Adaptive velocity strategy for swarm aggregation

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Collective behaviors of biological swarms have attracted significant interest in recent years, but much attention and efforts have been focused on constant speed models in which all agents are assumed to move with the same constant speed. One limitation of the constant speed assumption without attraction function is that global convergence is quite difficult or even practically impossible to achieve if the speed is relatively fast. In this paper, we propose an adaptive velocity model in which each agent not only adjusts its moving direction but also adjusts its speed according to the degree of direction consensus among its local neighbors. One important feature of the adaptive velocity model is that the speeds of all agents are adaptively tuned to the same maximum constant speed after a short transient process. The adaptive velocity strategy can greatly enhance the global convergence probability, and thus provides a powerful mechanism for coordinated motion in biological and technological multiagent systems.

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I. INTRODUCTION

The emergence of biological swarm is a beauty and wonder in nature [1]. It is common to see huge herds of animals, flocks of birds, or schools of fish moving as if they were a single living creature. Agents in these swarms usually do not share any global information and they often travel in the absence of any leaders or external stimuli. In recent years a variety of efforts have been devoted to modeling and exploring the dynamic properties of such self-propelled systems which can be roughly divided into three approaches: Lagrangian approach [2], Eulerian approach [3], and discrete approach [4–13].

In 1987, Reynolds introduced three heuristic rules—cohesion, separation, and alignment—to create the first computer model of flocking [4]. Later on, Vicsek *et al.* proposed a simplified model which mainly focused on emergence of directional alignment in self-driven particle systems [5]. In recent years, Vicsek model has been one of the frequently investigated swarm models [6–10]. For example, effects of noise and scaling behavior of the model were considered in Ref. [6]. Intermittency and clustering in self-driven particles and the onset of collective motion were studied in Refs. [7] and [8], respectively. Stability analysis of swarms revealed the relationship between network connectivity and stability [9,10]. There are some other models that capture the important rule of directional alignment. For example, Couzin *et al.* showed that the alignment actions together with attraction/repulsion functions between neighboring agents can lead to complex patterns of swarms and revealed the existence of major group-level behavioral transitions [11]. Effective leadership was investigated in Ref. [12] which indicated that the information owned by a few agents in a swarm can be transferred within the whole group. Self-driven many-particle systems with general network topologies such as vectorial network model (VNM) were investigated in Ref. [13].

All these researches assumed that all agents in a group move with the same constant speed (i.e., absolute value of velocity). However, in natural swarms, agents may not only adjust their moving directions but also adjust their speeds according to the behaviors of their neighbors. Indeed, when an agent finds itself in a surrounding of scattered moving agents, it may naturally feel at a loss to follow any direction and probably hesitates to move; in this dilemma case, it is safe for the agent to move with a slow speed. On the other hand, if a certain moving direction is dominant among the neighbors of an agent, the agent may take this direction without hesitation and thus moves relatively fast. A human scale example may be the rhythmic clapping in a concert hall after a good performance, which has been suggested to be formed by lowering the natural clapping frequency of each individual [14].

In this work, we propose an adaptive velocity swarm model in which each agent adjusts its direction and speed simultaneously according to the behaviors of its neighbors. Direction adjustment follows the same rule used in the Vicsek model. To design the speed adjustment rule, we introduce the concept of local order parameter of an agent to measure the local degree of direction consensus among its neighbors. At each time step, each agent moves along the average direction of its neighbors with a speed which is taken as the maximum possible speed scaled by a power-law function of the magnitude of its local order parameter. The adaptive velocity model reduces to the constant speed Vicsek model if the power-law exponent is zero. An important feature of the adaptive velocity model with a positive power-law exponent is that speeds of all agents are adaptively tuned to the same maximum constant speed after a quite short transient process. We show that global convergence probability can be greatly enhanced if the power-law exponent is sufficiently large.

The paper is organized as follows. In Sec. II, we describe in brief the constant speed Vicsek model and compare two order parameters to measure the phase transition phenomena of the swarm. In Sec. III, we propose the adaptive velocity model based on the concept of local order parameter. Simu-

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lation results and discussions are given in Sec. IV. Conclusions are given in the last section.

II. THE CONSTANT SPEED VICSEK MODEL

We first describe the original constant speed Vicsek model [5]. Consider N agents, labeled from 1 through N, all moving synchronously in a square shaped cell of linear size L with periodic boundary conditions. Each agent moves with the same speed v_0 but with different direction at different time step. Originally, agents' positions are randomly distributed in the cell with randomly distributed initial directions in $[0, 2\pi)$. At each time step, agent i adjusts its direction as the average moving direction $\langle \theta_i(k) \rangle_R$ of its neighbors perturbed by a random noise $\Delta\theta$:

$$\theta_i(k+1) = \langle \theta_i(k) \rangle_R + \Delta\theta, \quad i = 1,2,\ldots,N. \tag{1}$$

The neighbors of agent i are defined as those agents who fall in a circle of predefined sensing radius R that centered at the current position of agent i. One characteristic of this homogeneous model is that only by local interactions it shows phase transition through spontaneous symmetry breaking of the rotational symmetry. Different pattern behaviors, such as large-scale emergence, convergence and disordered disperse motion can be observed under different parameters by simulations [5]. The directional rule of local interactions together with constant speed assumption of agents has considerable influences [4–13].

The following order parameter has been widely adopted to measure the phase transition phenomena of the constant speed model from the initial zero net transport to emergence [5,7,11,13,15,16]:

$$\Phi_v(k) = \frac{1}{N v_0} \left| \sum_{i=1}^{N} \vec{v}_i(k) \right|, \quad 0 \le \Phi_v(k) \le 1. \tag{2}$$

Here $\vec{v}_i(k)$ is the velocity of agent i with the constant speed $|\vec{v}_i(k)| = v_0$ and the moving direction $\theta_i(k)$ at time step k. $\Phi_v(k)$ is a univocal physical parameter by definition—a scaled average momentum of the whole system. $\Phi_v = 0$ corresponds to the isotropy state of directional distribution and emergent behavior can be observed if $\Phi_v(k) \gg 0$.

Now suppose that different agents may have different speeds during the evolution. Let v_0 be the average value of all agents' possible maximum speeds, that is, $v_0 = (1/N)\Sigma_{i=1}^{N} v_{i0}$, where v_{i0} is the maximum possible speed of agent i, $i = 1,2,\ldots,N$. In this case, it is possible that $\Phi_v(k) > 0$ even if the moving directions of all agents are isotropic which corresponds to nonemergence state. Further, $\Phi_v(k) = 1$ does not necessarily mean linearly coherence, unless v_{i0} is the same value for all agents. Thus $\Phi_v(k)$ may not be very appropriate to measure the level of emergence.

Another order parameter that has been widely adopted especially for synchronous characteristic in networked phase oscillators is defined as [8,17]

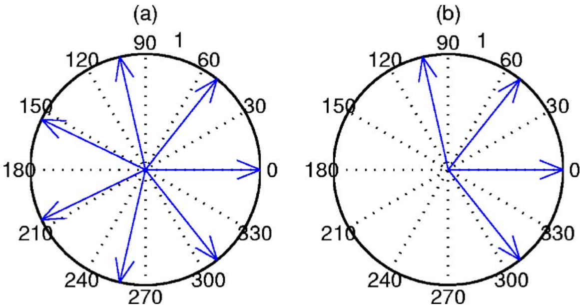

FIG. 1. (Color online) Illustration of local polarity ϕ_i of agent i. The arrows show the moving directional vectors of neighboring agents of agent i. For simplicity, these modular vectors are plotted with the same starting point located in the center of a circle. (a) The collection of agents moving scattered in the plane with no dominant direction, the order parameter $\phi_i \approx 0$ for this situation. (b) The agents with a relatively strong dominant direction, $\phi_i \neq 0$ for this situation. The polarity $\phi_i = 1$ if and only if all the agents in set $\Gamma_i(k)$ move in the same direction.

$$\Phi_\theta(k) = \frac{1}{N} \left| \sum_{i=1}^{N} e^{i\theta_i(k)} \right|, \quad 0 \le \Phi_\theta(k) \le 1. \tag{3}$$

This order parameter eliminates the influence of speed, with the expense of having no physical meaning of scaled average momentum. For constant speed Vicsek model, it is obvious that the two order parameters defined above are equivalent, that is, $\Phi_v \equiv \Phi_\theta$.

III. THE ADAPTIVE VELOCITY MODEL

In this section, we propose an adaptive velocity model in which each agent adjusts its direction and speed simultaneously according to the behaviors of its neighbors. To do so, we first define the complex-valued local order parameter of agent i at time step $(k+1)$ as follows:

$$\phi_i(k+1) e^{\mathrm{i}\theta_i(k+1)} = \frac{1}{n_i(k+1)} \sum_{j \in \Gamma_i(k+1)} e^{\mathrm{i}\theta_j(k)},$$

$$i = 1,2,\ldots,N; \; k = 0,1,2,\ldots, \tag{4}$$

where $e^{i\theta_j(k)}$ is the unit directional vector and $\Gamma_i(k+1)$ is the set of $n_i(k+1)$ neighbors of agent i at time step $(k+1)$. The magnitude (or local polarity) $\phi_i(k+1)$ of the local order parameter measures the local degree of direction consensus among the neighbors of agent i at time step $(k+1)$. Clearly, $0 \le \phi_i(k+1) \le 1$ and larger value of $\phi_i(k+1)$ implies higher degree of local direction consensus among neighbors of agent i (Fig. 1). The angle $\theta_i(k+1) \in [0, 2\pi)$ is the corresponding moving direction of agent i at time step $k+1$, which is the average direction of agents in the neighbor set $\Gamma_i(k+1)$. Computations using this vector form of expression can also avoid some undesired directional problems mentioned in Ref. [9].

Denote $\vec{X}_i(k)$ as the position vector of agent i on the complex plane at time step k. Agent i and agent j are neighbors at

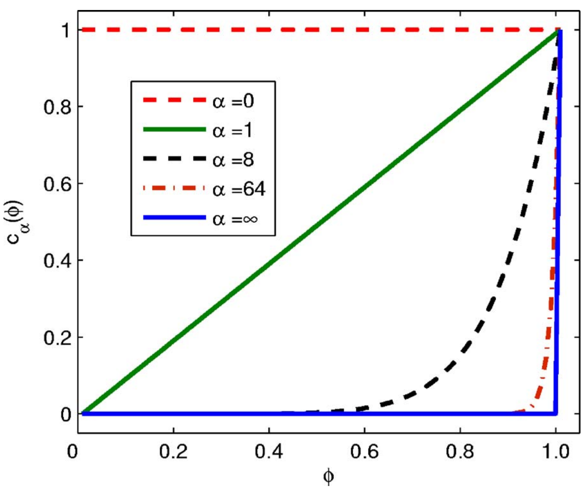

FIG. 2. (Color online) Scaled speed coefficient $c_\alpha(\phi)=\phi^\alpha$ as a power function of local polarity ϕ. $0\leq c_\alpha(\phi)\leq 1$. For any value of α, $c_\alpha(\phi)=1$ if $\phi=1$. For $\alpha=0$, $c_\alpha(\phi)\equiv 1$. For $0<\alpha<\infty$, $0<c_\alpha(\phi)<1$ if $0<\phi<1$. For $\alpha=\infty$, $c_\alpha(\phi)=0$ if $0<\phi<1$.

time step k if and only if $\|\vec{X}_i(k)-\vec{X}_j(k)\|\leq R$. In our adaptive velocity model, each agent does not only adjust its moving direction, but also adjusts its speed according to the degree of local direction consensus among its neighbors. Specifically, the speed of agent i at time step k is its maximum possible speed v_0 scaled by the power-law function $\phi_i^\alpha(k)$ with an exponent $\alpha\geq 0$ (Fig. 2). The adaptive velocity model can then be described mathematically as follows:

$$\vec{X}_i(k+1)=\vec{X}_i(k)+v_0\times\phi_i^\alpha(k)e^{i\theta_i(k)}\times\Delta t,$$

$$\phi_i(k+1)e^{i\theta_i(k+1)}=\frac{1}{n_i(k+1)}\sum_{j\in\Gamma_i(k+1)}e^{i\theta_j(k)},$$

$$i=1,2,\ldots,N;\ k=0,1,2,\ldots, \tag{5}$$

where Δt is the discrete time interval, and without loss of generality we take $\Delta t=1$. $\vec{v}_i(k)\equiv v_0\times\phi_i^\alpha(k)e^{i\theta_i(k)}$ represents the velocity of agent i at time step k with speed $|\vec{v}_i(k)|=v_0\times\phi_i^\alpha(k)$ and direction $\theta_i(k)$. Since $0\leq\phi_i^\alpha(k)\leq 1$ for any value of $\alpha\geq 0$, we have $0\leq|\vec{v}_i(k)|\leq v_0$.

The power-law exponent $\alpha\geq 0$ reflects the willingness of each agent to move faster or slower along the average direction of its neighbors based on the local degree of direction consensus. If $\alpha=0$, then $\phi_i^\alpha(k)\equiv 1$, the adaptive velocity model (5) reduces to the constant speed Vicsek model and each agent always moves with the maximum constant speed v_0 without any consideration about its local polarity. However, if $\alpha>0$, then an agent will move with the maximum constant speed if and only if complete local direction consensus is achieved among its neighbors. A large value of α implies that an agent will move with a slow speed in the face of a given value of local polarity. In the limit case we have

$$\lim_{\alpha\to+\infty}\phi_i(k)^\alpha=\begin{cases}0, & 0\leq\phi_i(k)<1,\\ 1, & \phi_i(k)=1.\end{cases} \tag{6}$$

It means that each agent will not move unless complete local direction consensus is achieved among its neighbors.

The adaptive velocity model satisfies four fundamental characteristics of the constant speed Vicsek model: (i) no leader, (ii) no external stimuli, (iii) only homogeneous agents, and (iv) only local interactions. However, we will show that the adaptive velocity model with a positive exponent α induces more intensified phase transition and symmetry-breaking from disordered to ordered state than the constant speed model.

The two-dimensional adaptive velocity model (5) can easily be generalized to the general M-dimensional Euclidean space case. Let $\vec{X}_i=[x_{i1},x_{i2},\ldots,x_{iM}]^T$ be the position vector of agent i, with $x_{ij}\in R$ for $i=1,2,\ldots,N$ and $j=1,2,\ldots,M$. The motion direction of agent i is represented by a unitary vector $\vec{d}_i=[d_{i1},d_{i2},\ldots,d_{iM}]^T$, which satisfies $\|\vec{d}_i\|=1$, $d_{ij}\in R$, $-1\leq d_{ij}\leq 1$, $j=1,2,\ldots,M$ for all i. Agent i and agent$j$ are neighbors if and only if the Euclidean norm $\|\vec{X}_i(k)-\vec{X}_j(k)\|\leq R$. Define the scalar order parameter as

$$r_i(k+1)=\frac{1}{n_i(k+1)}\left\|\sum_{j\in\Gamma_i(k+1)}\vec{d}_j(k)\right\|. \tag{7}$$

Of course, $0\leq r_i(k+1)\leq 1$. The M dimensional adaptive velocity model can be described as

$$\vec{X}_i(k+1)=\vec{X}_i(k)+v_0\times r_i^\alpha(k)\times\vec{d}_i(k)\times\Delta t,\quad k=0,1,2,\ldots, \tag{8}$$

$$\vec{d}_i(k+1)=\left(\sum_{j\in\Gamma_i(k+1)}\vec{d}_j(k)\right)\Bigg/\left\|\sum_{j\in\Gamma_i(k+1)}\vec{d}_j(k)\right\|,$$

$$k=0,1,2,\ldots\ . \tag{9}$$

IV. SIMULATIONS AND DISCUSSIONS

To illustrate the effect of adaptive velocity strategy, we consider N agents moving in the whole complex 2D plane without boundary restrictions instead of in a rectangle of open boundary or periodic boundary conditions [5]. The positions of N agents are initially randomly distributed on a region of $L\times L$ rectangle with random initial directions in the interval $[0,2\pi)$. Note that this rectangle does not represent the boundary for swarm motion, but only restricts the initial distribution of positions of agents. Recent studies have shown that for the swarm which move in the plane instead of in a rectangle of periodic conditions, convergence or emergence is due to the connectivity between agents [9,10], instead of long-range interactions [5,18].

Denote the initially distributed direction and position of agent i as θ_i and $\vec{X}_i(0)$, respectively. We compute the initial moving direction $\theta_i(0)$ and initial polarity $\phi_i(0)$ of agent i according to the local order parameter formula

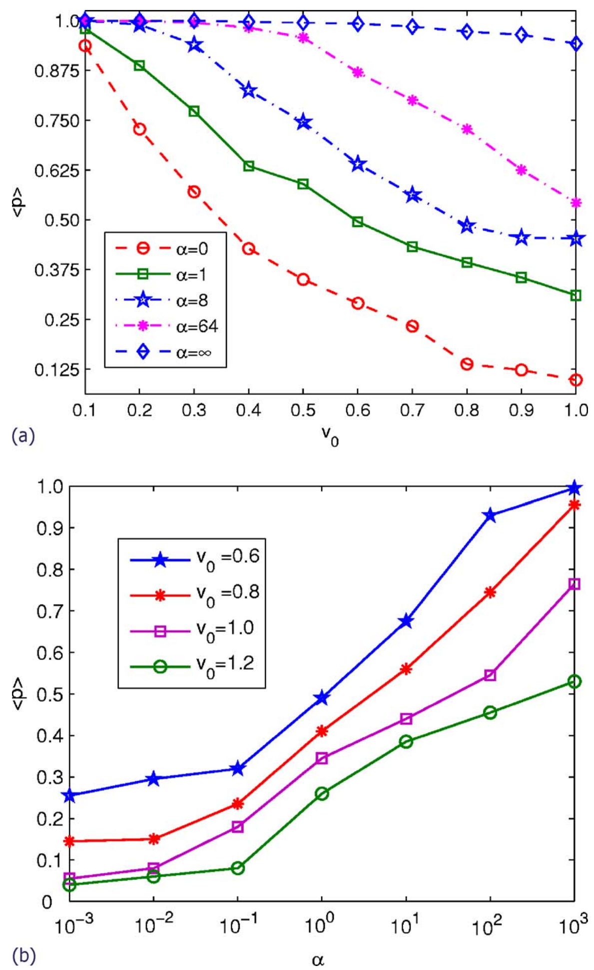

FIG. 3. (Color online) (a) Convergence probability p as a function of the maximum speed v_0 for five different values of α. (b) Convergent probability p as a function of the exponent α for five different values of v_0. All estimates are the results of averaging over 400 realizations.

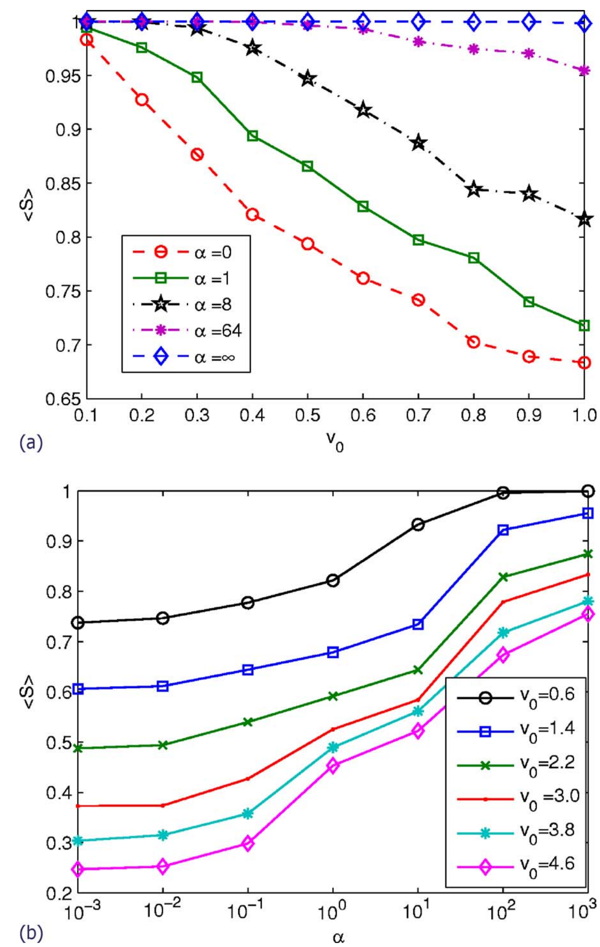

FIG. 4. (Color online) (a) The relative size S in steady state as a function of the maximum speed v_0 for different values of α. (b) The relative size S in steady state as a function of the exponent α for different values of v_0. All estimates are the results of averaging over 400 realizations.

$$\phi_i(0)e^{i\theta_i(0)} = \frac{1}{n_i(0)} \sum_{j \in \Gamma_i(0)} e^{i\theta_j}, \quad i = 1,2, \ldots, N.$$

This means that each agent moves with adaptive velocity strategy at the very beginning of its evolution. This beginning time step is denoted as step $k=0$ with the corresponding initial speed $v_0 \times \phi_i(0)$ for agent i.

We first investigate the influence of power-law exponent α in the adaptive velocity model on the convergence probability p which is defined as the probability that a group of N initially randomly distributed agents will finally all move along a global consensus direction with the same maximum speed v_0. In simulations, we take $N=300$, $L=5$, and $R=2$ if without further indication. Similar results can be derived for other values of these parameters. Figure 3(a) shows that for any given value of α, the convergence probability p is a decreasing function of the maximum speed v_0, but it decreases slower for larger value of α; while Fig. 3(b) shows that for any given value of v_0, the convergence probability p is an increasing function of the exponent α, and smaller v_0 leads to higher convergence probability. Therefore, if the constant speed v_0 is large enough, even though it is very difficult or even practically impossible to achieve global convergence in the original Vicsek model which corresponds to $\alpha=0$, the convergence probability may still be high for the adaptive velocity model with a sufficiently large α. In particular, the convergence probability is very close to 1 in the case of $\alpha=\infty$ for the present system parameters, even without any leader or other global information in the adaptive velocity model.

From the perspective of complex network theory [19,20], the swarm topology at time step k can be expressed as an undirected graph $G(k)-[V,E(k)]$: an agent i is represented by a vertex v_i in graph $G(k)$; an undirected edge between agent i and agent j means that they are neighbors. A component of a graph to which a vertex belongs is that set of vertices that can be reached from it by paths running along edges of the graph [19]. As time evolves, topology of the graph $G(k)$ may vary until the model evolves to a steady state. We are interested in the relative size S of the maximal component (i.e., largest cluster) in steady state, which is defined as the ratio of the number of agents within the maximal

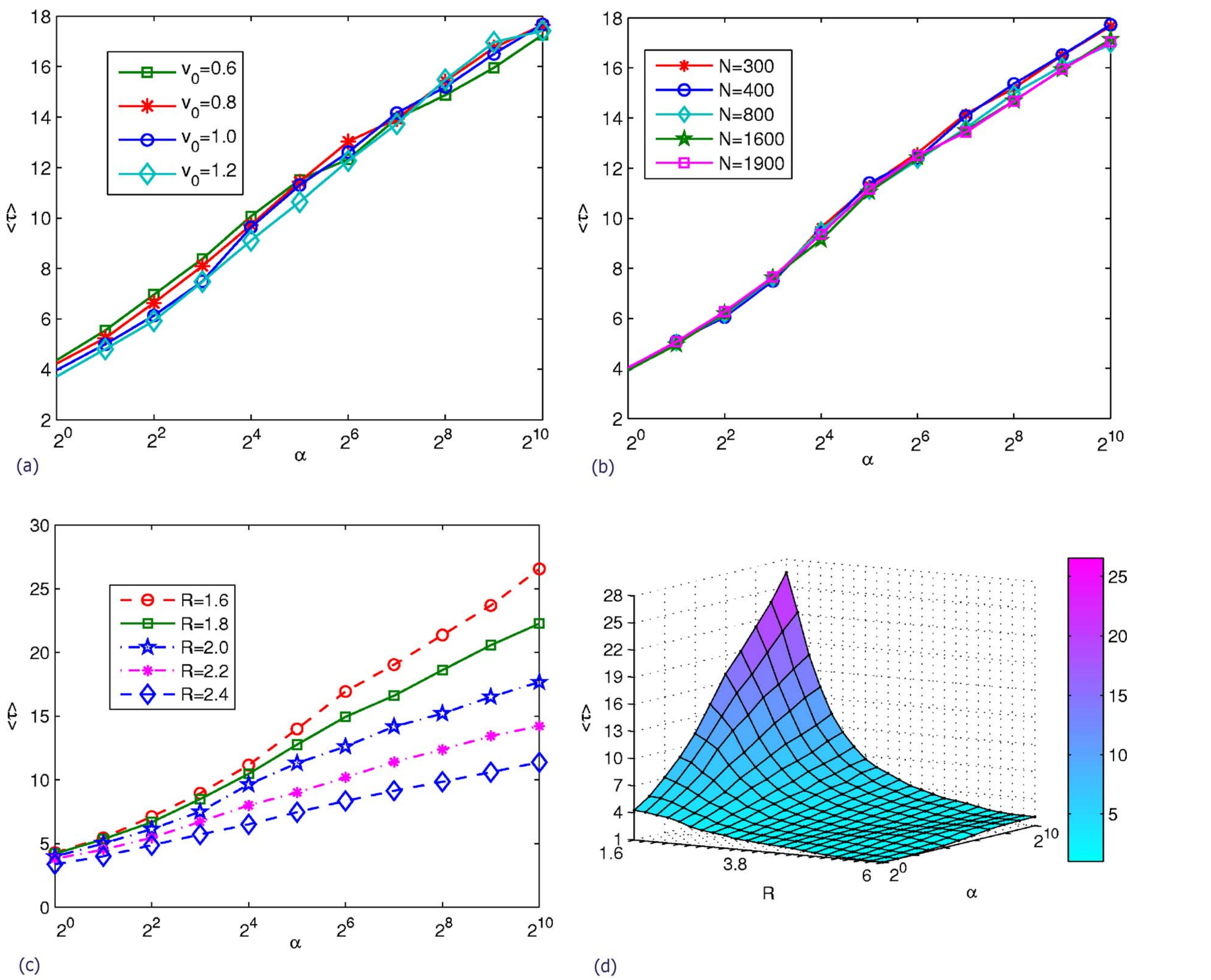

FIG. 5. (Color online) (a) The transient time τ as a function of α for different values of v_0. (b) The transient time τ as a function of α for different values of N, $v_0=1.0$. (c) The transient time τ as a function of α for different values of R, $v_0=1.0$. (d) The transient time τ as a function of R and α, $v_0=1.0$. All estimates are the results of averaging over 400 realizations.

component to the total number of agents in the whole group. Clearly, $0<S\leq 1$. $S\approx 0$ means all the agents disperse away without any apparent clusters. For $S\gg 0$, there exists a giant cluster in the swarm. Global convergence is achieved if and only if $S=1$. In this case, the whole graph consists of only one component.

For any given value of α, S is understandably a decreasing function of the maximum speed v_0, and it decreases much slower for larger value of α [Fig. 4(a)]; On the other hand, for any given value of v_0, S is an increasing function of the exponent α and smaller value of v_0 results in higher value of S [Fig. 4(b)]. Thus, in the case of a large maximum speed v_0, although it is quite difficult or even impossible to form a giant cluster in the constant speed Vicsek model which corresponds to $\alpha=0$, it is much easier to form a giant cluster for the adaptive velocity model if α is large enough.

Why is the convergence probability and size of the largest cluster enhanced as the exponent α increases in the adaptive velocity model? This may be due to the fact that the adaptive velocity strategy with large value of α tends to hold the local agents together to form a large cluster. Since the initial directions of agents are randomly distributed, most agents are in nearly isotropy regions at the beginning and their local order parameters $\phi\approx 0$, which implies that most agents are in a circumstance of neighbors with scattered moving directions and the speeds of these agents are relatively small according to the adaptive velocity strategy with $\alpha>0$. In fact, even for moderate or relatively large polarity, i.e., $0\ll\phi<1$, the speeds of these agents are still small for sufficiently large value of α. Thus transformations of these agents' positions are relatively indistinctive, which implies that neighbors tend to be also neighbors in next time step or even after, and communications for directional alignment actions between them continue to be held which are benefits to directional consensus.

Even though the directions of agents will reach global consensus only under certain conditions, speeds of all agents will always reach the same maximal value v_0 in steady state whether the swarm can finally converge or not. As stated above, in the case that the local polarity of an agent is large but less than one, the speed of the agent will still be small for sufficiently large value of α. Thus, one may guess that the

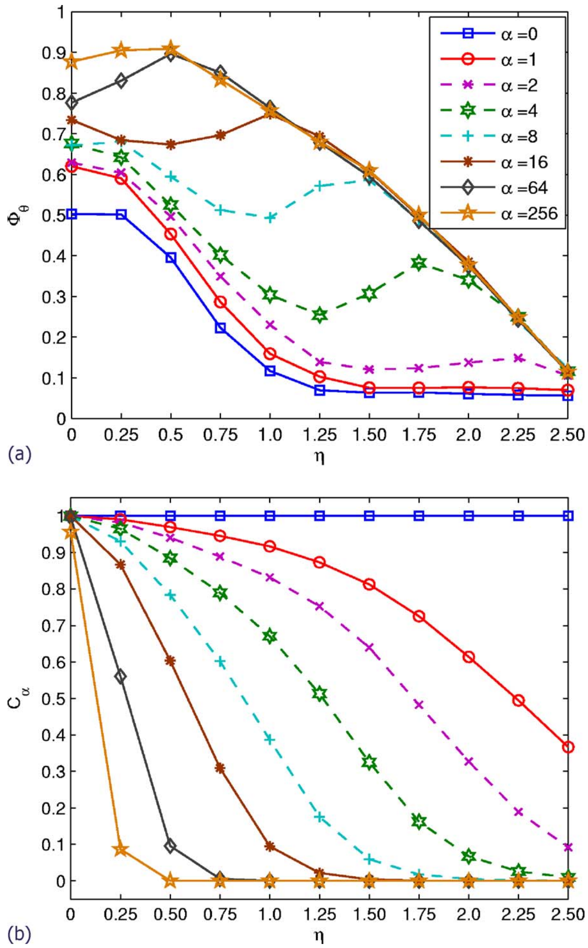

FIG. 6. (Color online). (a) The global order parameter Φ_θ as a function of noise amplitude η. (b) The average speed coefficient C_α as a function of noise amplitude η. $v_0=0.6$. $\eta \in [0,2.5]$(rad). All estimates are the results of averaging over 200 realizations.

required time for the group to achieve steady state may be quite long for large value of α. However, we find that the steady state can be achieved after a quite short transient process. Denote the transient time τ as the required time step that the average speed of all agents reaches 98% of maximum speed v_0. We find that, generically, τ increases logarithmically with α (Fig. 5):

$$\tau \approx \beta + \gamma \log_2 \alpha, \quad \alpha \geq 1. \tag{10}$$

Figures 5(a) and 5(b) show that τ does not distinctly depend on the maximum speed v_0 and the density N/L^2. In both cases, we have $\beta \approx 4.0$ and $\gamma \approx 1.4$. Therefore, even for a very high value of $\alpha = 1024$, most agents will move with nearly the maximum speed in just about 18 steps. Figures 5(c) and 5(d) indicate that the coefficients β and γ in Eq. (10) decrease as the sensing radius R increases. In particular, as R approaches 6, agents in the group are approximately globally coupled, β approaches 1 which implies that steady state can be achieved in almost one step.

We now briefly investigate the influence of noise via the global order parameter Φ_θ defined in Eq. (3) and the average speed coefficient C_α defined as

$$C_\alpha = \frac{1}{N}\sum_{i=1}^{N} c_\alpha(\phi_i) = \frac{1}{N}\sum_{i=1}^{N} \phi_i^\alpha. \tag{11}$$

C_α reflects the average ratio of all agents' speeds compared to the maximum speed v_0. Suppose that the moving direction of each agent is perturbed by a random number ξ chosen with a uniform probability from the interval $[-\eta, \eta]$. In this case, the local order parameter is computed as follows:

$$\phi_i(k+1)e^{i\theta_i(k+1)} = \frac{1}{n_i(k+1)} e^{i\xi} \sum_{j \in \Gamma_i(k+1)} e^{i\theta_j(k)},$$

$$k = 0,1,2, \ldots . \tag{12}$$

We can see from Fig. 6(a) that for large noise amplitude η and large exponent α, the global order parameter Φ_θ decreases along the same straight line. Even though the average speed coefficient C_α always equals one for $\alpha=0$, it decreases to zero with a faster speed as the noise amplitude η increases for larger value of $\alpha > 0$ [Fig. 6(b)].

V. CONCLUSIONS

As a generalization of the constant speed Vicsek model, we propose an adaptive velocity swarm model in which each agent not only adjusts its moving direction but also adjusts its speed based on the local degree of direction consensus among its neighbors at every time step. We show that high convergence probability can be achieved if each agent takes more consideration about its local polarity.

Some difficult yet important problems about the adaptive velocity model remain to be further investigated. For example, under what conditions can we derive a critical value of α such that above the value a given convergence probability or relative size of largest cluster can be guaranteed? Even if we can prove the existence of the critical value, how can each agent know this value? What if different agents have different values of α? Furthermore, stability analysis about the linearized Vicsek's model has been focused on the connectivity of the graph during the evolution process which is impossible to check from the initial condition [9]. Practical stability analysis of the adaptive velocity model needs to be explored.

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