Acoustic wave dispersion in a one-dimensional lattice of nonlinear resonant scatterers

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Nonlinear effects of acoustic wave propagation and dispersion are observed in a one-dimensional lattice made of Helmholtz resonators connected to a tube. These regularly spaced scatterers exhibit individually a wave frequency dependence, which induces a strong velocity dispersion. In addition, they exhibit a wave amplitude dependence (acoustic nonlinearity), which induces nonlinear effects on the dispersion relation of waves in the lattice. The usually observed forbidden frequency band gaps for the transmission coefficient through the lattice are shown to be amplitude dependent. Experimental results are compared to a developed model taking into account the nonlinear behavior of the Helmholtz resonator.

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Since the work of Brillouin in the middle of 20th century [1], the wave propagation and dispersion in ordered lattices have been widely studied. The solutions for the wave field were derived by the use of the Bloch theory [2], which allows one to find the dispersion relation of the periodic lattice by supposing a wave field based on functions that have the spatial periodicity of the lattice. The analysis of the obtained dispersion relation shows the existence of forbidden band gaps (corresponding to evanescent waves) and allowed band gaps (corresponding to propagative waves) in the transmission spectrum of the lattice. The first studies were performed in quantum physics where numerous works made use of Bloch waves [3,4]. For instance, the propagation of electrons through periodic potentials shows the existence of band gaps in the transmission spectrum [5,6]. More recently, some attention was brought to classical waves: the photonic crystals are now widely studied in particular for light filtering applications.

For elastic waves, the same fundamental properties of dispersion can be found. The string loaded by masses and springs [7], the granular beads in contact [8,9] or Helmholtz resonators connected to an acoustic waveguide [10–13] are some examples of applications where the Bloch waves have been observed.

In parallel, the studies of wave propagation through various disordered lattices allowed to observe the so-called Anderson localization [14] in quantum physics [15,16], in optics [17,18], and in mechanics [19–24]. Interestingly, the theory of wave transmission through nonlinear ordered lattice has also been developed [25–28]. The first theoretical studies on wave propagation through a nonlinear lattice were carried out using a dynamical approach [29–31]. These works show that the presence of localized nonlinearities introduces an interplay between nonlinear effects and spatial periodicity. Some experimental studies have been performed and revealed that the nonlinear effects may be strong enough to allow the propagation of waves even in the presence of a strong disorder [32–34]. However numerous questions have still to be answered. For example, what is the influence of the nonlinearities type (quadratic, cubic, …) on the propagation in a nonlinear ordered lattice?

In this work, we study the acoustic wave propagation through a periodic lattice made of Helmholtz resonator. This system of Helmholtz resonator periodically connected to an acoustic waveguide can be considered as a one-dimensional (1D) medium (when the acoustic wavelength is significantly larger than the guide cross section) with regularly spaced scatterers. Interestingly, the properties of this periodic lattice are not only due to the spacial periodicity but also to the frequency dependence of the Helmholtz resonator response, which influences dramatically the dispersion relation [11]. Another important feature of the Helmholtz resonator behavior, is its amplitude dependence, which in addition to the frequency dependence introduces important nonlinear effects on the acoustic wave propagation. Besides, these particular frequency and amplitude dependent behaviors have been used to generate acoustic solitons [35,36].

This study shows that localized nonlinearities due to the Helmholtz resonators of the periodic lattice can produce frequency band gaps with an amplitude dependent width.

In the first part, a simple analytical model is developed to take into account the effect of the nonlinear response of the resonators on the dispersion relation. The nonlinear effect is introduced through a "nonlinear" impedance of the Helmholtz resonators and the amplitude influence on the dispersion relation is shown.

In a second part, experimental results are presented and discussed. They are compared to the analytical study and are in good agreement with the predictions. The amplitude dependence of the transmission is clearly demonstrated and the influence of the frequency on the propagation is studied through the determination of the attenuation length of the lattice.

A short discussion finally presents the other physical processes that may possibly play a role in the presented results, and some extensions of the present study.

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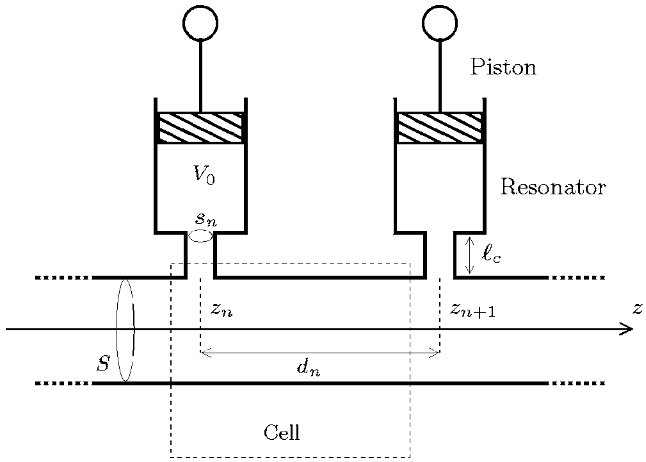

FIG. 1. Schematic representation of the Helmholtz resonator lattice.

I. MODEL OF THE NONLINEAR PROPAGATION OF ACOUSTIC WAVES IN A ONE-DIMENSIONAL LATTICE

A. Description of the lattice under study

We consider the problem of propagation of the lowest acoustic mode in a cylindrical waveguide of section S on which Helmholtz resonators are connected regularly each distance d_n. The Helmholtz resonators numbered by n are connected to the pipe through a pinpoint connection, the radius of the throat's cross sectional area s_n of the nth resonator being assumed to be small compared to the wavelength of the acoustic wave ($\sqrt{s_n}/\lambda \ll 1$). Each connection is located along the axis of the waveguide by its coordinate z_n, with axial spacing d_n between two consecutive points as shown in Fig. 1. A Helmholtz resonator is composed of (i) a neck of section s_n and length ℓ (ii) a cavity having a volume V_0, (iii) and in our case a piston which allows to adjust the volume V_0, and consequently the resonance frequency f_0 of the resonator.

B. Propagation equation and dispersion properties

In the part of the pipe between two consecutive connection points, the acoustic wave characterized by the pressure $p(z,t)$ is the solution of the plane wave equation

$$\frac{\partial^2 p(z,t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 p(z,t)}{\partial t^2} = 0, \qquad (1)$$

where c is the sound speed in free space.

At each connection point (denoted by z_n in Fig. 1), the boundary conditions require the conservation of acoustic flow and the continuity of acoustic pressure. By using the Euler equation, the propagation of acoustic waves through the lattice can be modeled by

$$\frac{\partial^2 p(z,t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 p(z,t)}{\partial t^2} = \sum_n \delta(z-z_n)\frac{-\rho s_n}{S}\frac{\partial v(z,t)}{\partial t}, \qquad (2)$$

where ρ is the air density at rest, v the acoustic wave particular velocity, and δ the Dirac function.

For a monochromatic acoustic wave with a frequency below the cutoff frequency of the waveguide (i.e., only the lowest mode can propagate), the acoustic pressure $p(z,t)$ and the acoustic velocity $v(z,t)$ along the waveguide are

$$p(z,t) = p(z)e^{j\omega t} \quad \text{and } v(z,t) = v(z)e^{j\omega t},$$

where $\omega = kc$ is the angular frequency and $j=\sqrt{-1}$. Using Eq. (2), the pressure $p(z)$ is given by the solution of the Helmholtz equation associated to the propagation equation (2)

$$\frac{\partial^2 p(z)}{\partial z^2} + k^2 p(z) = \sum_n \delta(z-z_n)\sigma_n p(z), \qquad (3)$$

where $\sigma_n = -j\omega\rho s_n/(SZ_n)$. In this relation, Z_n is the impedance of the nth resonator connected at the location $z=z_n$ and seen from the waveguide, and σ_n is the jump of the pressure derivative $\frac{\partial p}{\partial z}\big|_{z=z_n}$.

The matrix method is generally adequate to solve this type of problem and the case of an ordered 1D lattice is well known. Introducing a spatial periodicity condition on $p(z)$, a general dispersion equation is usually obtained [2] and can be written in our case as

$$\cos(qd) = \cos(kd) + \frac{\sigma}{2k}\sin(kd), \qquad (4)$$

where q is called the Bloch wave number, $k=\omega/c$ and $\sigma = \sigma_n$, $\forall n$, because all the resonators are considered as identical. This dispersion relation (4) exhibits the peculiar characteristic of filters with forbidden frequencies (or gaps, stopbands) and allowed frequencies (or passbands) in the frequency domain. In our case, the band gaps are the result of both the Helmholtz resonances and the periodic arrangement of the medium. Waves that obey the relation $|\cos(qd)| \leq 1$ are within a passband and travel almost freely in the duct and waves such that $|\cos(qd)| > 1$ are in a forbidden band and are quickly damped spatially (i.e., they are evanescent).

In Fig. 2(a), the quantity $\cos(qd)$ is plotted as a function of the frequency using the geometrical parameters of our system. As described above, the modulus of $\cos(qd)$ allows to know if the wave frequency is in an allowed or a forbidden band gap. Three forbidden band gaps, for which $|\cos(qd)| > 1$, are identified in the frequency range 0–2000 Hz. The largest band gap is the lowest in frequency and is associated with the Helmholtz resonator resonance. The band gap located around 1300 Hz is associated with the resonance of the cavity V_0. The third band gap, around 1800 Hz, is the so-called Bragg band gap and is associated only with the spatial periodicity of the medium. In Fig. 2(b), an experimental transmission coefficient recorded in the same system configuration than the results of Fig. 2(a) is plotted as a function of frequency. Forbidden band gaps are clearly seen as frequency bands where the transmission falls abruptly. They are observed at the same location than those indicated in black in Fig. 2 and obtained in the frame of Eq. (4).

Due to the fact that each band gap is associated with a given physical process, it is already possible to predict that the nonlinear behavior of the Helmholtz resonator will not

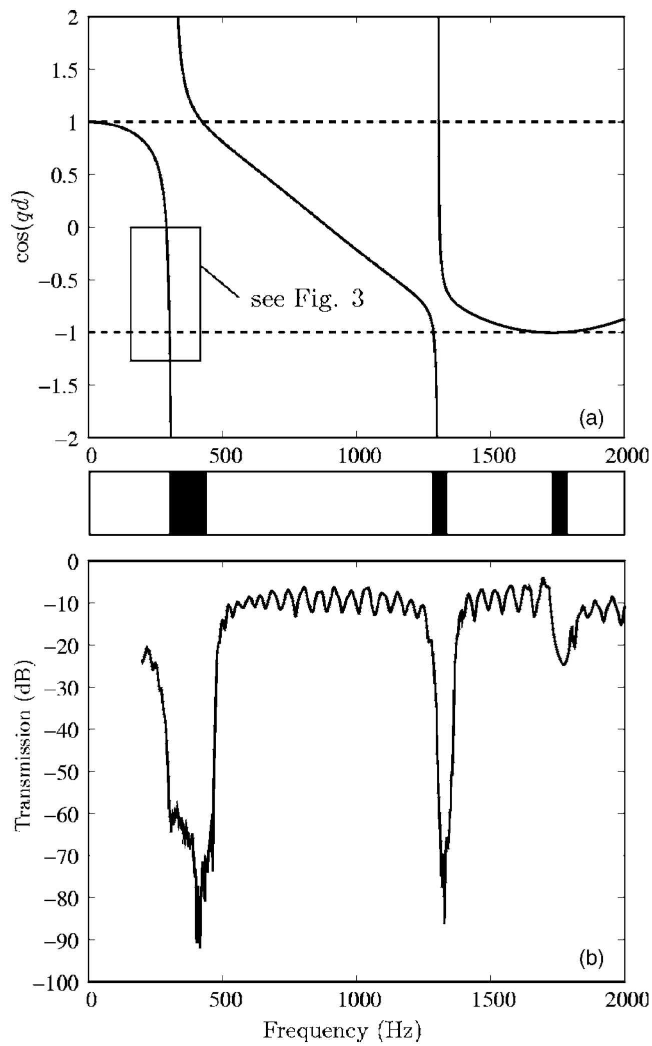

FIG. 2. Dispersion properties of a 1D lattice made up Helmholtz resonators. (a) Dispersion relation of the linear lattice obtained from Eq. (4). A zoom of the surrounded region is shown in Fig. 3. (b) Example of an experimental acoustic transmission through the corresponding one-dimensional lattice (in log-scale with arbitrary reference). The band gaps defined as $|\cos(qd)|>1$ are highlighted by the black regions.

have any effect on the band gap associated only to the spatial periodicity of the medium (the Bragg band gap). As seen in the following, the nonlinear acoustic behavior of the dispersion relation will only be visible on the Helmholtz band gap, due to the nonlinearity of the Helmholtz resonator.

C. Nonlinear model of Helmholtz resonator

A simple model of the Helmholtz resonator requires the following assumptions: (i) the pressure inside the cavity of volume V_0 is spatially uniform, (ii) the fluid in the neck moves like a solid piston. In this case, the air enclosed in the resonator acts as a spring for the lumped mass of air moving within the neck. A general description of the nonlinear behavior of the Helmholtz resonator may be derived by taking into account the quadratic term in the restoring force of the spring (see [37] and references therein).

The relative change of the pressure p_n in the cavity of the nth resonator due to a displacement x_n of the air in the neck induces a restoring force F_n that has the following form [11,37]:

$$F_n = p_n s_n = -\frac{\rho c^2 s_n^2}{V_0}[x_n - \alpha_n x_n^2 + o(x_n^3)],$$

where the nonlinear quadratic parameter $\alpha_n=(\gamma+1)s_n/(2V_0)$ and $\gamma\simeq 1.4$ is the specific heat ratio of air at normal conditions. The spring force is no longer linear and its stiffness is now described by two geometrical parameters s_n and V_0. For a monochromatic wave, the displacement x_n of the air in the neck is related to the acoustic velocity $v_n=v(z_n)$ by the relation $v_n=j\omega x_n$. The Euler relation applied to the air mass $m=\rho l_c' s_n$ (where l_c' is the effective neck length accounting for the acoustic radiation impedance at the end the neck, see [38,39] for details) submitted to the harmonic force $p_n/(\rho l_c')e^{j\omega t}$ reads

$$j\omega v_n + \omega_0^2\left[\frac{v_n}{j\omega} - \alpha_n\left(\frac{v_n}{j\omega}\right)^2 + o((v_n)^3)\right] = \frac{p_n}{\rho l_c'}, \qquad (5)$$

where $\omega_0^2=s_n c^2/(V_0 l_c')$ is the linear resonance frequency of the Helmholtz resonator, i.e., its resonance frequency for an infinitely small acoustic amplitude. The relation between the acoustic pressure and the velocity just outside the opening of the nth resonator is usually written for plane waves as

$$p_n = Z_n v_n. \qquad (6)$$

Note that this relation (6) can be nonlinear if Z_n depends on p_n or equivalently on v_n. By using the relation (6) in Eq. (5), a second degree equation on the nonlinear impedance Z_n is found:

$$Z_n^2 - j\omega\rho l_c'\left[1 - \frac{\omega_0^2}{\omega^2}\right]Z_n - \alpha_n \rho l_c' \frac{\omega_0^2}{\omega^2}p_n = 0. \qquad (7)$$

The solution physically allowed for Z_n defines an impedance of Helmholtz resonator with an amplitude dependent small correction.

Using the classical method of successive approximations in nonlinear acoustics [40], it is possible to write σ_n which depends on Z_n as $\sigma_n=\sigma_n^L+\sigma_n^{NL}$, where σ_n^L is the linear contribution (i.e., the value of σ_n when the acoustic amplitude is infinitely small) and σ_n^{NL} is the nonlinear correction arising from the nth Helmholtz resonator nonlinearity. It is important to note here that the other acoustic nonlinearities due to the intrinsic air behavior and due to the Eulerian description of the movement [40] are considered to be negligible compared to the nonlinearity of the Helmholtz resonators. In other words, the contribution of the localized Helmholtz resonators to the nonlinearity of the system is dominating the weak nonlinearity of air. Consequently, only the Helmholtz resonator nonlinearity is taken into account in this model. Once the expression for the nonlinear σ_n is found, it can be introduced as a small correction in the right-hand side of Eq. (3), and this equation should be solved. This can be interpreted as a correction to the Bloch wave number q due to nonlinear effects in the lattice. If one defines a Bloch wave phase ve-

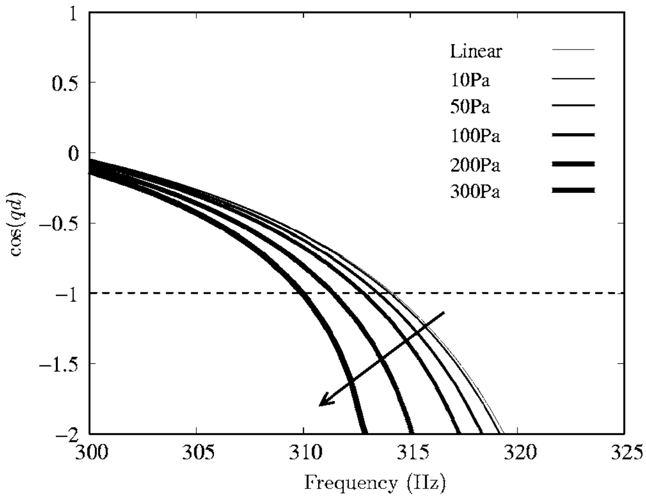

FIG. 3. Acoustic amplitude dependence of the dispersion relation around the low-frequency side of the first band gap. The arrow indicates the direction of the increasing acoustic amplitude.

locity in the lattice as $q=\omega/c_b$, an amplitude dependent c_b is obtained, similarly to classical nonlinear acoustics of homogeneous media with a quadratic nonlinearity for instance [40].

For the numerical calculations presented in the following, the complete expression for the nonlinear impedance Z_n has been introduced, and the expression for σ_n is consequently not written simply as $\sigma_n=\sigma_n^L+\sigma_n^{NL}$. The pressure amplitude p_n introduced in σ_n through the relation (7), should be dependent on the coordinate z (or equivalently dependent on n) but is taken constant in space ($p_n=p$ for all n). Consequently, in this assumption, the right-hand side of Eq. (3) remains in principle linear but takes into account the excitation amplitude dependent effects in the lattice, if one calculates the wave number q for two different p_n for instance. Equation (3) is then solved as previously for the linear case, and Eq. (4) with a new expression for $\sigma=\sigma_n$ is found. This assumption of constant wave amplitude for the amplitude dependent part of σ is reasonable in the case where self-action is studied, i.e., frequency ω is launched in the medium and frequency ω is detected. This would not be the case for frequency mixing processes, for instance, where the amplitude dependent right-hand side of Eq. (3) needs to contain new frequencies starting from a monochromatic initial pressure field. Figure 3 shows a zoom of the dispersion relation around the low frequency side of the first stopband for increasing acoustic amplitudes. The quantity $\cos(qd)$ is now amplitude dependent due to the presence of p_n in the expression of σ [see Eq. (4)]. The influence of the wave amplitude on the characteristic frequency f_c corresponding to $|\cos(qd)|=1$ is clearly visible. This characteristic frequency f_c defines the lower limit of the band gap as it is associated with the transition from a propagative behavior of the waves to an evanescent behavior. This transition takes place at a decreasing frequency when the acoustic level is increased. In Fig. 3, for the linear case of an infinitely small acoustic level, $f_c \simeq 315$ Hz, while for an acoustic amplitude of 300 Pa, $f_c \simeq 310$ Hz. The lower limit of the band gap is consequently sensitive to the resonance frequency shift of the Helmholtz resonators. Note that if the band gap limit is sufficiently steep as a function of frequency, a small shift of this limit can be responsible for a strong nonlinear induced attenuation of waves with frequencies $f \simeq f_c \simeq 315$ Hz. Concerning the upper limit of this Helmholtz band gap, there is no noticeable amplitude dependent effect. The resonance frequency shift of the Helmholtz resonators does not play an important role in this case. As a consequence the band gap width tends to increase with the acoustic amplitude, the lower limit is decreasing in frequency and the upper limit stays practically unchanged. Opacity of the medium is increased by the wave amplitude.

FIG. 4. (Color online) Picture of a part of the lattice under study.

II. EXPERIMENTAL RESULTS

Experimental setup. Figure 4 shows a picture of the lattice under study. It consists in a 8 m long cylindrical waveguide with an inner radius $r=2.5\times10^{-2}$ m and a 0.5 cm thick wall. This pipe is connected to an array of 60 Hemholtz resonators periodically distributed. The distance between two consecutive scatterers is $d=0.1$ m. Each resonator is composed by a neck (cylindrical tube with an inner area $s=7.85\times10^{-5}$ m^2 and a length $l_c=2$ cm) and a variable length cavity (cylindrical tube with an inner area $S=7.85\times10^{-3}$ m^2 and a maximum length $l=16.5$ cm) as described in Fig. 1. The sound source is a compression chamber, designed to deliver high acoustic power level, connected to one end of the main tube. At the other end of the main tube, an anechoic termination made of a 10 m long waveguide partially filled with porous plastic foam suppresses the back propagative waves. A microphone designed for high acoustic powers is used to measure the pressure at the beginning of the lattice close to the acoustic source. Another microphone mounted on a translation system measures the pressure field inside the lattice at each resonator location. The data acquisition is performed by means of a spectrum analyzer.

Results and discussions. As it is shown in the first section, the nonlinear effects associated with the Helmholtz resonator are present at the Helmholtz resonance frequency. Consequently, we choose to study the propagation of acoustic

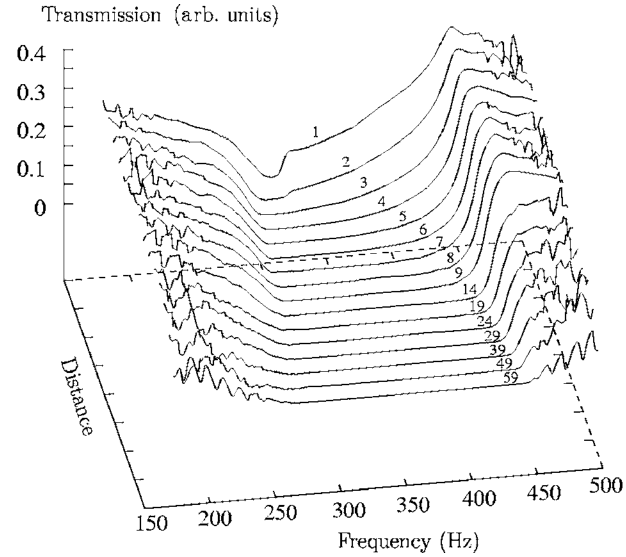

FIG. 5. Acoustic transmission through the lattice as a function of the frequency for different lattice lengths. The corresponding number of resonators is indicated on each curve.

waves through the lattice around the first band gap (due to the Helmholtz resonance of the scatterers). This forbidden band is found between approximately 200 and 400 Hz (see Fig. 2).

The transfer function of the medium is measured for different lattice lengths by changing the location of the microphone. Figure 5 shows the evolution of the transmission for 15 different amounts of resonators (from 1 to 50) in a frequency range around the first band gap. For a short lattice made of few resonators (1–3), the band gap is not well defined yet. The strong frequency dependence due to the Helmholtz resonance is however clearly visible, even for only one resonator, and precludes the further apparition of the band gap. After a distance corresponding to ~10 resonators, the band gap is well defined and its shape does not change qualitatively at larger distance, except for a general diminution of the transmission due in particular to acoustic energy thermoviscous absorption at all frequencies.

Figure 6 shows the influence of the wave amplitude on the transmission through the lattice in the first forbidden band. Only the lowest side of the band gap is influenced by the wave amplitude excitation level. The theoretical results presented above have the same tendency (see Fig. 3) and the same magnitude is found for the shift of the characteristic frequency of the band gap lowest side (of the order of 10 Hz). This observation can be interpreted as a nonlinear conversion of the propagative waves toward evanescent waves. This phenomenon is found to be in agreement with the introduced quadratic type of nonlinearity for the Helmholtz resonator behavior. This observation is different from several other works considering cubic nonlinearity, where the lattice exhibits a self-transparency, i.e., an enhanced transmission with an increased amplitude [13].

A different presentation of the same phenomenon is performed in Fig. 7. The spatial distribution of the wave level in the lattice is plotted as a function of the distance from the first resonator for different frequencies located around the first band gap. As expected, when the frequency is in a passband, there is almost no attenuation and the wave is well transmitted through the lattice. When the frequency is located in the middle of the band gap or inside the band gap but close to its upper-frequency side, the wave is quickly damped in the lattice whatever the amplitudes of the excitation are. In contrary, when the frequency is close to the low-frequency side of the band gap (for instance $f=250$ Hz), the spatial attenuation is amplitude dependent: the greater the excitation amplitude is, the higher the attenuation is. This effect on the attenuation length confirms the frequency localized character of the observed nonlinear manifestation.

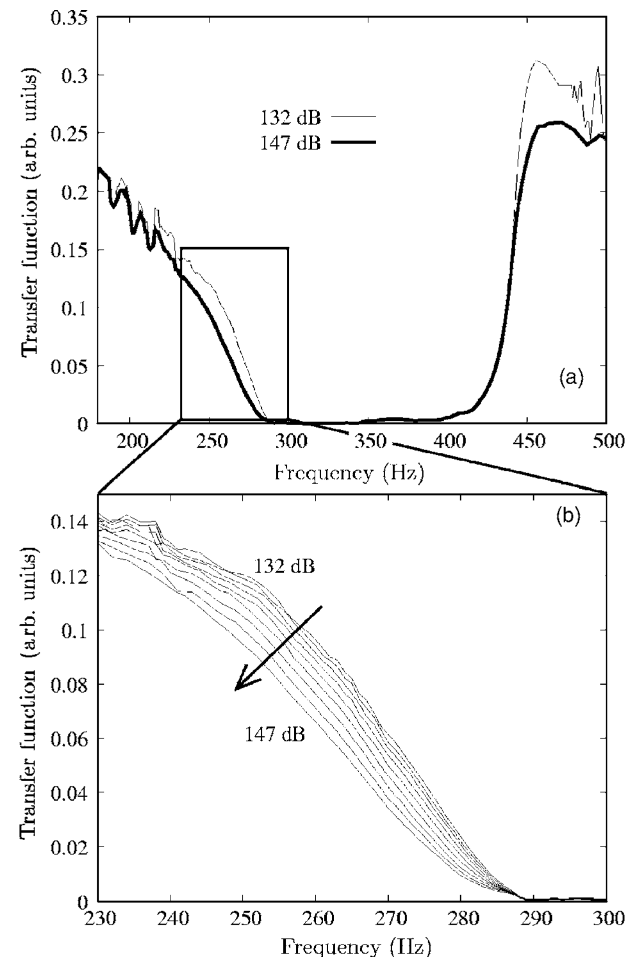

FIG. 6. Amplitude dependence of the band gap width. (a) Acoustic transfer function measured by a microphone located at the 9th resonator for two acoustic excitation levels [132 dB (thin line) and 147 dB (thick line)]. (b) Detail of the low frequency side of the first band gap. Acoustic transfer function for 9 acoustic excitation levels (from 132 to 147 dB).

Focusing now on the low-frequency side of the band gap under study (Fig. 8), where the nonlinear effects are the most visible, it is possible to plot the attenuation length as a function of the excitation amplitude, for different frequencies covering the band gap side. Here, the attenuation length has been defined as the distance from the source where the level is decreased by 10 dB. The points under −40 dB (around the noise level) have been removed to perform the least square

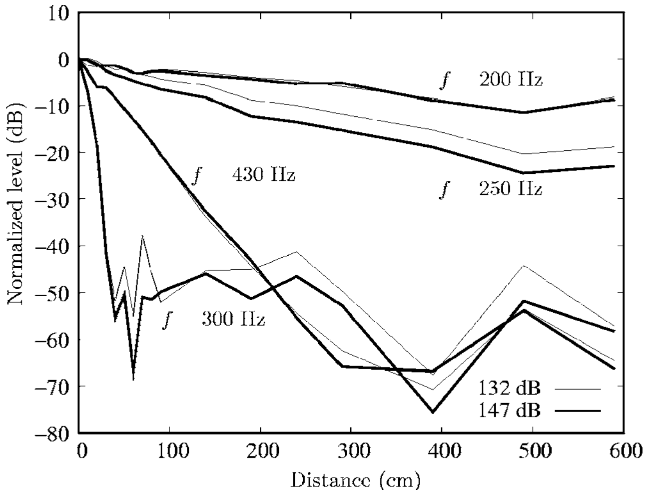

FIG. 7. Normalized level of the acoustic wave for different characteristic frequencies (f=200, 250, 300, 430 Hz) as a function of the distance from the first resonator. Two acoustic excitation levels are compared.

linear fit of the acoustic level as a function of distance used in the derivation of the attenuation length.

When the attenuation length is of the order of the lattice size (600 cm), the medium is considered as transparent and the wave propagates without significant attenuation. In contrast, when the attenuation length is close to several centimeters, the waves are strongly attenuated over distances lower than the wavelength and can consequently be considered as evanescent. Figure 8 shows that the attenuation length is dependent on the frequency but also on the excitation amplitude. The most important is that the amplitude dependence is clearly localized inside a limited frequency range corresponding to the side of the band gap, i.e., from 235 Hz to 285 Hz.

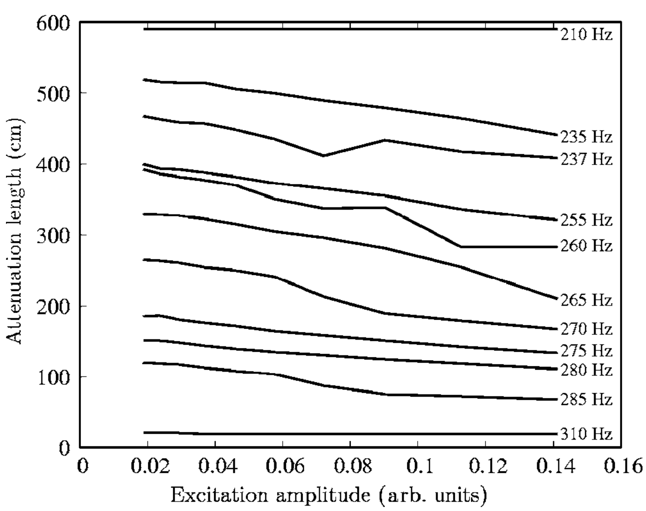

FIG. 8. Attenuation length in the lattice for different frequencies as a function of the acoustic excitation amplitude in a linear scale. The acoustic excitation amplitudes correspond to acoustic excitation levels ranging from 130 to 147 dB.

Discussion. The developed model based on a quadratic elastic nonlinearity for the Helmholtz resonator is in good agreement with the presented experimental results and interpretations. Both the magnitude of the frequency shift and its location only on the low-frequency side of the band gap confirm the validity of the model. However, other physical processes that are usually involved in the nonlinear acoustic wave propagation in fluids may play a (minor) role in the observed amplitude dependencies. Further studies should consider the role of the nonlinear wave absorption due to Helmholtz resonator, which was not considered in the present study. This idea is suggested by the amplitude dependence of the transfer function observed for frequencies above the studied band gap (above roughly 430 Hz, see the top of Fig. 6), and not described in the frame of the nonlinear model developed here. We attribute this particular effect to the presence of the Helmholtz resonators because the experiment was carried out in the same condition on a tube without any resonator and a maximum saturation of the transfer function of −0.6 dB (corresponding to a ratio 0.93) was observed. This is less than the ratio $0.25/0.3 \simeq 0.83$ observed in Fig. 6. This particular experimental observation is not described correctly by the developed model either because the model assumptions are too strong (the assumption that p_n is constant in the lattice for instance), either because there is an additional source of nonlinear attenuation in the Helmholtz resonators, not taken into account in the model. For a large acoustic particular velocity, formation of vorticity at the neck of the resonators can be a source of nonlinear energy dissipation.

Improvements of the model dedicated to this study can be performed if future experimental results make it necessary, such as considering different incident acoustic amplitudes on the resonators as a function of distance from the first resonator. The introduction of nonlinear absorption of acoustic energy in the comportment law of the Helmholtz resonator is a further possible improvement of the present model.

A possible application of such observed effect, is for instance a nonlinear acoustic filter, that saturates the acoustic wave level transmitted through the lattice when the input acoustic energy is increased. This effect may be optimized by modifying strongly the geometrical parameters of the Helmholtz resonators, because the effective quadratic parameter of nonlinearity α_n depends on these parameters (see Sec. I C).

III. CONCLUSION

In this paper, we have investigated the nonlinear properties of the acoustic transmission through a one-dimensional lattice with localized nonlinear scatterers, the Helmholtz resonators. Due to the frequency dependence of the Helmholtz resonator response to an acoustic solicitation, it is possible to observe a low-frequency band gap associated with the resonance frequency of the Helmholtz resonator. This low-frequency band gap exhibits a nonlinear behavior on its low-frequency side, induced by the nonlinear behavior of the resonator resonance. A possible explanation of such observation is the downward frequency shift of the resonance frequency of the resonators. A good agreement is obtained between the presented experimental results and the developed model to describe the nonlinear behavior of the Helmholtz resonator.

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[1] L. Brillouin and P. Parodi, *Propagation des ondes dans les milieux périodiques* (Masson-Dunod, Paris, 1956).
[2] F. Bloch, Z. Phys. **52**, 555 (1928).
[3] D. J. Griffiths and N. F. Taussig, Am. J. Phys. **60**, 883 (1992).
[4] D. W. L. Sprung and Hau Wu, Am. J. Phys. **61**, 1118 (1993).
[5] F. Bentalosa, J. Phys. C **15**, 7119 (1982).
[6] P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B **22**, 3519 (1980).
[7] S. Parmley, Am. J. Phys. **63**, 547 (1995).
[8] C. Coste, E. Falcon, and S. Fauve, Phys. Rev. E **56**, 6104 (1997).
[9] V. Tournat, V. E. Gusev, and B. Castagnède, Phys. Rev. E **70**, 056603 (2004).
[10] N. Sugimoto, J. Acoust. Soc. Am. **97**, 1446 (1995).
[11] O. Richoux, C. Depollier, and J. Hardy, Acta. Acust. Acust. **88**, 934 (2002).
[12] O. Richoux and V. Pagneux, Europhys. Lett. **59**, 34 (2002).
[13] O. Richoux, C. Depollier, and J. Hardy, Phys. Rev. E **73**, 026611 (2006).
[14] P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[15] H. Schmidt, Phys. Rev. **105**, 425 (1957).
[16] F. J. Dyson, Phys. Rev. **92**, 1331 (1953).
[17] B. A. Tiggelen and A. Lagendijk, Phys. Rev. B **45**, 12233 (1992).
[18] E. N. Economou and C. M. Soukoulis, Phys. Rev. B **40**, 7977 (1989).
[19] A. Figotin and A. Klein, Commun. Math. Phys. **180**, 439 (1996).
[20] C. H. Hodges, J. Sound Vib. **114**, 411 (1982).
[21] P. Sheng, *Scattering and Localization of Classical Waves in Random Media* (World Scientific, Singapore, 1990).
[22] C. M. Soukoulis, S. Datta, and E. N. Economou, Phys. Rev. B **49**, 3800 (1994).
[23] T. R. Kirkpatrick, Phys. Rev. B **31**, 5746 (1985).
[24] E. N. Economou, Physica A **167**, 215 (1990).
[25] C. E. Bradley, Ph.D. thesis, The University of Texas at Austin, 1994.
[26] D. Hennig, H. Gabriel, and G. P. Tsironis, Phys. Rep. **307**, 333 (1999).
[27] D. Hennig, H. Gabriel, G. P. Tsironis, and M. Molina, Appl. Phys. Lett. **64**, 2934 (1994).
[28] Q. Li, C. T. Chan, K. M. Ho, and C. M. Soukoulis, Phys. Rev. B **53**, 15577 (1996).
[29] Y. Wan and C. M. Soukoulis, Phys. Rev. A **41**, 800 (1990).
[30] P. Hawrylak, M. Grabowski, and P. Wilson, Phys. Rev. B **40**, 6398 (1989).
[31] M. Grabowski and P. Hawrylak, Phys. Rev. B **41**, 5783 (1990).
[32] M. J. McKenna, R. L. Stanley, and J. D. Maynard, Phys. Rev. Lett. **69**, 1807 (1992).
[33] Yu S. Kivshar, S. A. Gredeskul, A. Sanchez, and L. Vazquez, Phys. Rev. Lett. **64**, 1693 (1990).
[34] E. Cota, J. V. Jose, J. Maytorena, and G. Monsivais, Phys. Rev. Lett. **74**, 3302 (1995).
[35] N. Sugimoto, J. Acoust. Soc. Am. **99**, 1971 (1996).
[36] N. Sugimoto, M. Masuda, J. Ohno, and D. Motoi, Phys. Rev. Lett. **83**, 4053 (1999).
[37] R. R. Boullosa and F. O. Bustamante, Am. J. Phys. **60**, 722 (1992).
[38] M. Bruneau and T. Scelo, *Fundamentals of Acoustics* (ISTE, UK and USA, 2006).
[39] P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (Princeton University Press, Princeton, NJ, 1987).
[40] K. Naugolnykh and L. Ostrovky, *Nonlinear Wave Processes in Acoustics* (Cambridge texts in applied mathematics, New York, 1998).