Generalization of the reaction-diffusion, Swift-Hohenberg, and Kuramoto-Sivashinsky equations and effects of finite propagation speeds

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(Received 12 May 2006; revised manuscript received 7 November 2006; published 23 February 2007)

The work proposes and studies a model for one-dimensional spatially extended systems, which involve nonlocal interactions and finite propagation speed. It shows that the general reaction-diffusion equation, the Swift-Hohenberg equation, and the general Kuramoto-Sivashinsky equation represent special cases of the proposed model for limited spatial interaction ranges and for infinite propagation speeds. Moreover, the Swift-Hohenberg equation is derived from a general energy functional. After a detailed validity study on the generalization conditions, the three equations are extended to involve finite propagation speeds. Moreover, linear stability studies of the extended equations reveal critical propagation speeds and unusual types of instabilities in all three equations. In addition, an extended diffusion equation is derived and studied in some detail with respect to finite propagation speeds. The extended model allows for the explanation of recent experimental results on non-Fourier heat conduction in nonhomogeneous material.

I. INTRODUCTION

The propagation of activity in spatially extended systems has attracted much attention in the last centuries, starting from celestial mechanics in the sixteenth century to its study in complex physical, chemical, or biological systems [1,2] in recent decades. In such systems the communication between the system subunits plays a decisive role. These subunits may be coupled locally to their next neighbors or may reveal longer-distance connections. In the latter case, the finite speed of interactions, i.e., the finite propagation speed, may yield temporal delays which affect the space-time dynamics of the system. Such effects have been found in several systems, e.g., in neural networks [3–6], quantum devices [7], porous and nonhomogeneous media [8,9], metals irradiated by laser pulses [10], hot plasma [11], and electronic circuits [12,13] like the internet or computational networks.

In order to describe spatial systems mathematically, partial differential equations (PDEs) have been applied widely. However, most PDE models neglect effects caused by finite propagation speeds in the system. To consider these effects in PDE models, various approaches have been followed such as, e.g., the introduction of additional temporal derivatives [14] or temporal constant delays [15]. However, propagation delays depend on the distance between two locations and thus represent space-dependent delays. In contrast to the PDE formulation, a natural way to consider these delays is integral-differential equations (IDEs) which sum up all activity in a spatial domain and easily consider propagation delay; see, e.g., [16,17]. Moreover, the strong connection between PDE and IDE models is well known [18]. The present work proposes an IDE model involving finite propagation delays which generalizes well-studied PDE models in one spatial dimension, namely, the reaction-diffusion, Swift-Hohenberg, and Kuramoto-Sivashinsky models. Moreover, re-calling the mathematical description of neuronal populations by IDEs, the proposed model represents a generic pattern-forming

model with a vast range of possible applications.

In addition to the generalization described above, the IDE model allows for the extension of PDE models in order to consider propagation delay effects. The present work shows that the results obtained in IDE models involving finite propagation speeds can be applied easily to PDE models. Hence this work extends previous studies [19,20] as it clearly distinguishes local dynamics from nonlocal spatial interactions and elicits the strong connection between well-known PDE models of pattern-forming systems and the IDE models in neural field theory [21–23]. We derive extensions of the diffusion equation, the Swift-Hohenberg equation (SHE), and the Kuramoto-Sivashinsky equation now involving finite propagation speeds. These extensions are important if the speed of the propagating activity in the system approaches the propagation speed of the system. Such ultrafast phenomena have been observed experimentally in solids [24,25], in plasma [11], and on solid and fluid surfaces [26,27].

The present work is structured as follows. Section II introduces the integral-differential equation model and elicits its formal relation to partial differential equations. Then the subsequent section shows the reduction procedure to the PDE models and investigates the validity of that reduction. Eventually Sec. IV studies the linear behavior of three specific models and examines their dependence on the finite propagation speed, while the last section closes the work by a brief discussion of the obtained results.

II. THE GENERAL MODEL

The present work considers the general evolution equation

$$\hat{T}V(x,t) = h[V(x,t)] + \int_\Omega dy K(x-y)f[V(y,t-\tau(x-y))]$$

$$+ \int_\Omega dy L(x-y)g[V(y,t-\tau(x-y))] \quad (1)$$

with the scalar field variable $V(x,t)$ and the spatial domain

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Ω, which is assumed being large but finite. The temporal operator $\hat{T}=\hat{T}(\partial/\partial t)$ represents the temporal linear dynamics of an uncoupled element at spatial location x. For instance, a damped oscillator would be modeled as $\hat{T}V=\partial^2 V/\partial t^2 + \partial V/\partial t + V$. In the following, we specify the operator by $\hat{T}(0)=1$. In addition, the self-interaction $h[V(x,t)]$ may represent an additional nonlinear driving. Considering the previous example, the nonlinear damped oscillators may be modeled by $\hat{T}V=V-\sin(V)$, i.e., $h[V]=V-\sin(V)$. Moreover, to extend the system by spatial coupling, $K(x-y)$ and $L(x-y)$ in Eq. (1) represent the different coupling functions between elements at spatial locations x and y, and the two coupling functions belong to two different spatial interaction types. For instance, in neural nets neurons may be coupled by excitation and by inhibition, which is modeled typically by an excitatory net and an inhibitory net with $K(x-y)>0$ and $L(x-y)<0$, respectively. In the next section, this distinction turns out to be very important in the context of the Kuramoto-Sivashinsky equation. In addition, the functionals $f[V]$ and $g[V]$ allow for various (nonlinear) interaction types of the corresponding spatial interactions. Finally the propagation delay $\tau(x-y)=|x-y|/c$ with the propagation speed c takes into account the finite time it takes the signal to propagate from one spatial location x to another location y.

We point out that Eq. (1) is similar to previous IDE models of neuronal populations [28,29], which can be derived from basic properties in neural tissue [19]. In addition, it extends the previous IDE models by considering local interactions and an additional spatial interaction type and thus renders the model more realistic for physical systems. As we show in the subsequent section, this extension proves to be very powerful.

Finally recall the relation of IDEs to PDEs. Some previous studies remarked that IDEs generalize PDEs [18,19] by the identity

$$\int_{-\infty}^{\infty} dy\, K(x-y)S[V(y)] = \sum_{n=0}^{\infty} (-1)^n K_n \frac{\partial^n S[V(x)]}{\partial x^n} \qquad (2)$$

with a nonlinear functional S and the kernel moments $K_n = \int d\eta\, K(\eta)\eta^n/n!$ $\forall\ n \in \mathbf{N}_0$. The expansion (2) represents an order expansion of spatial interactions whose order n indicates the spatial interaction range. Hence K_n represent the contribution of spatial interactions of order n. Typically local or short-range PDE models involve spatial derivatives of up to second order, while PDE models with spatial derivatives of up to fourth order involve nonlocal or long-range interactions. The expansion (2) extends this classification to spatial derivatives of arbitrary order.

III. REDUCTION TO SPECIFIC MODELS

This section shows that Eq. (1) generalizes reaction-diffusion systems and the SHE and the Kuramoto-Sivashinsky equation in one-dimensional spatial systems. Since these equations assume an infinite propagation speed in the system, we set $\tau(x-y)=0$.

A. Reaction-diffusion equations

First let us focus on reaction-diffusion models, which allow for the mathematical description of diverse spatiotemporal phenomena. We mention heat propagation or phase transitions in solids [30,31], chemical reactions in fluids [32,33], and pattern formation in biological systems [2,18]. Reaction-diffusion equations are obtained from (1) by choosing $\hat{T} = \partial/\partial t$, $g[V]=0$, and $f[V]=f_1 V$. Further, the kernel $K(x)$ is symmetric, i.e., $K_{2n+1}=0$, and exhibits short-ranged spatial interactions with $K_n \to 0$ for $n>2$. Then the application of Eq. (2) yields

$$\frac{\partial V(x,t)}{\partial t} = \bar{h}[V(x,t)] + D\frac{\partial^2}{\partial x^2}V(x,t) \qquad (3)$$

with $\bar{h}(V)=h[V]+K_0 f_1 V$ and $D=K_2 f_1$. This reaction-diffusion equation [33] accounts for (non)linear self-interactions represented by $\bar{h}[V]$ and considers local diffusive interactions with diffusion constant D. Hence, reaction-diffusion systems neglect spatial interactions of order $n \geqslant 4$.

B. Swift-Hohenberg equation

In their famous work on fluctuations near the onset of Rayleigh-Bénard convection [34,35], Swift and Hohenberg derived an order parameter equation for the temperature and fluid velocity dynamics of the convection. This work has attracted much attention in the last decades and the order parameter equation has been interpreted as a model system for pattern formation [33]. For instance, Lega *et al.* modeled the spatiotemporal pattern formation in large aspect ratio lasers [36,37] by the SHE and Brazovskii [38] discussed the SHE in the context of the condensation of liquid crystal fluids. However, in spite of its broad applicability, the SHE fails to model some experimental results [33] and several previous studies extended the equation by adding some terms [39,40]. Hence the application of the SHE is not limited to the original physical problem and it thus represents a generic model for the spatiotemporal dynamics of spatially extended systems [40].

To gain the SHE from Eq. (1), we choose the temporal operator as $\hat{T}=\partial/\partial t$ and define $h[V]=aV-bV^3$, $g[V]=0$, $f[V]=f_1 V$, $f_1<0$. Subsequently we obtain

$$\frac{\partial V(x,t)}{\partial t} = h[V(x,t)] + f_1 \int_{\Omega} dy\, K(x-y)V(y,t). \qquad (4)$$

In addition the kernel $K(x)$ is symmetric and exhibits longer-ranged spatial interactions, i.e., $K_n \to 0$ for $n>4$. The subsequent application of Eq. (2) yields

$$\frac{\partial V(x,t)}{\partial t} = aV(x,t) - bV^3(x,t) + K_0 f_1 V(x,t) + K_2 f_1 \frac{\partial^2}{\partial x^2}V(x,t)$$
$$+ K_4 f_1 \frac{\partial^4}{\partial x^4}V(x,t).$$

By choosing $b=K_2^2|f_1|/4K_4>0$ and rescaling time and space according to $t \to (4K_4/K_2^2|f_1|)t$ and $x \to \sqrt{2K_4/K_2}x$, respectively, we find the SHE [33]

$$\frac{\partial V(x,t)}{\partial t} = \varepsilon V(x,t) - V^3(x,t) - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 V(x,t) \qquad (5)$$

with $\varepsilon = 4K_4 a/K_2^2 |f_1| - 4K_4 K_0/K_2^2 + 1$. Hence, systems obeying the SHE neglect spatial interactions of order $n \geq 6$.

Moreover, Eq. (5) maybe derived from an energy functional $\mathcal{F}[V]$ by the generalized Ginzburg-Landau equation

$$\frac{\partial V}{\partial t} = -\frac{\delta \mathcal{F}[V]}{\delta V}. \qquad (6)$$

Since the present work aims to introduce nonlocal interactions to physical models in a generalized way, we follow the approach of Ginzburg and Landau and attempt to model the spatiotemporal dynamics of spatially extended systems by a generalized energy functional. Taking into account the spatial interactions of elements in a system, the interaction energy in a spatial field between two spatial locations x and y at time t reads $K(x-y)V(x,t)V(y,t)$. Here the term $K(x-y)$ quantifies the interaction strength between both elements [41]. Considering local interactions $h[V]$ and their corresponding energy contributions $W[V]$ additionally, then the field energy is the space integral over all spatial locations and all interactions:

$$\mathcal{F} = \int_\Omega dx\, W[V(x,t)] - \frac{1}{2}\int_\Omega dx \int_\Omega dy\, K(x-y)V(x,t)V(y,t)$$

$$= \int_\Omega dx\, W[V(x,t)] - \frac{1}{2}V(x,t)\sum_{n=0}^{\infty}(-1)^n K_n \frac{\partial^n V(x)}{\partial x^n}$$

$$= \int_\Omega dx\, W[V(x,t)] - \frac{1}{2}V(x,t)$$

$$\times \left(K_0 + K_2 \frac{\partial^2 V(x)}{\partial x^2} + K_4 \frac{\partial^4 V(x)}{\partial x^4} + \cdots\right) \qquad (7)$$

with $-\delta W/\delta V = h[V]$ and Ω denoting the real axis. In the third line, we applied the identity (2). Then the specific choices applied in the previous paragraph yields the well-known energy functional of the SHE [34,35,42]

$$\mathcal{F} = \int_\Omega dx \frac{1}{4} V^4(x,t)$$

$$+ \frac{1}{2}\int_\Omega dx\, V(x,t)\left[-\varepsilon + \left(1 + \frac{\partial^2}{\partial x^2}\right)^2\right]V(x,t). \qquad (8)$$

Corresponding to this derivation of the SHE from general physical considerations, we conclude that the SHE represents a specific description of a general spatial system. This view angle on the equation is supported by its various motivations in physics mentioned above. In addition to the previous discussion, Eq. (7) is the Lyapunov functional of Eq. (4) as

$$\frac{d}{dt}\mathcal{F} = -\int_\Omega dx \left(\frac{\partial V(x,t)}{\partial t}\right)^2 < 0.$$

Then the question arises whether there exists an energy functional in the case of finite propagation speeds. Corresponding to the previous treatment, the first ansatz for a pos-

sible energy functional would contain the terms $K(z)V(x,t)V(x-z,t-\tau(z))$ with the distance $z=x-y$ between two spatial locations and the propagation delay $\tau(z) \neq 0$ taken from Eq. (1). Then applying the identity (2) in the time domain, we obtain

$$K(z)V(x,t)V(x-z,t-\tau(z))$$

$$= K(z)V(x,t)\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{|z|}{c}\right)^n \frac{\partial^n V(x-z,t)}{\partial t^n}.$$

We observe that the energy of a pair of elements at distance z, whose interaction is delayed by the finite propagation speed c, depends on the temporal derivatives of the elements. Subsequently the energy functional of the system $\mathcal{F}[V, \partial V/\partial t, \ldots]$ would depend on the temporal derivatives of the field and the generalized Ginzburg-Landau equation (6) is not applicable. Preliminary results on the different derivation of Eq. (4) including propagation delays by another variational principle seems promising. However, the discussion of these results would exceed the major aim of the present work and we refer the reader to forthcoming work.

C. Kuramoto-Sivashinsky equation

Finally we focus on the Kuramoto-Sivashinsky equation [33,43] which allows for the study of various phenomena in fluids and solids such as, e.g., the phase turbulence in fluids [44], the thermal diffusive instabilities of flame fronts [45], the directional solidification in alloys [46], or the interface instability during the application of industrial beam cutting techniques [47]. Here, the terms in Eq. (1) are chosen as $\hat{T} = \partial/\partial t$, $g(V) = g_1 V$, $f(V) = f_2 V^2$ with $g_1, f_2 < 0$. Now the kernel $K(x)$ is nonsymmetric and short ranged, i.e., $K_n \to 0$, $n > 1$, while the kernel $L(x)$ is chosen symmetric. Hence the corresponding kernel moments obey $L_{2n+1} = 0$ and $L(x)$ describes long-range interaction with $L_n \to 0$, $n > 4$. Then the application of Eq. (2) leads to

$$\frac{\partial V(x,t)}{\partial t} = (L_0 g_1 - h_0)V(x,t) + L_2 g_1 \frac{\partial^2}{\partial x^2}V(x,t) + L_4 g_1 \frac{\partial^4}{\partial x^4}V(x,t)$$

$$+ 2K_1 f_2 V\frac{\partial V}{\partial x}$$

while $h(V)$ has been chosen as $h(V) = -K_0 f_2 V^2 - h_0 V$, $h_0 \geq 0$. Finally the rescaling $t \to (L_4/L_2^2|g_1|)t$ and $x \to \sqrt{L_4/L_2}x$ yields

$$\frac{\partial V(x,t)}{\partial t} = -\eta V - \frac{\partial^2}{\partial x^2}V(x,t) - \frac{\partial^4}{\partial x^4}V(x,t) - V\frac{\partial V}{\partial x} \qquad (9)$$

with $\eta = |g_1|L_0 + h_0$ and the additional condition $f_2 = L_2\sqrt{L_2/L_4}/2K_1|g_1|$. Equation (9) is called the generalized Kuramoto-Sivashinsky equation [33] and $\eta = 0$ represents the original Kuramoto-Sivashinsky equation. Hence systems obeying the Kuramoto-Sivashinsky equation involve symmetric spatial interactions up to order $O(\sigma^5)$ and nonsymmetric spatial interactions of order $O(\sigma)$. It is interesting to note that this model includes both nonsymmetric and symmetric spatial interactions while the reaction-diffusion equa-

tion and the Swift-Hohenberg equation involve symmetric spatial interactions only.

D. Validity study

After the previous examinations, the question arises which specific kernel functions may satisfy the appropriate conditions in the three models. To this end, we apply the Fourier transform to the nonlinear function $S(V(x))$ in (2) and obtain the spatial mode expansion

$$\int_{-\infty}^{\infty} dy \, K(x-y) S(V(y,t)) = \int_{-\infty}^{\infty} dk \, \tilde{S}(k,t) e^{ikx} \sum_{n=0}^{\infty} f_n(k) \tag{10}$$

with the sequence $f_n(k)=(-ik)^n K_n$. In the following, we abbreviate $f_n=f_n(k)$ for convenience. As we will see later, $f_n(k)$ represents the contribution of interactions at order n to the spatial modes of the system.

To obtain convergence of the series expansion in (2), the ratio of subsequent sequence terms $|f_n|$ has to be smaller than unity. This means this condition reads $|f_{n+2}|/|f_n|<1$ for n even and $|f_{n+1}|/|f_n|<1$ for all n for symmetric and nonsymmetric kernels, respectively.

To be more specific let us focus on a specific symmetric kernel function. Since the kernel represents the probability density function of the spatial interactions between two locations, the central limit theorem supports the choice of the Gauss function $K(x)=\exp(-x^2/2\sigma^2)/\sqrt{2\pi}\sigma$ with the spatial range constant σ. Then previous convergence condition leads to $\sigma^2<(n+2)/k^2$, $n=2,4,6,\ldots$. If there is a maximum spatial frequency $|k_m|$, i.e., the Fourier transform $\tilde{S}(|k|)$ is negligible for $|k|>|k_m|$, then the spatial interaction range σ is limited from above by $\sqrt{n+2}/|k_m|$. Subsequently, the spatial interaction range σ is delimited to the interval $0<\sigma^2<1/k_m^2$ which guarantees the Taylor expansion convergence in (2). We point out that the limitation to spatial modes is present in various physical systems. For instance, spatial systems near a phase transition exhibit a hierarchy of time scales of spatial modes with corresponding prominent spatial frequencies. In this case the time evolution of the Fourier transforms $\tilde{S}(k,t)$ at spatial frequencies far from the prominent ones are negligible and the prominent frequencies define the maximum spatial frequency $|k_m|$. However, if there is no maximum spatial frequency, i.e., $|k_m|\to\infty$, it is $\sigma\to0$ and the convergence condition yields $K(x)\to\delta(x)$. Hence for an unlimited range of spatial frequencies, the expansion (2) is valid for local interactions only and $K_n\to0$ for $n>0$.

Now let us interpret the latter results. It turns out that the convergence condition for Gaussian kernels leads to the dependence of the maximum spatial range on the interaction order n. For instance if the expansion in Eq. (2) approximates local interactions but no higher orders of nonlocality (cf. the previous discussion of the reaction-diffusion equation), it is $|K_{n+2}/K_n|\ll1$ for $n=2,4,\ldots$ and, subsequently, $\sigma^2<(2+2)/k^2=4/k^2$. Hence, a maximum spatial frequency $|k_m|$ yields the validity condition $0<\sigma^2<4/k_m^2$. In case of spatial interactions at higher orders of nonlocality (cf. the

previous discussion of the Swift-Hohenberg and Kuramoto-Sivashinsky equations), it is $|K_{n+2}/K_n|\ll1$ for $n=4,6,\ldots$. This means that the validity condition now reads $0<\sigma^2<(4+2)/k_m^2=6/k_m^2$. Thus the spatial interaction range σ may be larger than in the diffusion model.

Since the Kuramoto-Sivashinsky equation involves nonsymmetric interactions, let us also examine briefly the kernel function $K(x)=\delta(x-x_0)$ with the nonsymmetric shift $x_0\neq0$. We find $K_n=x_0^n/n!$ and the convergence condition reads $x_0<(n+1)/|k_m|$. Subsequently, the Kuramoto-Sivashinsky equation is an approximation of (1) if $x_0<2/|k_m|$ and if a maximum spatial frequency exists.

IV. LINEAR ANALYSIS

Now let us examine the stationary state and its linear behavior for a finite propagation speed c. In case of the stationary state $V(x,t)=V_0$, Eq. (1) recasts to

$$T(0)V_0 = h(V_0) + \kappa_K f(V_0) + \kappa_L g(V_0),$$

$$\kappa_K = \int_{-\infty}^{\infty} K(x)dx, \quad \kappa_L = \int_{-\infty}^{\infty} L(x)dx \tag{11}$$

with $\kappa_K=\int K(x)dx$, $\kappa_L=\int L(x)dx$. Considering linear deviations $u(x,t)=V(x,t)-V_0\sim e^{\lambda t+ikx}$, Eq. (1) reads

$$T(\lambda) = s_h + \int_{-\infty}^{\infty} dz \, M(z) e^{-\lambda|z|/c} e^{-ikz} \tag{12}$$

with $M(z)=s_f K(z)+s_g L(z)$ and $s_h=\delta h/\delta V$, $s_f=\delta f/\delta V$, $s_g=\delta g/\delta V$ computed at $V=V_0$, while $\delta/\delta V$ denotes the functional derivative. In the following, the propagation delay c is assumed large but finite and the approximation $\exp(-\lambda|z|/c)\approx1-\lambda|z|/c+\lambda^2(|z|/c)^2$ holds. For the characteristic spatial scale of the system σ, we introduce the characteristic propagation delay $\tau=\sigma/c$ which represents the delay time caused by the finite propagation speed. Thus the previous approximation holds for small propagation delays $\tau\ll1/\lambda$. Subsequently expanding the exponential $\exp(-ikz)$ into a power series we obtain

$$T(\lambda) + \frac{\lambda}{c}\sum_{n=0}^{\infty}(-ik)^n P_n - \frac{\lambda^2}{c^2}\sum_{n=0}^{\infty}(-ik)^n Q_n = s_h + \sum_{n=0}^{\infty}(-ik)^n M_n \tag{13}$$

with the kernel moments M_n defined previously and $P_n=\int_\Omega dz \, M(z)|z|z^n/n!$, $Q_n=\int_\Omega dz \, M(z)|z|^2 z^n/n!$. Preliminary computations of M_n, P_n, and Q_n for Gaussian kernels yield $M_n\sim O(\sigma^n)$, $P_n\sim O(\sigma^{n+1})$, and $Q_n\sim O(\sigma^{n+2})$

Equation (13) defines the stability condition for general spatial interactions and short propagation delays. Additional more detailed studies show that large propagation speeds c with $1/c^2\approx0$ and vanishing odd kernel moments $M_{2n+1}=0$, $P_{2n+1}=$, $Q_{2n+1}=0$ lead to real Lyapunov exponents and thus may yield stationary bifurcations. In contrast oscillatory bifurcations may emerge for nonvanishing odd moments only. By recalling the definitions of these moments, we find that symmetric and nonsymmetric spatial interactions may yield
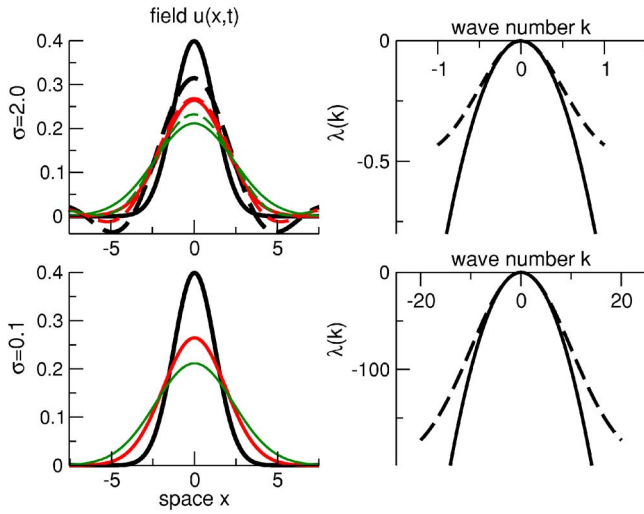
FIG. 1. (Color online) The field of the diffusion equation and the Lyapunov exponent for different spatial ranges σ. The dashed lines represent the results from the diffusion PDE, while the solid lines denote the results from the IDE model. In case of $\sigma=0.1$, the reconstructed field $u(x,t)$ obtained from the IDE cannot be distinguished from the field of the original diffusion equation. In addition the colors in the left panel denote subsequent times $t=0$ (black, very thick line), 3 (red, thick line), and 6 (green, thin line). That is, the field $u(x,t)$ flattens at increasing time.

stationary and nonstationary bifurcations, respectively. In case of smaller propagation speeds $1/c^2 \neq 0$, this classification no longer holds and oscillatory bifurcations may occur for both symmetric and nonsymmetric kernels.

A. The reaction-diffusion equation

To illustrate the latter results, first let us study a specific reaction-diffusion equation, namely, the homogeneous diffusion equation also known as Fick's second law. According to the previous discussion, Eq. (3) represents the diffusion equation with $\bar{h}=0$ yielding $h(V)=-f_1 V$. Further, Eq. (11) yields the stationary solution $V_0=0$ and the parameters $s_f=f_1$, $s_h=-f_1$, while the spatial interactions are given by $K(z) \neq 0$ and $L(z)=0$, i.e., $M(z)=f_1 K(z)$.

According to the previous discussion of (10), Eq. (2) represents a spatial mode expansion in orders of the characteristic spatial scale σ. That is, in the case of the diffusion equation, the order expansion is truncated at $O(\sigma^3)$ yielding $M_0=f_1 \sim O(1)$, $M_2=f_1 \sigma^2/2 \sim O(\sigma^2)$ while all odd and higher kernel moments M_{2n} vanish with $n \geq 2$. Further, we find $P_0 = f_1 \sqrt{2} \sigma / \sqrt{\pi}$, $P_1=0$, $P_2=f_1 \sqrt{2} \sigma^3 / \sqrt{\pi}$ with $P_n \approx 0 \ \forall \ n > 3$ and $Q_0=f_1 \sigma^2$, $Q_n \approx 0 \ \forall \ n > 0$. Here it is $f_1=2D\sigma^2$ and D represents the diffusion constant. Subsequently, in the case of $c \to \infty$ Eq. (13) yields $\lambda(k) \approx -Dk^2$, which is the well-known dispersion relation of the diffusion equation. Figure 1 compares λ and λ_{RD} and the corresponding resulting field activity for two values of σ. We observe that the IDE model is a good approximation to the diffusion equation for $\sigma=0.1$, while the approximation is much worse for $\sigma=2$. This finding confirms our previous results on the convergence condition. Here the maximum value of $|k|$ is $2/\sigma$ according to the convergence

condition $\sigma < 2/k$ derived in the previous paragraphs.

Now let us turn to finite propagation speeds $c < \infty$. Considering the corresponding orders of M_n, P_n, and Q_n, Eq. (13) reads

$$\lambda + \frac{\lambda}{c}(P_0 - P_2 k^2) - \frac{\lambda^2}{c^2} Q_0 = s_h + M_0 - M_2 k^2. \quad (14)$$

For small propagation delays $(\lambda \tau)^2 \to 0$ we find the Lyapunov exponent

$$\lambda_{RD}(k) = \frac{-Dk^2}{1 + \alpha \tau (1 - \sigma^2 k^2)} \quad (15)$$

with $\alpha=f_1 \sqrt{2/\pi}$. By virtue of the wave number limit $|k| < 2/\sigma$, there is a critical propagation delay $\tau_{th}=1/3\alpha$ and a corresponding critical propagation speed $c_{th}=3\alpha\sigma$. Then small delays $\tau < \tau_{th}$ retain the field stability, while large delays $\tau > \tau_{th}$ yield linear unstable modes k with $\lambda_{RD}(k) > 0$. Figure 2 shows the corresponding stability diagram of the system.

Moreover, Eq. (15) may be interpreted as if it originates from the modified diffusion equation

$$\frac{\partial u(x,t)}{\partial t} + \frac{\alpha\sigma}{c}\left(1 + \sigma^2 \frac{\partial^2}{\partial x^2}\right)\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2}. \quad (16)$$

This equation is an extension of the standard diffusion equation considering finite propagation speeds. It applies in media whose propagation delay on the typical length scale of the system is not negligible and, for instance, which show delayed spread of space-time activity. We mention results from fast hot pulses in plasma [11], whose investigation indicates nonlocal effects in the reaction-diffusion mechanism. Moreover, there is the debate on the non-Fourier heat conduction in nonhomogeneous materials which show such delayed temporal activity at measurement points [8,9,48]. In order to compare our model to the results found in experiments [9], Fig. 3 shows the temporal activity at a spatial point. We observe a prominent delay of the activity at large delays, while $\tau \to 0$ yields the Fourier case. This finding is irrespective of the spatial location used and coincides qualitatively to experimental findings. Moreover, the derived model (16) contrasts with the Cattaneo equation [14,49], which represents a telegraphic equation and extends the parabolic diffusion equation to a hyperbolic partial differential equation by an additional second temporal derivative and
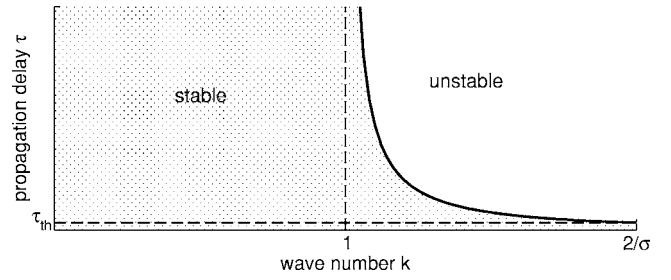

FIG. 2. Illustrated stability diagram of the extended diffusion equation. The vertical axis $k=1$ represents the asymptote of the instability regime.
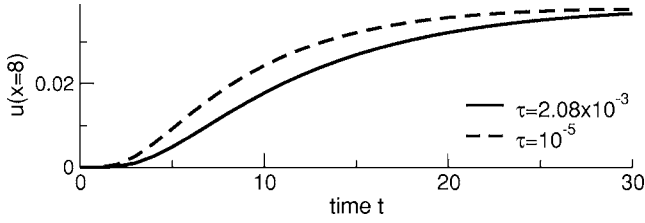
FIG. 3. Temporal activity of the modified diffusion equation at spatial location $x=8$ for two different propagation delays, i.e., propagation speeds. The solutions are computed analytically with initial condition $u(x,t=0)=\exp(-x^2/2\times 0.8^2)/\sqrt{2\pi 0.8^2}$.

thus involves finite propagation speed. Originally the Cattaneo extension was proposed according to heuristic arguments while our model is derived from the explicit treatment of propagation delay.

Finally, let us briefly focus on the case of larger propagation delays $(\lambda\tau)^3\to 0$ yielding the implicit equation $-\lambda^2\tau^2 2D/\sigma^2+\lambda[1+\alpha D\tau(1-\sigma^2k^2)]-Dk^2=0$. Simple calculus reveals $\mathrm{Re}(\lambda)>0$ for some k, and larger delays destabilize the field similarly to the previous case of small delays.

B. The Swift-Hohenberg equation

Now let us turn to the Swift-Hohenberg model which involves long-range spatial interactions. It has been studied as a generic model for spatial pattern formation in various physical systems [36,38,40]. By virtue of its broad applicability, this section aims to extend it by finite propagation delay effects. The stationary state is given by Eq. (11) and we find one or three solutions. The single solution $V_0=0$ leads to the parameters $s_f=-|f_1|$ and $s_h=a$. Further the spatial interactions are given by $K(z)\neq 0$ and $L(z)=0$ yielding $M(z)=-|f_1|K(z)$.

According to the discussions in Sec. III D, Eq. (2) breaks off at order $O(\sigma^5)$ and we obtain the expansion terms

$$M_0=-|f_1|, \quad M_2=-|f_1|\sigma^2/2, \quad M_4=-|f_1|\sigma^4/8,$$

$$P_0=-|f_1|\sqrt{\frac{2}{\pi}}\sigma, \quad P_2=-|f_1|\sqrt{\frac{2}{\pi}}\sigma^3, \quad P_4=-|f_1|\frac{\sqrt{2}}{3\sqrt{\pi}}\sigma^5,$$

$$Q_0=-|f_1|\sigma^2, \quad Q_2=-|f_1|\frac{3}{2}\sigma^4,$$

while all odd and higher terms M_n, P_n, Q_n vanish. Subsequently Eq. (13) yields for $c\to\infty$

$$\lambda_\infty=\frac{|f_1|}{2}(\varepsilon-1+2l^2-l^4)$$

with the scaled wave number $l=\sigma k/\sqrt{2}$ and ε taken from Sec. III B. Here λ_∞ represents the Lyapunov exponent of the linearized original Swift-Hohenberg equation (5) and thus defines the linear stability of the underlying spatial system. To be more detailed, for $\varepsilon<0$ the system is linearly stable with $\lambda_\infty<0$ for all l, while $\varepsilon>0$ yields $\lambda_\infty>0$ for $1-\sqrt{\varepsilon}<l<1+\sqrt{\varepsilon}$. In other words, a Turing instability

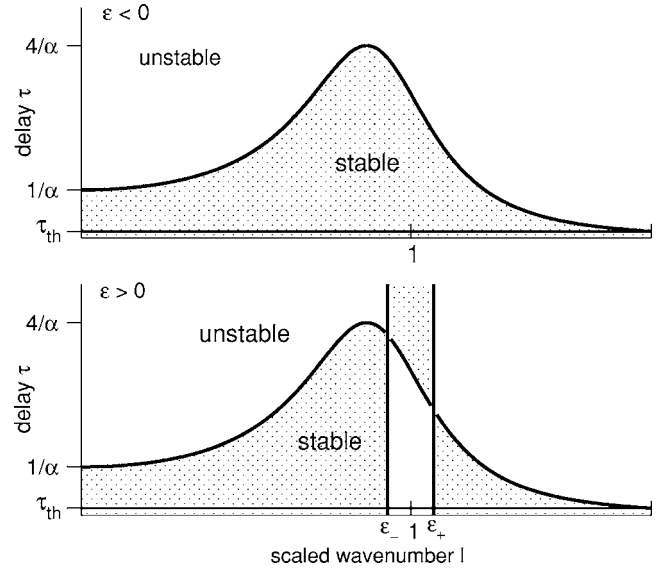FIG. 4. Illustrated stability diagram of the extended Swift-Hohenberg equation involving propagation delays. For $\varepsilon<0$ (top panel) the underlying system is stable for $\tau<1/\alpha(1-2l^2+4l^4/3)$ and unstable otherwise. For $\varepsilon>0$ (bottom panel) the stability is the same as for $\varepsilon<0$ apart from the wave number band $\varepsilon_-<1<\varepsilon_+$, $\varepsilon_\pm=\pm\sqrt{\varepsilon}$. For short delays $\tau<1/\alpha(1-2l^2+4l^4/3)$ the underlying system is linearly unstable in this wave number band, while the modes in this band are stable otherwise. In both the latter cases $\tau<\tau_{th}$ guarantees the linear stability of the underlying system.

emerges for $\varepsilon>0$ at scaled wave numbers $|l|\approx 1$.

Now let us turn to the case $c<\infty$. For small propagation delays $(\lambda\tau)^2\to 0$ we find

$$\lambda_{SH}=\frac{\lambda_\infty}{1-\alpha(1-2l^2+4l^4/3)\tau} \tag{17}$$

with $\alpha=|f_1|\sqrt{2/\pi}$. Here λ_{SH} represents the Lyapunov exponent of the Swift-Hohenberg equation subject to large propagation speeds. It is important to mention that the validity criteria $0<\sigma^2<6/k_m^2$ derived in Sec. III D delimits the scaled wave number to $-\sqrt{3}<l<\sqrt{3}$. Taking into account this constraint, closer examinations of Eq. (17) reveal the critical propagation delay $\tau_{th}=1/7\alpha$ and the corresponding critical propagation speed $c_{th}=7\alpha\sigma$. For $\tau<\tau_{th}$, the denominator of λ_{SH} is positive for all valid wave numbers and thus λ_∞ defines the stability of the system. Hence the propagation delay does not affect the stability of the system.

In contrast, large propagation delays $\tau>\tau_{th}$ yield a negative denominator for a half band of wave numbers $|l|>l_{th}$ and thus change the sign of the Lyapunov exponent. Here the critical wave number l_{th} represents the root of the denominator in Eq. (17). Figure 4 shows the stability diagram of Eq. (17). We observe the unstable half band for $\tau>\tau_{th}$ for all ε, while $\tau>1/\alpha>\tau_{th}$ yields an additional unstable band of wave numbers around $l=0$. For $\varepsilon<0$ the stable band of spatial modes vanishes for large propagation delays $\tau>4/\alpha$ and the system is totally destabilized, while for $\varepsilon>0$ increased propagation delays stabilize the band of unstable Turing modes. This stabilization by propagation delays represents an

unusual effect near Turing instabilities. In addition, recall that the previous treatment around l_{th} reflects $|\lambda| \to \infty$ and thus conflicts with the previous assumption $(\lambda\tau)^2 \approx 0$. Sequentially higher polynomial orders in λ are necessary for closer examination.

Finally Eq. (17) may be interpreted as if it originates from the extended Swift-Hohenberg equation

$$\frac{\partial V(x,t)}{\partial t} - \frac{\alpha\sigma}{c}\left(1 + 2\frac{\partial^2}{\partial x^2} + \frac{4}{3}\frac{\partial^4}{\partial x^4}\right)\frac{\partial V(x,t)}{\partial t}$$
$$= \varepsilon V(x,t) - V^3(x,t) - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 V(x,t) \qquad (18)$$

with the rescaled time $t \to 2t/|f_1|$. This equation represents an extension of the well-known Swift-Hohenberg equation involving a finite propagation speed. In the case of infinite propagation speed $c \to \infty$ the additional second term on the left-hand side of Eq. (18) vanishes and the original model equation is regained. Further, the finite propagation speed leads to the coupling term between spatial and temporal derivatives in Eq. (18) reflecting the space-dependent delay in Eq. (1).

C. The Kuramoto-Sivashinsky equation

Finally let us investigate the Kuramoto-Sivashinsky equation by following the same steps as in the previous examples. Now Eq. (13) reads for small propagation delays $(\lambda\tau)^2 \approx 0$

$$\lambda + \frac{\lambda}{c}(P_0 - k^2 P_2) = s_h + M_0 - ikM_1 - k^2 M_2 + k^4 Q_2$$

with $M_0 = s_f + s_g$, $M_1 = s_f x_0$, $M_2 = s_g \sigma^2/2$, $M_4 = s_g \sigma^4/8$, $P_0 = s_f x_0 + s_g \sqrt{2}\sigma/\sqrt{\pi}$, $P_2 = s_g \sqrt{2}\sigma^3/\sqrt{\pi}$, and $M_n, P_m \to 0$ for $n > 4$, $m > 2$. The stationary solution is taken from (11) and reads $V_0 = 0$ yielding $s_f = 0$, $s_g = g_1$, and the Lyapunov exponent

$$\lambda_{KS} = \frac{-\eta + l^2 - l^4}{1 - \tau\alpha(1 - 4l^2)} \qquad (19)$$

with $\alpha = |g_1|\sqrt{2/\pi}$, $l = \sigma k/2$, and $\lambda_{KS} = \lambda/2|g_1|$. We point out that the rescalings $k \to l$ and $\lambda \to \lambda_{KS}$ are identical to the rescalings of space and time in Sec. III C.

Now let us examine (19) in some more detail. For a vanishing propagation delay $\tau = 0$ λ_{KS} becomes the Lyapunov exponent well known for the original Kuramoto-Sivashinsky model. This means the stationary solution is linear unstable at wave numbers $-1 < l < 1$ while it is linearly stable otherwise. In the case of $\tau \neq 0$, the propagation delay changes the Lyapunov exponent and may destabilize the system. It turns out that there is a critical propagation delay $\tau_{th} = 1/\alpha$ and a corresponding critical propagation speed $c_{th} = \alpha\sigma$. For small propagation delays $\tau < \tau_{th}$, the Lyapunov exponents are modified but the system stability is retained (see Fig. 5). In contrast large propagation delays $\tau > \tau_{th}$ lead to a sign inversion of the Lyapunov exponent for $-l_c < l < l_c$ with $l_c^2 = 1 - 1/\alpha\tau$. Figure 6 summarizes the stability conditions in a diagram. However $\tau \to \tau_{th}$ yields $\lambda \to \infty$ for $l \to l_c$ and thus does not satisfy the previous condition $(\lambda\tau)^2 \approx 0$.
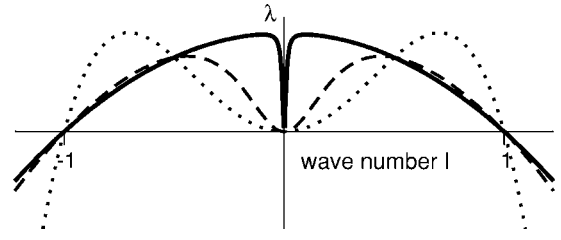

FIG. 5. The Lyapunov exponent of the Kuramoto-Sivashinsky equation subject to the scaled wave number l. The different line styles encode the different propagation delays $\tau = 0$ (dotted line), 2.0 (dashed line), and 2.499 (solid line). $\tau_{th} = 2.5$.

Finally, Eq. (19) may be interpreted as if it originates from the PDE

$$\frac{\partial V(x,t)}{\partial t} - \frac{\sigma\alpha}{c}\left(1 + 4\frac{\partial^2}{\partial x^2}\right)\frac{\partial V(x,t)}{\partial t} = -\eta V - \frac{\partial^2}{\partial x^2}V(x,t)$$
$$- \frac{\partial^4}{\partial x^4}V(x,t) - V\frac{\partial V}{\partial x}. \qquad (20)$$

This equation represents an extension of the Kuramoto-Sivashinsky equation now involving large propagation speeds c. We observe that $c \to \infty$ yields the original Kuramoto-Sivashinsky equation, while the finite propagation speed leads to the additional second term on the left-hand side.

V. CONCLUSION

The present work has introduced a nonlocal integral-differential equation model which generalizes three important partial differential equations. Moreover, we have shown how to consider large propagation speeds in these models and have quantified the propagation delay as the fraction of spatial interaction range and propagation speed. The subsequent linear stability analysis turns out to be dependent on
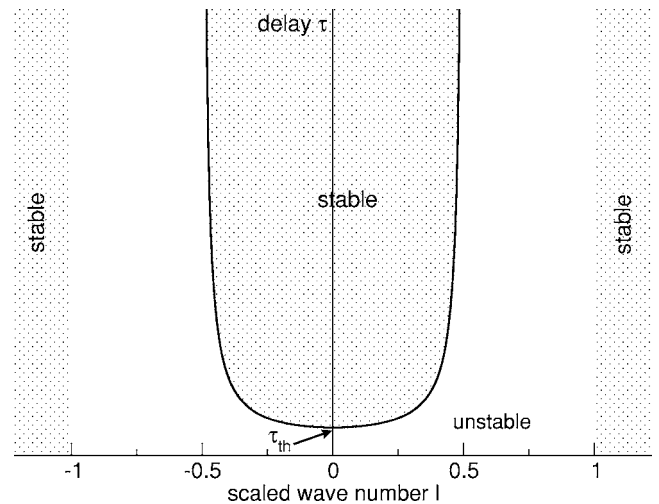

FIG. 6. Illustrated stability diagram of the extended Kuramoto-Sivashinsky equation subject. The center stability region does not exceed the interval $[-0.5, 0.5]$ on the horizontal axis.

this quantity and reveals critical propagation delays and corresponding propagation speeds. We find that large propagation speeds slow down the activity spread in diffusion systems and thus allows for the explanation of non-Fourier behavior in nonhomogeneous systems. Further finite propagation speeds may destabilize the dynamics of the Swift-Hohenberg equation in the stable regime of the original equation, while it may stabilize an occurring Turing instability. Finally, finite propagation speeds may stabilize the dynamics of the Kuramoto-Sivashinsky equation. For all three models partial differential equations have been formulated which incorporate the propagation delay effects. We are convinced that propagation delays play an important role in media showing ultrafast phenomena and that the obtained results will allow for their modeling.

ACKNOWLEDGMENT

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