rho$, and

$$\int_0^{\infty}dx\frac{x^{\gamma\rho}-(1+x^{\rho})^{\gamma}}{x^{\gamma\rho+1-m/2}(1+x^{\rho})^{\gamma}}=\frac{1}{\rho}\frac{\Gamma\left(\frac{m}{2\rho}\right)}{\Gamma\left(1-\frac{n}{2\sigma}\right)}\Gamma\left(1-\frac{n}{2\sigma}-\frac{m}{2\rho}\right). \tag{A21}$$

By substitution of Eq. (A17) into Eq. (A18), taking into account the small-argument expansion Eq. (A20), for the gap equation (3.8) we obtain the scaling form (4.4).

APPENDIX B: GENERALIZED MITTAG-LEFFLER FUNCTIONS

Let us formulate some necessary properties of the generalized Mittag-Leffler functions. It might be useful to note the relationship

$$-\frac{d}{dz}E^{\gamma}_{\alpha,1}(-z^{\alpha})=\gamma z^{\alpha-1}E^{\gamma}_{\alpha,\beta}(-z^{\alpha}), \tag{B1}$$

which follows from the power-series representation. In obtaining Eq. (A13) we have taken into account the identity

$$\int_0^{\infty}dx x^{\gamma\rho-1}E^{\gamma}_{\rho,\gamma\rho}(-x^{\rho})=1,\quad \rho>0 \tag{B2}$$

that follows by integration of Eq. (B1) over z from zero to infinity. Next, by subtracting and adding $1/\Gamma(\alpha\gamma)$ to the function $E^{\gamma}_{\rho,\gamma\rho}$ we obtain

$$\int_0^{\infty}dte^{-zt}t^{\alpha\gamma-1}\left[E^{\gamma}_{\alpha,\alpha\gamma}(-t^{\alpha})-\frac{1}{\Gamma(\alpha\gamma)}\right]=\frac{z^{\alpha\gamma}-(1+z^{\alpha})^{\gamma}}{(1+z^{\alpha})^{\gamma}z^{\alpha\gamma}}. \tag{B3}$$

APPENDIX C: DERIVATION OF EQ. (5.1)

First, we represent the integral in Eq. (A11) as a sum of three terms ($m=1$). The first term is given by

$$(y^{-1/\rho})^{\gamma\rho}\int_0^\infty dt t^{\rho\gamma-1}\left[E^{\gamma}_{\rho,\rho\gamma}(-t^\rho)-\frac{1}{\Gamma(\rho\gamma)}\right]\left[A\left(\frac{4\pi^2 t}{y^{1/\rho}}\right)-1\right] \equiv S_{\rho,\gamma}(y^{1/\rho}), \tag{C1}$$

the second term is

$$-\frac{1}{2\sqrt{\pi}y^{\gamma-1/2\rho}}\int_0^\infty dt t^{\rho\gamma-3/2}\left[E^{\gamma}_{\rho,\rho\gamma}(-t^\rho)-\frac{1}{\Gamma(\rho\gamma)}\right] \equiv -\frac{1}{y^{\gamma-1/2\rho}}C_{\gamma,\rho}, \tag{C2}$$

and the third one equals the constant (provided $1>2\gamma\rho$)

$$\frac{1}{\Gamma(\rho\gamma)}\frac{1}{(2\pi)^{\gamma 2\rho}}\int_0^\infty dx x^{\gamma\rho-1}\left[A(x)-1-\left(\frac{\pi}{x}\right)^{1/2}\right] \equiv F^{\gamma}_{1,2\rho}(0). \tag{C3}$$

Let us now calculate the function $S_{\rho,\gamma}(y^{1/\rho})$ and the constant $C_{\gamma,\rho}$. Making use of the identity (B3), we represent Eq. (C1) as

$$S_{\rho,\gamma}(y^{1/\rho}) = 2\sum_{l=1}^{\infty}\frac{(4\pi^2 l^2)^{\rho\gamma}-[y+(4\pi^2 l^2)^\rho]^\gamma}{(4\pi^2 l^2)^{\rho\gamma}[y+(4\pi^2 l^2)^\rho]^\gamma}. \tag{C4}$$

To calculate $C_{\gamma,\rho}$, in Eq. (C2) we first write

$$t^{-1/2} = \frac{1}{\pi^{1/2}}\int_0^\infty dx x^{-1/2}e^{-tx}, \tag{C5}$$

then, by using the identity (B3) we take the integral over t, and then

$$C_{\gamma,\rho} = \frac{1}{2\pi}\int_0^\infty dx\frac{x^{\rho\gamma}-(1+x^\rho)^\gamma}{(1+x^\rho)^\gamma x^{\rho\gamma+1/2}}, \tag{C6}$$

i.e., $C_{\gamma,\rho}=-a(n,1;\rho,\sigma)$. Collecting the results for Eqs. (C1)–(C3) for Eq. (A11) we get Eq. (5.1).

[1] J. Cardy, *Scaling and Renormalization in Statistical Physics* (Cambridge University, Cambridge, England, 1996), p. 58.
[2] B. Schmittman and R. K. Zia in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, New York, 1995), Vol. 17.
[3] M. Henkel, Nucl. Phys. B **641**, 405 (2002).
[4] A strong motivation for studying anisotropic systems comes from the physics of driven diffusive systems. See Ref. [2], Sec. IV for additional comments and corresponding definitions.
[5] E. Brézin, J. Phys. (France) **43**, 15 (1982).
[6] E. Brézin, Ann. N.Y. Acad. Sci. **410**, 339 (1983).
[7] J. G. Brankov, D. M. Danchev, and N. S. Tonchev, *The Theory of Critical Phenomena in Finite-Size Systems–Scaling and Quantum Effects* (World Scientific, Singapore, 2000).
[8] H. Chamati and N. S. Tonchev, Mod. Phys. Lett. B **17**, 1187 (2003).
[9] X. S. Chen and V. Dohm, Phys. Rev. E **70**, 056136 (2004); **71**, 059901(E) (2005).
[10] K. Binder and J.-S. Wang, J. Stat. Phys. **55**, 87 (1989).
[11] K. T. Leung, Phys. Rev. Lett. **66**, 453 (1991).
[12] K. Leung, Int. J. Mod. Phys. C **3**, 367 (1992).
[13] H. Chamati and N. S. Tonchev, J. Phys. A **33**, 873 (2000).
[14] H. Chamati, D. M. Danchev, and N. S. Tonchev, Eur. Phys. J. B **14**, 307 (2000).
[15] M. Henkel and U. Schollwöck, J. Phys. A **34**, 3333 (2001).
[16] A. Hucht, J. Phys. A **35**, L481 (2002).
[17] S. Caracciolo, A. Gambassi, M. Gubinelli, and A. Pellissetto, Eur. Phys. J. B **34**, 205 (2003).
[18] H. Chamati and N. S. Tonchev, J. Phys. A **39**, 469 (2006).
[19] J. G. Brankov, J. Stat. Phys. **56**, 309 (1989).
[20] H. Chamati, Physica A **212**, 357 (1994).
[21] N. S. Tonchev, JINR Dubna, Russia, E17–2005–148, 2005 (unpublished).
[22] H. Chamati, E. S. Pisanova, and N. S. Tonchev, Phys. Rev. B **57**, 5798 (1998).
[23] T. R. Prabhakar, Yokohama Math. J. **19**, 7 (1971).
[24] R. K. Saxena, A. M. Mathai, and H. J. Haubold, Astrophys. Space Sci. **290**, 299 (2004).
[25] A. A. Kilbas, M. Saigo, and R. K. Saxena, Integral Transforms Spec. Funct. **15**, 31 (2004).
[26] D. M. Danchev and J. Rudnick, Eur. Phys. J. B **21**, 251 (2001).
[27] H. Chamati and N. S. Tonchev, J. Phys. A **33**, L167 (2000).
[28] Compare with Eq. (5.40) in [7]. Formally, if one sets $\gamma=1$ in Eq. (4.4) both equations exactly coincide. Notice that in Eq. (5.40) of Ref. [7] a slightly different definition of the function (4.6) is used. In order to compare both results one needs to set $\sigma=2\rho$, $d=m$, and to change y with $y^{1/\rho}$ in Eq. (5.40).
[29] M. A. Shpot, Yu. M. Pis'mak, and H. D. Diehl, J. Phys.: Condens. Matter **17**, S1947 (2005).
[30] M. Weigel and W. Janke, Phys. Rev. B **62**, 6343 (2000).