Skyrmion-like states in two- and three-dimensional dynamical lattices

P. G. Kevrekidis,[1] R. Carretero-González,[2] D. J. Frantzeskakis,[3] B. A. Malomed,[4] and F. K. Diakonos[3]
[1]*Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts 01003-4515, USA*
[2]*Nonlinear Dynamical Systems Group,* * *Department of Mathematics and Statistics, and Computational Science Research Center,*†
San Diego State University, San Diego California, 92182-7720, USA
[3]*Department of Physics, University of Athens, Panepistimiopolis, Zografos, Athens 15784, Greece*
[4]*Department of Interdisciplinary Studies, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel*
(Received 5 April 2006; revised manuscript received 6 November 2006; published 8 February 2007)

We construct, in discrete two-component systems with cubic nonlinearity, stable states emulating Skyrmions of the classical field theory. In the two-dimensional case, an analog of the *baby Skyrmion* is built on the square lattice as a discrete vortex soliton of a complex field [whose vorticity plays the role of the Skyrmion's winding number (WN)], coupled to a radial "bubble" in a real lattice field. The most compact quasi-Skyrmion on the cubic lattice is composed of a nearly planar complex-field discrete vortex and a three-dimensional real-field bubble; unlike its continuum counterpart which must have WN=2, this stable discrete state exists with WN =1. Analogs of Skyrmions in the one-dimensional lattice are also constructed. Stability regions for all these states are found in an analytical approximation and verified numerically. The dynamics of unstable discrete Skyrmions (which leads to the onset of lattice turbulence) and their partial stabilization by external potentials are explored too.

DOI: 10.1103/PhysRevE.75.026603 PACS number(s): 05.45.Yv, 03.75.Lm, 03.75.Mn

I. INTRODUCTION

Intrinsic localized modes in nonlinear lattices have drawn much attention [1] due to their relevance to various physical systems, including optical waveguide arrays [2], photonic crystals [3], Bose-Einstein condensates (BECs) trapped in deep optical lattices [4], and Josephson-junction ladders [5]. A wide variety of species of these modes have been predicted and observed, such as bright and dark optical discrete solitons [6,7] in arrays of semiconductor waveguides [7], multidimensional solitons in photonic lattices [8], discrete vortices, supervortices, and multipoles in two-dimensional (2D) [9–11] and three-dimensional (3D) [12] settings, lattice dipoles [13], multicomponent solitons [14], soliton trains [15], necklace solitons [16], discrete gap solitons [17], twisted localized modes (TLMs) [18], and others.

These developments raise the question whether counterparts of more complex structures known in continuum media within field-theoretical contexts can be constructed in dynamical lattices. Challenging objects of this type are three-dimensional (3D) Skyrmions, proposed in the field theory as models of nucleons [19]. Their 2D counterparts, *baby Skyrmions* in the sigma model [20], may account for the disappearance of antiferromagnetism and the onset of high-T_c superconductivity [21], as well as formation of the ground state in quantum-Hall ferromagnets [22]. Skyrmion-like objects have also been predicted in BECs [23]. On the other hand, lattice Skyrmions were considered in the Heisenberg model of magnetism and as electron spin textures in quantum Hall systems [24].

In this work, we construct Skyrmion-emulating discrete structures as stable 2D and 3D lattice solitons with topological properties resembling those of Skyrmions in the continuum theory. The newly found states may be relevant not only to lattice media *per se*, but also to the quantization of the original Skyrme model, as its nonrenormalizability [25] makes it necessary to put it on a lattice.

We aim to construct discrete quasi-Skyrmions in the standard nonlinear-lattice model, viz., the discrete nonlinear Schrödinger (DNLS) equation. DNLS provides a universal envelope equation for nonlinear Klein-Gordon lattice models, and serves as a direct model for BECs trapped in strong lattices [4] and crystals built of microresonators [26]; in addition, the two-dimensional DNLS equation is a model of optical waveguide arrays [2,6]. Based on the *hedgehog ansatz* (HA) [19], static 3D Skyrmions necessarily involve three scalar fields, hence their dynamical description requires, at least, two complex fields. Therefore in this work we study a two-component DNLS equation. This model is by itself directly relevant to waveguide arrays [14] when the light is carried by different polarizations or frequencies, and to BEC mixtures of two different spin states [27].

After introducing the model, we construct 2D analogs of *baby Skyrmions* on the square lattice and their 1D counterparts, and then extend them into *toroidal* quasi-Skyrmions on the cubic lattice, which are the most compact (hence most experimentally relevant) 3D objects with the Skyrmion-emulating topology. Stability regions for all the states are found in an analytical approximation and verified via numerical computation of stability eigenvalues. Simulations of the evolution of unstable lattice quasi-Skyrmions reveals an onset of lattice turbulence. We also show that a parabolic external trapping potential, which is a necessary ingredient of the BEC setting, may partially stabilize the lattice objects of the Skyrmion type.

II. THE MODEL AND ANALYTICAL CONSIDERATIONS

We introduce a vectorial DNLS equation for $\phi \equiv \{\phi^{(1)}, \phi^{(2)}\}$ on the cubic/square lattice,

*URL: http://nlds.sdsu.edu/
†URL: http://www.csrc.sdsu.edu/

$$i\dot{\phi}_{\mathbf{n}}^{(1)} = -C\Delta\phi_{\mathbf{n}}^{(1)} + |\phi_{\mathbf{n}}^{(1)}|^2\phi_{\mathbf{n}}^{(1)} + \beta|\phi_{\mathbf{n}}^{(2)}|^2\phi_{\mathbf{n}}^{(1)},$$

$$i\dot{\phi}_{\mathbf{n}}^{(2)} = -C\Delta\phi_{\mathbf{n}}^{(2)} + |\phi_{\mathbf{n}}^{(2)}|^2\phi_{\mathbf{n}}^{(2)} + \beta|\phi_{\mathbf{n}}^{(1)}|^2\phi_{\mathbf{n}}^{(2)}, \tag{1}$$

where the overdot stands for time derivative (or derivative with respect to the propagation distance, in the optical-waveguide model), $\mathbf{n}=\{n_{1,2,3}\}$ or $\{n_{1,2}\}$ is the vectorial discrete coordinate in the 3D and 2D lattices, C is a coupling constant, the discrete D-dimensional Laplacian is $\Delta\phi_{\mathbf{n}} \equiv \Sigma_{|\mathbf{m}-\mathbf{n}|=1}\phi_{\mathbf{m}} - 2D\phi_{\mathbf{n}}$, and β is a relative strength of the onsite interspecies interaction. Stationary solutions are looked for as

$$\phi_{\mathbf{n}}^{(1)} = u_{\mathbf{n}} \exp(-i\Lambda t),$$

$$\phi_{\mathbf{n}}^{(2)} = v_{\mathbf{n}} \exp(-i\Lambda t),$$

where Λ is the frequency (chemical potential, in the context of BECs), and stationary lattice fields obey the equations

$$\Lambda u_{\mathbf{n}} = C\Delta u_{\mathbf{n}} - (|u_{\mathbf{n}}|^2 + \beta|v_{\mathbf{n}}|^2)u_{\mathbf{n}},$$

$$\Lambda v_{\mathbf{n}} = C\Delta v_{\mathbf{n}} - (|v_{\mathbf{n}}|^2 + \beta|u_{\mathbf{n}}|^2)v_{\mathbf{n}}.$$

Note that the nonlinear terms in Eqs. (1) imply onsite self-repulsion, while self-attraction can be transformed into the present form by the usual staggering transformation [4,18], $\phi_{\mathbf{n}} \equiv (-1)^{n_1+n_2+n_3}\tilde{\phi}_{\mathbf{n}}$ for $D=3$, or its counterpart for $D=2$.

We start by constructing Skyrmion-emulating lattice configurations in the anticontinuum (AC) limit, $C=0$; in terms of the BEC, this bears similarities to the limit case of a *Mott insulator* with fully confined atoms [28], lending support (alongside the earlier works indicating the relevance of discrete descriptions of such dynamical superfluid-insulator transitions [29] and their quantitative experimental verification [30]), to the validity of such a discrete model. We then verify a possibility to continue solution branches to $C>0$, by means of fixed-point iterations (i.e., by using the Newton method to identify solutions of the ensuing system of nonlinear algebraic equations for given C). While arbitrary structures can be introduced in the AC limit, only true solutions admit continuation to finite C.

The linear stability analysis is then performed for a perturbed solution,

$$\phi_{\mathbf{n},\mathrm{pert}}^{(1)} = (u_{\mathbf{n}} + \mathbf{a}_{\mathbf{n}}e^{\lambda t} + \mathbf{b}_{\mathbf{n}}e^{\lambda^* t})e^{-i\Lambda t},$$

$$\phi_{\mathbf{n},\mathrm{pert}}^{(2)} = (v_{\mathbf{n}} + \mathbf{c}_{\mathbf{n}}e^{\lambda t} + \mathbf{d}_{\mathbf{n}}e^{\lambda^* t})e^{-i\Lambda t}, \tag{2}$$

where $(\mathbf{a}_{\mathbf{n}}, \mathbf{b}_{\mathbf{n}}, \mathbf{c}_{\mathbf{n}}, \mathbf{d}_{\mathbf{n}})$ constitute an eigenmode of infinitesimal perturbations, and λ is the corresponding eigenvalue [computed by the application of a standard eigenvalue solver to the matrix formed by substitution of the perturbed solution in Eqs. (1)]. The stationary solution is unstable if at least one pair of the eigenvalues has $\mathrm{Re}(\lambda) \neq 0$.

We first consider the 2D case and look for a representation of the vector field in the form of the above-mentioned hedgehog ansatz for the *baby Skyrmion* in the continuum field theory [20]. In the polar coordinates, r and θ, this ansatz reads

$$\mathbf{\Psi} = \begin{pmatrix} \sin[f(r)]\cos(q\theta) \\ \sin[f(r)]\sin(q\theta) \\ \cos[f(r)] \end{pmatrix}, \tag{3}$$

with boundary conditions $\lim_{r\to 0} f(r)=0$, and $\lim_{r\to\infty} f(r) = \pi N$, with q and N being integers. The latter may be combined into a single topological charge, alias *winding number*, $\mathrm{WN}=[1-(-1)^N]\mathrm{q}/2$. We aim to construct a lattice ansatz emulating the continuum ansatz in Eq. (3). In particular, combining the first two components of the ansatz as $\Psi_1 + i\Psi_2$, we obtain a complex field with vorticity q. To consider a lattice analog of the fundamental baby Skyrmion, we set $q=1$ and $N=1$, hence WN=1. In this case, the complex field represents a localized discrete vortex, which carries the WN in the form of its vorticity, S (although the latter is not related to a conserved angular momentum, which does not exist on a lattice, S can be unambiguously defined in lattice fields for $D=2$ [9] and $D=3$ [12]), while the remaining real field takes the form of a *bubble* in the quasiradial direction on the lattice (see, e.g., Ref. [31]),which helps to support the vortex. In fact, the hedgehog ansatz (3) demonstrates that the WN of the baby Skyrmion may also be interpreted as the usual vorticity in the continuum space. Thus the 2D continuum hedgehog ansatz of Eq. (3) is directly transferred onto the 2D lattice, with correspondence $\Psi_1 + i\Psi_2 \to u$ and $\Psi_3 \to v$. We note that a definition of a Skyrmion on the 2D lattice through these asymptotic features was proposed in a different context in Ref. [32].

The construction of the desired lattice structure for $D=2$ proceeds by setting up a discrete vortex (alias *vortex cross*) at $C=0$. To this end, we assign the complex field, u, a nonzero absolute value, $\sqrt{\Lambda}$, and phases 0, $\pi/2$, π, $3\pi/2$, at four sites surrounding the origin, $(n_1,n_2)=(1,0)$, $(0,1)$, $(-1,0)$, and $(0,-1)$ [9]. In the same AC limit, the radial bubble in the real field is seeded by setting $v_{0,0}=\sqrt{\Lambda}$, $v_{n_1,n_2}=0$ at the above-mentioned set of four sites, and $v_{n_1,n_2}=-\sqrt{\Lambda}$ elsewhere. The thus constructed lattice ansatz indeed closely emulates the baby Skyrmion in the continuum. Note that we have chosen the minimal spatial size for it; the size may be made larger in the AC limit, but this would adversely affect the stability [33], and, generally, the possibility to create the pattern in the experiment. It is easy to construct higher-order lattice baby Skyrmions, whose core will be a planar discrete vortex with $S=2,3,\ldots$. However, the minimum size of such an object is essentially larger than that its fundamental counterpart, with $S=1$ [10], while the stability region is narrower, hence it would be harder to observe it in the experiment. We also note in passing that, while arbitrary configurations can be chosen at the anticontinuum limit of $C=0$, only a small subset of these, satisfying stringent Lyapunov-Schmidt conditions (developed for one-component systems in [33,34]; an example for multicomponent systems is given in [35]) will survive for nonzero couplings.

A cross section of the 2D seed ansatz also suggests a possibility to build an analog of the lattice baby Skyrmion

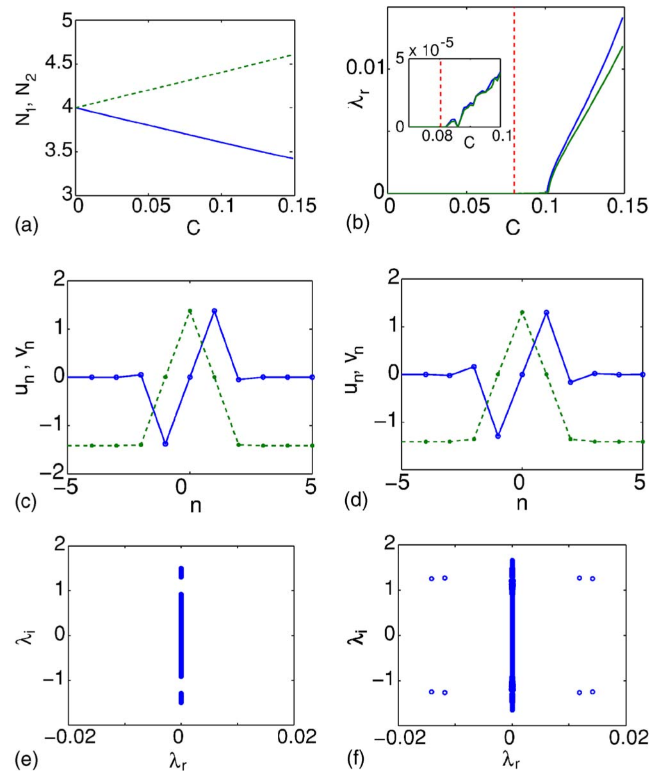

FIG. 1. (Color online) The one-dimensional discrete analog of the baby Skyrmion. Left and right panels in the top row show norms $N_1=\Sigma_{\mathbf{n}}|u_{\mathbf{n}}|^2$ (solid line) and $N_2=\Sigma_{\mathbf{n}}(\Lambda-|v_{\mathbf{n}}|^2)$ (dashed line), as well as the two most unstable eigenvalues vs coupling C (the vertical dashed line is the analytically predicted instability onset). Left and right panels in the middle and bottom rows display, respectively, the solutions u_n (solid line) and v_n (dashed line) for $C=0.05$ and 0.15, and the corresponding spectral planes (λ_r,λ_i) of the eigenvalues, $\lambda=\lambda_r+i\lambda_i$. In this figure and below, results are presented for $\beta=1/4$ and $\Lambda=2$. This case is generic, as revealed by systematic simulations.

for $D=1$, in terms of 1D complex and real lattice fields, u_n and v_n, by adopting $u_n=\sqrt{\Lambda}(\delta_{n,1}-\delta_{n,-1})$, $v_0=\sqrt{\Lambda}$, $v_1=v_{-1}=0$, and $v_n=-\sqrt{\Lambda}$ for $|n|>1$, in the AC limit. This 1D structure essentially consists of a twisted localized mode (TLM) [18] in the complex field, coupled to a bubble [31] in the real one (see Fig. 1). This 1D state is possible *only* in the lattice setting, as TLMs do not exist in the continuum.

Proceeding to the search for discrete analogs of Skyrmions for $D=3$, we notice that a variety of such structures can be generated from the baby Skyrmion constructed, as described above, on the 2D lattice. We limit the consideration to the simplest (*most compact*, hence easiest for the experimental realization) seed pattern for $D=3$, which is defined, in the AC limit, by taking the complex field precisely as in the above planar configuration in the central plane, $n_3=0$, and zero elsewhere. At finite C, this seed continues into a true 3D lattice solution, in which the complex field, $u_{\mathbf{n}}$, is a flat 3D vortex (alias vortical torus [12]), while the real field, $v_{\mathbf{n}}$, is shaped as a 3D bubble. In this 3D configuration, the role of the WN, which keeps value 1, inherited from the quasi-2D seed, is played by the vorticity of the 3D vortex. Comparing this lattice solution with known types of Skyrmions in the 3D field theory, we conclude that it emulates not the fundamental state, which is based on the spherically symmetric hedgehog ansatz [19], but rather the *toroidal* di-Skyrmion, which is a stable solution in the continuum model for $D=3$ (it is interpreted as a model of the deuteron [36]). However, in the continuum limit the toroidal Skyrmion exists only with WN=2 (this WN may also be interpreted as the topological charge of the 3D vortex, which lies at the core of the di-Skyrmion), while our results demonstrate that its *more compact* stable toroidal analog with WN=1 exists on the 3D lattice, thus being the most favorable target for experiments (with binary BECs in deep 3D optical lattices or with microresonator crystals). In fact, it is easy to construct higher-order toroidal lattice quasi-Skyrmions with WN≥2, whose core will be a higher-order 3D lattice vortex; however, we expect that, as well as in the case of $D=2$, the size of such objects will be larger and the stability region smaller than in the case of WN=1.

Proceeding to the stability analysis for the lattice quasi-Skyrmions, we first examine it through the dispersion relation of Eqs. (1) linearized around such solutions (i.e., we aim to find the continuous spectrum of small perturbations, from which unstable eigenvalues may emerge). Accordingly, far from the center, the perturbation with infinitesimal amplitudes $(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$ [see Eq. (2)] is

$$\phi_1=\mathbf{a}e^{i(\omega t+\mathbf{k}\cdot\mathbf{n})}+\mathbf{b}e^{-i(\omega^\star t+\mathbf{k}\cdot\mathbf{n})}$$

$$\phi_2=-\sqrt{\Lambda}+\mathbf{c}e^{i(\omega t+\mathbf{k}\cdot\mathbf{n})}+\mathbf{d}e^{-i(\omega^\star t+\mathbf{k}\cdot\mathbf{n})},$$

in terms of Eq. (2), $\lambda\equiv i\omega$. Then, for $\mathbf{k}$ oriented along a lattice axis (i.e., for 1D plane waves with $\mathbf{k}\cdot\mathbf{n}=kn$), the linearization of Eqs. (1) yields

$$\omega=\pm[\Lambda(1-\beta)-4C\sin^2(k/2)]$$

and

$$\omega=\pm\sqrt{[\Lambda+4C\sin^2(k/2)]^2-\Lambda^2}.$$

From these dispersion relations and their straightforward extensions for $D=2$ and $D=3$, we find the spectral bands of the real excitation frequencies

$$|\omega|<\sqrt{(\Lambda+4CD)^2-\Lambda^2}$$

for the second component and

$$\Lambda(1-\beta)-4CD<|\omega|<\Lambda(1-\beta)\quad\text{for }\beta<1,$$

$$\Lambda(\beta-1)<|\omega|<\Lambda(\beta-1)+4CD\quad\text{for }\beta>1$$

for the first component. These two bands have opposite *Krein signatures* [37], hence their merger, which occurs with the increase of C, at

$$C_{\mathrm{cr}}^{(D)}=\frac{\Lambda}{4D}\begin{cases}(1-\beta)^2/[2(2-\beta)] & \text{for }\beta<1,\\ \sqrt{\beta^2-2\beta+2}-1 & \text{for }\beta>1\end{cases}\tag{4}$$

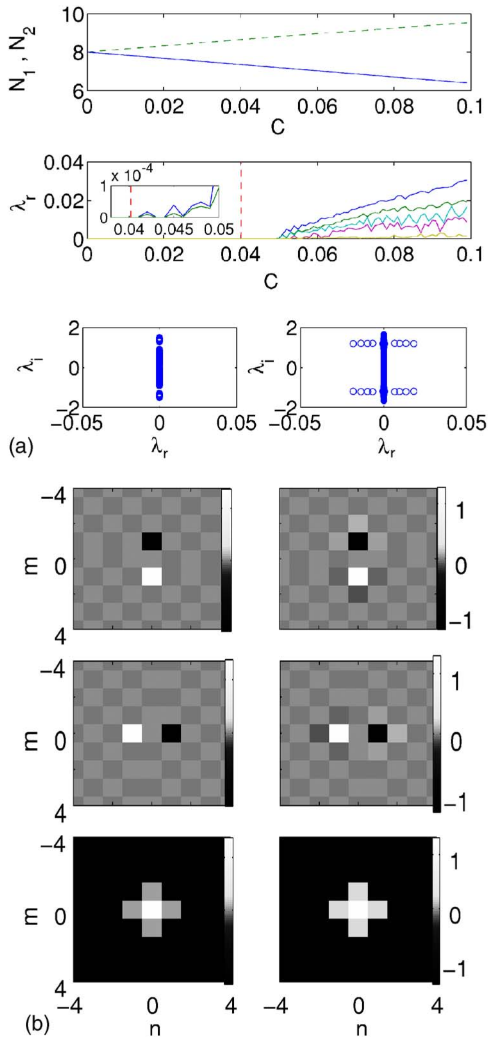

FIG. 2. (Color online) Same as in Fig. 1 for the lattice analog of the two-dimensional (*baby*) Skyrmion. Left and right paired panels pertain, respectively, to $C=0.025$ and 0.075, displaying, respectively, stable and unstable solutions. The top, middle, and bottom rows of the solution profiles display contours of real and imaginary parts of the complex field, and the real field, respectively.

(recall D is the dimension of the lattice), generates a set of *unstable* complex eigenvalues. Thus the lattice quasi-Skyrmions are predicted to be stable in the interval of $0 \leq C < C_{\mathrm{cr}}^{(D)}$ [note that the stability interval is absent if $\beta=1$, in which case Eq. (1) is a discrete version of the Manakov's system].

III. NUMERICAL RESULTS

Findings produced by the numerical computations for $D=1$ are summarized in Fig. 1, which shows norms of the two components of the solution defined with respect to the boundary conditions, $N_1=\Sigma_{\mathbf{n}}|u_{\mathbf{n}}|^2$ and $N_2=\Sigma_{\mathbf{n}}(\Lambda-|v_{\mathbf{n}}|^2)$, and most unstable eigenvalues (computed on the 1D lattice with 600 sites). In this case, Eq. (4) predicts $C_{\mathrm{cr}}^{(D=1)} \approx 0.0804$, while the numerical finding is 0.083 (the destabilization occurs via the collision of two spectral bands, as expected). It is worthwhile to note here that the oscillatory instability which sets in at $C=C_{\mathrm{cr}}^{(D=1)}$ remains extremely weak at $C \lesssim 0.1$, as shown in the inset of Fig. 1. Examples of stable and unstable 1D solutions are included too, for $C=0.05$ and 0.15, respectively.

Numerically found results for the analogs of the baby Skyrmion on the 2D lattice are displayed in Fig. 2, for which the prediction of Eq. (4) is $C_{\mathrm{cr}}^{(D=2)} \approx 0.0402$, while the respective numerical value is ≈ 0.042 (see the inset in Fig. 2).

Figure 3 shows an example of the toroidal lattice structure emulating the 3D Skyrmion, constructed as described above, with the complex field carrying the vorticity in the horizontal plane, and the real field featuring a 3D radial bubble. Here, the analytical prediction is $C_{\mathrm{cr}}^{(D=3)} \approx 0.027$. The example in Fig. 3 shows a weakly unstable solution, for $C=0.05$.

The next step is to simulate the evolution of unstable solutions. With a cascade of secondary instabilities produced by collisions between the spectral bands beyond the primary instability threshold [given by Eq. (4)], one may expect that the corresponding multitude of unstable eigenmodes leads to dynamical chaos ("lattice turbulence"), especially because the critical eigenmodes, belonging to the continuous spectrum, are delocalized at the instability thresholds. This expectation is borne out by the simulations, as shown in Fig. 4 for $C=0.149 > C_{\mathrm{cr}}^{(D=1)}$ for $D=1$. The weakly unstable configurations remain undisturbed for a while, but, at sufficiently long times ($t \approx 650$ in Fig. 4), the instability generates spatial chaos in the real-field component and breathings in its complex counterpart. This dynamics persists indefinitely long (unless effects of boundary conditions come into the play). A transition to chaotic behavior is also observed for $D=2$, as well as in simulations on the 3D lattice.

We also considered the influence of external potentials, which is relevant to BEC, as explained above. The latter amounts to adding a term $(\Omega^2/2)\mathbf{n}^2\phi$ to Eqs. (1). As seen in Fig. 5, the real-field pattern acquires a finite size in this case, as per the Thomas-Fermi approximation [38]. The main novel feature induced by the trap is the appearance of *gaps* in

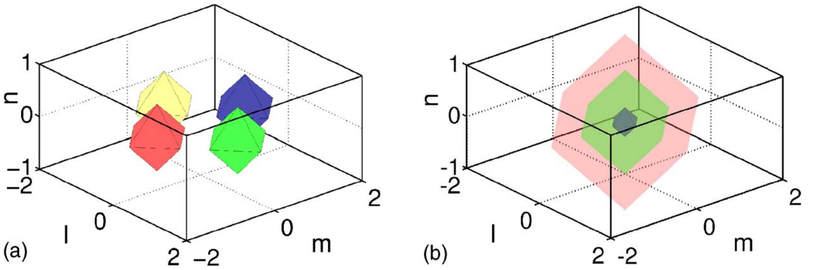

FIG. 3. (Color online) A three-dimensional lattice structure emulating the toroidal Skyrmion, for $C=0.05$. The left and right panels show, respectively, contours of the complex field, at $\mathrm{Re}(u)=\pm 1$ (blue/red in the color version and dark gray/gray in black and white) and $\mathrm{Im}(u)=\pm 1$ (green/yellow in the color version and light gray/very light gray in black and white), and of the real field, at $v=(1,0,-1)$ (blue, green, and red, i.e., contours from the inside out, respectively).

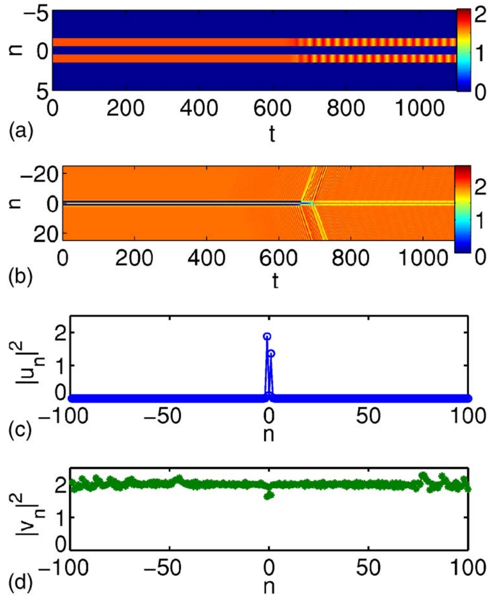

FIG. 4. (Color online) Development of the instability of one-dimensional lattice quasi-Skyrmions. The top two panels show space-time contours of the absolute value of the complex field and the real one. The two bottom panels show the spatial distribution of the respective fields at $t=1100$.

the linearization spectrum. The overall shape of the respective dependence of the largest unstable eigenvalue on C (top panel of Fig. 5) traces its counterpart in Fig. 1, which pertains to the spatially uniform system, but the gaps lead to *restabilization* of the discrete quasi-Skyrmions in certain intervals. For instance, the 1D solution is stable for $0.083 \leq C \leq 0.099$ and $0.113 \leq C \leq 0.115$, where it is unstable without the trapping potential. Similar restabilization mechanisms can be found also for $D=2$ and 3 (not shown here).

IV. CONCLUSIONS

We have constructed lattice structures that emulate 2D and 3D Skyrmions on the dynamical lattice with the cubic nonlinearity, and also provide for a 1D lattice counterpart of the Skyrmions. The discrete analogs of 2D (*baby*) Skyrmions were built following the hedgehog ansatz, and, accordingly, their structure resembles that of 2D Skyrmions in the continuum, combining a vortex soliton in the complex field and a radial bubble in a real field, that support each other. Also similar to the 2D continuum Skyrmions, the winding number (WN) of the 2D quasi-Skyrmion is represented by the vorticity of the complex field. The 3D lattice analog of the Skyrmion was built as a toroidal one, which is the most compact 3D pattern of the Skyrmion type, i.e., the one which is most relevant to the experiment. The WN of the discrete toroidal quasi-Skyrmion is represented (as well as in the toroidal Skyrmions in the 3D continuum field theory) by the topological charge of the 3D complex-field vortex, which constitutes its core. However, unlike its continuum counterpart which may only exist with WN=2, stable toroidal quasi-Skyrmions on the lattice exist with WN=1 (higher-order states, with WN≥2, can be readily constructed too). We

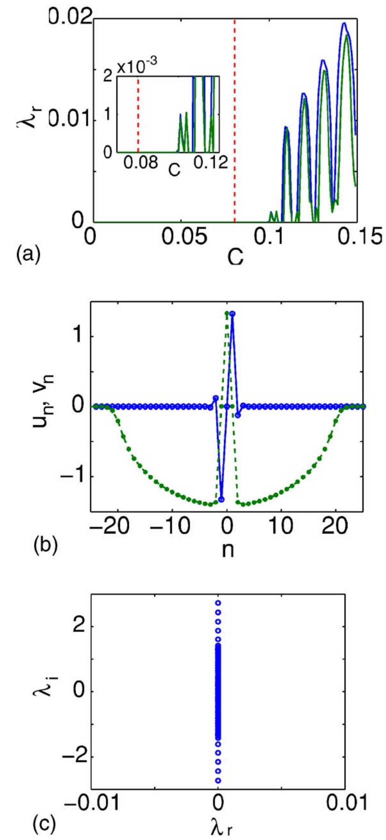

FIG. 5. (Color online) The most unstable eigenvalues for the one-dimensional lattice quasi-Skyrmion (top), and an example of a *restabilized* field configuration, u_n (solid line) and v_n (dashed line), and the respective eigenvalues for $C=0.115$ (middle and bottom) in the presence of the trapping potential with $\Omega=0.1$.

have investigated the stability of the 1D, 2D, and 3D discrete quasi-Skyrmions in an analytical approximation which is the first example of a direct analytical approach to the stability problem for Skyrmion-like structures in any setting, and verified it by numerical computation of the eigenvalues. We have also demonstrated that the evolution of unstable discrete Skyrmions leads to the onset of lattice turbulence. A possibility of further stabilization of the Skyrmions by means of external confining potentials was highlighted too. More sophisticated types of 2D and 3D lattice quasi-Skyrmions can be built too. In particular, the nearly planar vortex lying at the core of the 3D state may be replaced by a nonplanar *vortex cube* [12]. Moreover, there is a possibility to construct *zero*-WN lattice quasi-Skyrmions by placing a planar quadrupole [10] or cubic octupole [12] at the core. Results obtained in these directions will be reported elsewhere.

ACKNOWLEDGMENTS

P.G.K. acknowledges support from NSF-DMS-0204585 and NSF-CAREER. R.C.G. and P.G.K. also acknowledge support from NSF-DMS-0505663. The work of B.A.M. was supported, in part, by the Israel Science Foundation through the Center-of-Excellence Grant No. 8006/03.

[1] S. Aubry, Physica D **103**, 201 (1997); S. Flach and C. R. Willis, Phys. Rep. **295**, 181 (1998); D. K. Campbell, S. Flach, and Yu. S. Kivshar, Phys. Today **47**, 43 (2004).
[2] U. Peschel, R. Morandotti, J. M. Arnold, J. S. Aitchison, H. S. Eisenberg, Y. Silberberg, T. Pertsch, and F. Lederer, J. Opt. Soc. Am. B **19**, 2637 (2002); A. A. Sukhorukov, Yu. S. Kivshar, H. S. Eisenberg, and Y. Silberberg, IEEE J. Quantum Electron. **39**, 31 (2003).
[3] S. F. Mingaleev and Yu. S. Kivshar, Phys. Rev. Lett. **86**, 5474 (2001).
[4] A. Trombettoni and A. Smerzi, Phys. Rev. Lett. **86**, 2353 (2001); F. Kh. Abdullaev, B. B. Baizakov, S. A. Darmanyan, V. V. Konotop, and M. Salerno, Phys. Rev. A **64**, 043606 (2001); F. S. Cataliotti, S. Burger, C. Fort, P. Maddaloni, F. Minardi, A. Trombettoni, A. Smerzi, and M. Inguscio, Science **293**, 843 (2001); G. L. Alfimov, P. G. Kevrekidis, V. V. Konotop, and M. Salerno, Phys. Rev. E **66**, 046608 (2002); R. Carretero-González and K. Promislow, Phys. Rev. A **66**, 033610 (2002); M. A. Porter, R. Carretero-González, P. G. Kevrekidis, and B. A. Malomed, Chaos **15**, 015115 (2005).
[5] P. Binder, D. Abraimov, A. V. Ustinov, S. Flach, and Y. Zolotaryuk, Phys. Rev. Lett. **84**, 745 (2000); E. Trías, J. J. Mazo, and T. P. Orlando, *ibid.* **84**, 741 (2000).
[6] D. N. Christodoulides, F. Lederer, and Y. Silberberg, Nature (London) **424**, 817 (2003); P. G. Kevrekidis, K. Ø. Rasmussen, and A. R. Bishop, Int. J. Mod. Phys. B **15**, 2833 (2001).
[7] H. S. Eisenberg, Y. Silberberg, R. Morandotti, A. R. Boyd, and J. S. Aitchison, Phys. Rev. Lett. **81**, 3383 (1998); R. Morandotti, H. S. Eisenberg, Y. Silberberg, M. Sorel, and J. S. Aitchison, *ibid.* **86**, 3296 (1999).
[8] J. W. Fleischer, T. Carmon, M. Segev, N. K. Efremidis, and D. N. Christodoulides, Phys. Rev. Lett. **90**, 023902 (2003); Nature (London) **422**, 147 (2003); D. Neshev, E. Ostrovskaya, Yu. S. Kivshar, and W. Krolikowski, Opt. Lett. **28**, 710 (2003); H. Martin, E. D. Eugenieva, Z. Chen, and D. N. Christodoulides, Phys. Rev. Lett. **92**, 123902 (2004).
[9] B. A. Malomed and P. G. Kevrekidis, Phys. Rev. E **64**, 026601 (2001).
[10] P. G. Kevrekidis, B. A. Malomed, Z. Chen, and D. J. Frantzeskakis, Phys. Rev. E **70**, 056612 (2004); H. Sakaguchi and B. A. Malomed, Europhys. Lett. **72**, 698 (2005).
[11] D. N. Neshev, T. J. Alexander, E. A. Ostrovskaya, Yu. S. Kivshar, H. Martin, I. Makasyuk, and Z. Chen, Phys. Rev. Lett. **92**, 123903 (2004); J. W. Fleischer, G. Bartal, O. Cohen, O. Manela, M. Segev, J. Hudock, and D. N. Christodoulides, *ibid.* **92**, 123904 (2004).
[12] P. G. Kevrekidis, B. A. Malomed, D. J. Frantzeskakis, and R. Carretero-González, Phys. Rev. Lett. **93**, 080403 (2004); R. Carretero-González, P. G. Kevrekidis, B. A. Malomed, and D. J. Frantzeskakis, *ibid.* **94**, 203901 (2005).
[13] J. Yang, I. Makasyuk, A. Bezryadina, and Z. Chen, Opt. Lett. **29**, 1662 (2004).
[14] J. Meier, J. Hudock, D. Christodoulides, G. Stegeman, Y. Silberberg, R. Morandotti, and J. S. Aitchison, Phys. Rev. Lett. **91**, 143907 (2003).
[15] Z. Chen, H. Martin, E. D. Eugenieva, J. Xu, and A. Bezryadina, Phys. Rev. Lett. **92**, 143902 (2004).
[16] J. Yang, I. Makasyuk, P. G. Kevrekidis, H. Martin, B. A. Malomed, D. J. Frantzeskakis, and Z. Chen, Phys. Rev. Lett. **94**, 113902 (2005).
[17] B. Z. Essimbi and I. V. Barashenkov, J. Phys. Soc. Jpn. **71**, 2061 (2002); A. A. Sukhorukov and Y. S. Kivshar, Opt. Lett. **27**, 2112 (2002); P. G. Kevrekidis, B. A. Malomed, and Z. Musslimani, Eur. Phys. J. D **67**, 013605 (2003).
[18] S. Darmanyan, A. Kobyakov, and F. Lederer, JETP **86**, 682 (1998).
[19] T. Skyrme, Proc. R. Soc. London, Ser. A **260**, 127 (1961); G. S. Adkins, C. R. Nappi, and E. Witten, Nucl. Phys. B **228**, 552 (1983).
[20] A. Kudryavtsev, B. Piette, W. J. Zakrzewski, Nonlinearity **11**, 783 (1998); T. Weidig, *ibid.* **12**, 1489 (1999).
[21] F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990).
[22] D.-H. Lee and C. L. Kane, Phys. Rev. Lett. **64**, 1313 (1990); S. L. Sondhi, A. Karlhede, S. A. Kivelson, and E. H. Rezayi, Phys. Rev. B **47**, 16419 (1993).
[23] J. Ruostekoski and J. R. Anglin, Phys. Rev. Lett. **86**, 3934 (2001); U. Al. Khawaja and H. T. C. Stoof, Nature (London) **411**, 918 (2001); R. A. Battye, N. R. Cooper, and P. M. Sutcliffe, Phys. Rev. Lett. **88**, 080401 (2002); C. M. Savage and J. Ruostekoski, *ibid.* **91**, 010403 (2003).
[24] R. S. Ward, Lett. Math. Phys. **35**, 385 (1995); C. Timm, S. M. Girvin, and H. A. Fertig, Phys. Rev. B **58**, 10634 (1998).
[25] A. J. Schramm and B. Svetitsky, Phys. Rev. D **62**, 114020 (2000).
[26] J. E. Heebner and R. W. Boyd, J. Mod. Opt. **49**, 2629 (2002); P. Chak, J. E. Sipe, and S. Pereira, Opt. Lett. **28**, 1966 (2003).
[27] D. S. Hall, M. R. Matthews, J. R. Ensher, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **81**, 1539 (1998).
[28] M. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch, and I. Bloch, Nature (London) **415**, 39 (2002).
[29] A. Smerzi, A. Trombettoni, P. G. Kevrekidis, and A. R. Bishop, Phys. Rev. Lett. **89**, 170402 (2002).
[30] F. S. Cataliotti, L. Fallani, F. Ferlaino, C. Fort, P. Maddaloni, and M. Inguscio, New J. Phys. **5**, 71 (2003).
[31] I. V. Barashenkov and E. Yu. Panova, Physica D **69**, 114 (1993).
[32] G. Seibold, Phys. Rev. B **58**, 15520 (1998).
[33] D. E. Pelinovsky, P. G. Kevrekidis, and D. Frantzeskakis, Physica D **212**, 20 (2005).
[34] D. E. Pelinovsky, P. G. Kevrekidis, and D. Frantzeskakis, Physica D **212**, 1 (2005).
[35] P. G. Kevrekidis and D. E. Pelinovsky, Proc. R. Soc. London, Ser. A **462**, 2671 (2006).
[36] E. Braaten and L. Carson, Phys. Rev. D **39**, 838 (1989).
[37] T. Kapitula, P. G. Kevrekidis, and B. Sandstede, Physica D **195**, 263 (2004).
[38] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).