Feynman perturbation expansion for the price of coupon bond options and swaptions in quantum finance. I. Theory

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European options on coupon bonds are studied in a quantum field theory model of forward interest rates. Swaptions are briefly reviewed. An approximation scheme for the coupon bond option price is developed based on the fact that the volatility of the forward interest rates is a small quantity. The field theory for the forward interest rates is Gaussian, but when the payoff function for the coupon bond option is included it makes the field theory nonlocal and nonlinear. A perturbation expansion using Feynman diagrams gives a closed form approximation for the price of coupon bond option. A special case of the approximate bond option is shown to yield the industry standard one-factor HJM formula with exponential volatility.

I. INTRODUCTION

Coupon bonds and interest rate swaps are primary financial instruments. Options on coupon bonds and swaps are amongst the most widely traded of financial instruments. The Bank of International Settlements (Switzerland) estimated that in 2001 the notional value of the swap market was approximately 40 trillion dollars and that of the combined interest rate caps and swaptions market was about 9 trillion dollars.

Quantum finance [1] refers to the application of the formalism of quantum mechanics and quantum field theory to finance. The pricing of European options on coupon bonds is studied in some detail using the approach of quantum finance. The volatility of the forward interest rates is a small quantity, of the order of 10^{-2}/year, and hence provides a small parameter for obtaining an approximate value for the option price. The approximation is based on a field theory formulation of the forward interest rates [1]. A perturbation expansion using Feynman diagrams is developed for the option price as a power series in the forward interest rates' volatility. The expansion is convergent, and does not have any of the divergences that usually appear in quantum field theory [2] since the propagators for the Feynman diagrams are all nonsingular at short distances.

In Sec. II the field theory formulation of the forward interest rates is briefly reviewed. Section III discusses the general properties of coupon bonds options. Section IV discusses interest rate swaps and swaptions. In Sec. V the bond option price is expressed as a path integral.

Section VI contains the main result of this paper; a self-consistent perturbation expansion using Feynman diagrams is developed for coupon bond options, and shown to fulfill the consistency relation required by put-call parity. In Sec. VII the results of the perturbation expansion are compared with the special case of the zero coupon bond, and shown to give the correct result. The approximate value of the option price is numerically compared with the exact value of the option for a zero coupon bond, and shown to be very accu-

rate. In Sec. VIII the HJM limit of the field theory result is discussed in some detail. The industry standard one-factor HJM price for a coupon bond option is seen to be a special and nontrivial limit of the approximate field theory option price. In Sec. IX some conclusions are discussed. The Appendixes contain details of the more mathematical derivations.

II. FIELD THEORY MODEL OF FORWARD INTEREST RATES

The quantum field theory of forward interest rates is a general framework for modeling the interest rates that provides a particularly transparent and computationally tractable formulation of interest rate instruments.

A Treasury Bond is defined to be an instrument that gives a predetermined payoff of say $1 when it matures at some fixed time T; it's price at earlier time $t < T$ is given by $B(t, T)$. The price of a Treasury Bond is given by *discounting* the payoff of $1, paid at time T, to present time t by using the prevailing forward interest rates.[1] Forward interest rates $f(t, x)$ are the interest rates, fixed at time t, for an instantaneous loan at future times $x > t$; $f(t, x)$ has the dimensions of 1/time. Discounting the $1 payoff, paid at maturity time T, is obtained by taking infinitesimal backward time steps ϵ from T to present time t, and yields[2]

$$B(t, T) = e^{-\epsilon f(t, t+\epsilon)} e^{-\epsilon f(t, t+2\epsilon)} \cdots e^{-\epsilon f(t, x)} \cdots e^{-\epsilon f(t, T)} \$ 1$$

$$\Rightarrow B(t, T) = \exp\left(-\int_t^T dx f(t, x)\right).$$

Suppose a Treasury Bond $B(t_*, T_i)$ is going to be issued at some future time $t_* > t_0$, and expires at time T_i; the forward price of the Treasury Bond is the price that one pays at time

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[1]The term discounting is fundamental to finance. Consider the interest on a fixed deposit that is rolled over; this leads to an exponential compounding of the initial fixed deposit. Discounting, the inverse of the process of compounding, is the procedure that yields the present day value of a future prefixed sum of money.

[2]The fixed payoff $1 is assumed and is not written out explicitly.
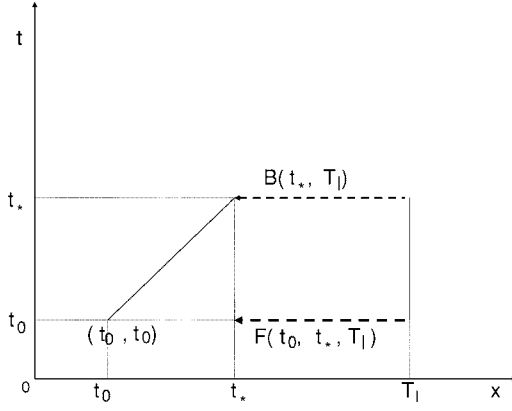
FIG. 1. The forward interest rates, indicated by the dashed lines, that define a Treasury Bond $B(t_*, T_i)$ and it's forward price $F(t_0, t_*, T_i)$.

t to lock-in the delivery of the bond when it is issued at time t_*, and is given by

$$F(t_0, t_*, T_i) = \exp\left(-\int_{t_*}^{T_i} dx f(t_0, x)\right)$$

$$= \frac{B(t_0, T)}{B(t_0, t_*)} : \text{forward bond price.} \quad (1)$$

Treasury Bond $B(t_*, T_i)$, to be issued at time t_* in the future, is graphically represented in Fig. 1, together with its (present day) forward price $F(t_0, t_*, T_i)$ at $t_0 < t_*$.

Forward interest rates $f(t, x)$ are related to the two-dimensional stochastic (random) field $\mathcal{A}(t, x)$ that drives the time evolution of the forward interest rates, and is given by

$$\frac{\partial f(t, x)}{\partial t} = \alpha(t, x) + \sigma(t, x)\mathcal{A}(t, x). \quad (2)$$

The drift of the forward interest rates $\alpha(t, x)$ is fixed by a choice of numeraire [1,3], and $\sigma(t, x)$ is the volatility function that is fixed from the market [1].

The value of all financial instruments are given by averaging the stochastic field $\mathcal{A}(t, x)$ over all it's possible values. This averaging procedure is formally equivalent to a quantum field theory in imaginary (Euclidean) time and hence, in effect, $\mathcal{A}(t, x)$ is equivalent to a two-dimensional quantum field.

Integrating Eq. (2) yields

$$f(t, x) = f(t_0, x) + \int_{t_0}^{t} dt' \alpha(t', x) + \int_{t_0}^{t} dt' \sigma(t', x)\mathcal{A}(t', x), \quad (3)$$

where $f(t_0, x)$ is the initial forward interest rate that is specified by the market.

One is free to choose the dynamics of the quantum field $\mathcal{A}(t, x)$. Following Baaquie and Bouchaud [4,5], the Lagrangian that describes the evolution of instantaneous for-

ward rates is defined by three parameters μ, λ, η and is given by[3]

$$\mathcal{L}(A) = -\frac{1}{2}\left[\mathcal{A}^2(t, z) + \frac{1}{\mu^2}\left(\frac{\partial \mathcal{A}(t, z)}{\partial z}\right)^2 + \frac{1}{\lambda^4}\left(\frac{\partial^2 \mathcal{A}(t, z)}{\partial^2 z}\right)^2\right], \quad (4)$$

where market (psychological) future time is defined by $z = (x - t)^\nu$.

A more general Gaussian Lagrangian, which will be useful in studying the empirical behavior of swaptions, is nonlocal in future time z and has the form

$$\mathcal{L}(A) = -\frac{1}{2}\mathcal{A}(t, z)N^{-1}(t, z, z')\mathcal{A}(t, z'). \quad (5)$$

The action $S[\mathcal{A}]$ of the Lagrangian is defined as

$$S[\mathcal{A}] = \int_{t_0}^{\infty} dt \int_0^{\infty} dz dz' \mathcal{L}(A). \quad (6)$$

The market value of all financial instruments based on the forward interest rates are obtained by performing a path integral over the (fluctuating) two-dimensional quantum field $\mathcal{A}(t, z)$. The expectation value for an instrument, say $F[\mathcal{A}]$, is denoted by $\langle F[\mathcal{A}]\rangle \equiv E[F[\mathcal{A}]]$ and is defined by the functional average over all values of $\mathcal{A}(t, z)$, weighted by the probability measure e^S/Z. Hence

$$\langle F[\mathcal{A}]\rangle \equiv E[F[\mathcal{A}]] \equiv \frac{1}{Z}\int D\mathcal{A}F[\mathcal{A}]e^{S[\mathcal{A}]}, \quad Z = \int D\mathcal{A}e^{S[\mathcal{A}]}. \quad (7)$$

The quantum theory of the forward interest rates is defined by the generating (partition) function [1] given by

$$Z[h] = E(e^{\int_{t_0}^{\infty} dt \int_0^{\infty} dz h(t, z)\mathcal{A}(t, z)})$$

$$\equiv \langle e^{\int_{t_0}^{\infty} dt \int_0^{\infty} dz h(t, z)\mathcal{A}(t, z)}\rangle$$

$$\equiv \frac{1}{Z}\int D\mathcal{A}e^{S[\mathcal{A}] + \int_{t_0}^{\infty} dt \int_0^{\infty} dz h(t, z)\mathcal{A}(t, z)}$$

$$= \exp\left(\frac{1}{2}\int_{t_0}^{\infty} dt \int_0^{\infty} dz dz' h(t, z)D(z, z'; t)h(t, z')\right) \quad (8)$$

which follows from the correlator of the $\mathcal{A}(t, x)$ quantum field given by

$$\langle \mathcal{A}(t, z)\mathcal{A}(t', z')\rangle = E[\mathcal{A}(t, z)\mathcal{A}(t', z')] = \delta(t - t')D(z, z'; t). \quad (9)$$

For simplicity of notation $\langle F[\mathcal{A}]\rangle$ will be used for denoting expectation values and only the case of $\nu = 1$ will be considered; all integrations over z are replaced with those over future time x. For $\nu = 1$ from Eq. (6) the dimension of

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[3]More complicated nonlinear Lagrangians have been discussed in Ref. [1].

the quantum field $\mathcal{A}(t,x)$ is 1/time and from Eq. (3) the volatility $\sigma(t,x)$ of the forward interest rates also has dimension of 1/time.

The expression for $D(x,x';t)$ for the Lagrangian given in Eq. (5) has been studied in Refs. [4,5], and provides a very accurate description of the correlation of the forward interest rates. In the present paper the explicit value of the propagator $D(x,x';t)$ will not be used. Instead only the Gaussian property of the Lagrangian will be used. The effective market propagator, which describes the market behavior of swaptions, is given by $M(x,x';t) = \sigma(t,x)N(x,x';t)\sigma(t,x')$; for empirical studies carried out in Ref. [6] $M(x,x';t)$ is evaluated directly from the market data.

III. EUROPEAN COUPON BOND OPTIONS

Consider a coupon bond on a principal L that matures at time T, and pays fixed dividends (coupons) a_i at times T_i, $i = 1, 2, \ldots, N$. The value of the coupon bond at time $t_* < T_i$ is given by

$$\sum_{i=1}^{N} a_i B(t_*, T_i) + LB(t_*, T) = \sum_{i=1}^{N} c_i B(t_*, T_i), \quad (10)$$

where for simplicity of notation the final payment is included in the sum by setting $c_i = a_i$; $c_N = a_N + L$, and with the time of maturity of the coupon bond given by $T = T_N$.

The payoff function $S(t_*)$ of a European call option maturing at time t_*, for strike price K, is given by

$$S(t_*) = \left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K \right)_+, \quad (11)$$

where

$$(a - b)_+ \equiv (a - b)\Theta(a - b) \quad (12)$$

and the Heaviside step function $\Theta(x)$ is defined by

$$\Theta(x) \equiv \begin{cases} 1, & x > 0, \\ \frac{1}{2}, & x = 0, \\ 0, & x < 0. \end{cases} \quad (13)$$

The price of a European call option at time $t_0 < t_*$ is given by discounting the payoff $S(t_*)$ from time t_* to time t. Any measure that satisfies the martingale property can be used for this discounting [1]; in particular the money market numeraire is given by $\exp[\int r(t)dt]$ where $r(t) = f(t,t)$ is the spot interest rate. Discounting the payoff function by the money market numeraire yields the following price of a European call and put options:

$$C(t_0, t_*, K) = E[e^{-\int_{t_0}^{t_*} dt r(t)} S(t_*)]$$

$$= E\left[e^{-\int_{t_0}^{t_*} dt r(t)} \left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K \right)_+ \right],$$

$$P(t_0, t_*, K) = E\left[e^{-\int_{t_0}^{t_*} dt r(t)} \left(K - \sum_{i=1}^{N} c_i B(t_*, T_i) \right)_+ \right]. \quad (14)$$

In particular, Treasury Bonds are martingales for the money market numeraire; hence

$$E[e^{-\int_{t_0}^{t_*} dt r(t)} B(t_*, T)] = B(t_0, T) \quad (15)$$

and yields the money market drift for the forward interest rates [see Eq. (2)] given by [1]

$$\alpha_M(t,x) = \int_t^x dx' M(x,x';t) : \text{money market drift,}$$

$$M(x,x';t) \equiv \sigma(t,x)N(x,x';t)\sigma(t,x'). \quad (16)$$

Put-call parity is a relationship between a call and a put option on any instrument, and is independent of how one models the price of an instrument or its derivative. Put-call parity follows directly from the market prices of traded instruments. The fact that the prices of the traded instruments do not allow for arbitrage can be shown to imply the model-independent put-call parity relationship.

Put-call parity for Libor derivatives and swaps have been discussed in Ref. [3]; the general results are applied to the specific case of the coupon bond option.

The derivation of put-call parity hinges on the identity, which follows from Eq. (13), that

$$\Theta(x) + \Theta(-x) = 1 \quad (17)$$

since it yields

$$(a - b)_+ - (b - a)_+ = (a - b)\Theta(a - b) - (b - a)\Theta(b - a)$$

$$= a - b. \quad (18)$$

The difference in the call and put payoff functions for the coupon bond option, using Eq. (18), satisfies

$$\left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K \right)_+ - \left(K - \sum_{i=1}^{N} c_i B(t_*, T_i) \right)_+$$

$$= \sum_{i=1}^{N} c_i B(t_*, T_i) - K.$$

Multiplying both sides by $\exp[-\int_{t_0}^{t_*} dt r(t)]$ and taking the expectation value yields the put-call parity relation given by

$$C(t_0, t_*; K) - P(t_0, t_*; K)$$

$$= E\left[e^{-\int_{t_0}^{t_*} dt r(t)} \left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K \right) \right]$$

$$= \sum_{i=1}^{N} c_i B(t_0, T_i) - KB(t_0, t_*).$$

The martingale condition for the money market measure given in Eq. (15) has been used to obtain the last line above. Put-call parity yields the expected result that

$$C(t_0, t_*; K) - P(t_0, t_*; K)$$

$$= \sum_{i=1}^{N} c_i B(t_0, T_i) - KB(t_0, t_*) : \text{put-call parity.} \quad (19)$$

The right-hand side is difference, at time t_0, between the

value of the coupon bond portfolio and the discounted value of the strike price K.

The necessity of satisfying put-call parity for the European coupon bond option puts constraints on any perturbation expansion for the price of the options: the expansion needs to satisfy put-call parity order by order.

IV. SWAPS

An interest rate swap is contracted between two parties. Payments are made at fixed times T_n and are separated by time intervals ℓ, which is usually 90 or 180 days. The swap contract has a notional principal V, with a prefixed period of total duration and with the last payment being made at time T_N. One party pays, on the notional principal V, a fixed interest rate denoted by R_S and the other party pays a floating interest rate based on the prevailing market rate, or vise versa. The floating interest rate is usually determined by the prevailing value of Libor (see below) at the time of the floating payment.

In the market, the usual practice is that floating payments are made every 90 days whereas fixed payments are made every 180 days; for simplicity of notation we will only analyze the case when both fixed and floating payments are made on the same day.

The forward interest rates $f(t,x)$ for swaps and swaptions are determined from Libor, the London Interbank Offer Rate [1]. Libor is a simple interest rate derived from US\$ deposits outside the United States. The minimum deposit for Libor is a par value of US\$1,000,000. Libor are interest rates for which commercial banks are willing to lend or borrow funds in the interbank market.

Let $L(t,T_n)$ be the Libor (simple interest rate), at time t, for a $\ell = 90$ days loan of US\$1,000,000 starting at future time T_n. The forward interest rates $f(t,x)$ are given by the following relation:[4]

$$L(t,T_n) = \frac{e^{\int_{T_n}^{T_n+\ell} dx f(t,x)} - 1}{\ell}.$$

A swap of the first kind, namely swap$_I$, is one in which a party pays at fixed rate R_S and receives payments at the floating rate [7]. Hence, at time T_n the value of the swap is the difference between the floating payment received at the rate of $L(t,T_n)$, and the fixed payments paid out at the rate of R_S. All payments are made at time $T_n+\ell$, and hence need to be discounted by the bond $B(T_0,T_n+\ell)$ for obtaining its value at time T_0. Similarly, swap$_{II}$—a swap of the second kind—is one in which the party holding the swap pays at the floating rate and receives payments at fixed rate R_S.

Consider a swap that starts at time T_0 and ends at time $T_N=T_0+N\ell$, with payments being made at times $T_0+n\ell$, with $n=1,2,\ldots,N$. The value of the swaps are given by [3,7]

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[4]Forward interest rates derived from Libor carry a small element of credit risk that is not present in the forward interest rates derived from zero risk US Treasury Bonds; in this paper the difference is considered negligible and ignored.

$$S_I(T_0,R_S) = V\left(1 - B(T_0,T_0+N\ell) \right.$$
$$\left. - \ell R_S \sum_{n=1}^{N} B(T_0,T_0+n\ell) \right),$$

$$S_{II}(T_0,R_S) = V\left(\ell R_S \sum_{n=1}^{N} B(T_0,T_0+n\ell) + B(t,T_0+N\ell) - 1 \right),$$
$$(20)$$

where holder of S_I receives at the floating interest rate and pays at a fixed interest rate of R_S, and similarly for S_{II}. Note that, since $S_I+S_{II}=0$, an interest swap is a zero sum game, with the gain of one party being equal to the loss of the other party.

The par value of the swap when it is initiated at time T_0 is zero; hence the par fixed rate R_P, from Eq. (24), is given by

$$S_I(T_0,R_P) = 0 = S_{II}(T_0,R_P) \Rightarrow \ell R_P = \frac{1 - B(T_0,T_0+N\ell)}{\sum_{n=1}^{N} B(T_0,T_0+n\ell)}.$$

The forward swap or a deferred swap, similar to the forward price of a Treasury Bond, is a swap entered into at time $t_0 < T_0$, and it's price is given by [3]

$$S_I(t_0;T_0,R_S) = V\left(B(t_0,T_0) - B(t_0,T_0+N\ell) \right.$$
$$\left. - \ell R_S \sum_{n=1}^{N} B(t_0,T_0+n\ell) \right). \qquad (21)$$

A deferred swap matures at time T_0.

At time t_0 the par value for the fixed rate of the deferred swap, namely $R_P(t_0)$, is given by [3]

$$S_I(t_0;T_0,R_P(t_0))$$
$$= 0 = S_{II}(t_0;T_0,R_P(t_0))$$
$$\Rightarrow \ell R_P(t_0) = \frac{B(t_0,T_0) - B(t_0,T_0+N\ell)}{\sum_{n=1}^{N} B(t_0,T_0+n\ell)}. \qquad (22)$$

A swaption, denoted by C_I and C_{II}, is an option on S_I and S_{II}, respectively; suppose the swaption matures at time T_0; it will be exercised only if the value of the swap at time T_0 is greater than its par value of zero; hence, the payoff function is given by

$$C_I(T_0;R_S) = V\left(1 - B(T_0,T_N) - \ell R_S \sum_{n=1}^{N} B(T_0,T_0+n\ell) \right)_+$$

and a similar expression for C_{II}. The value of the swaption at an earlier time $t < T_0$ is given for the money market numeraire by

$$C_I(t,R_S) = V\langle e^{-\int_t^{T_0} r(t')dt'} C_I(T_0;R_S)\rangle$$

$$= V\left\langle e^{-\int_t^{T_0} r(t')dt'}\left(1 - B(T_0,T_N)\right.\right.$$

$$\left.\left. - \ell R_S \sum_{n=1}^{N} B(T_0,T_0+n\ell)\right)_+ \right\rangle \quad (23)$$

and similarly for $C_{II}(t,R_S)$.

One can see that a swap is equivalent to a specific portfolio of coupon bonds, and all techniques that are used for coupon bonds can be used for analyzing swaptions.

Equation (18), together with the martingale property of zero coupon bonds under the money market measure given in Eq. (15) that $\langle e^{-\int_t^{T_0} r(t')dt'} B(T_0,T_n)\rangle = B(t,T_n)$, yields the put-call parity for the swaptions as [3]

$$C_I(t,R_S) - C_{II}(t,R_S)$$

$$= V\left\langle e^{-\int_t^{T_0} r(t')dt'}\left(1 - B(T_0,T_0+N\ell)\right.\right.$$

$$\left.\left. - \ell R_S \sum_{n=1}^{N} B(T_0,T_0+n\ell)\right)\right\rangle$$

$$= V\left(B(t,T_0) - B(t,T_0+N\ell) - \ell R_S \sum_{n=1}^{N} B(t,T_0+n\ell)\right)$$

$$= S_I(t;T_0,R_S), \quad (24)$$

where recall $S_I(t;T_0,R_S;t)$ is the price at time t of a deferred swap that matures at time $T_0 > t$. Note Eq. (19) is the general expression for put-call parity for coupon bond options and the put-call parity for swaptions given in Eq. (24) being a special case.

The price of swaption C_{II}, in which the holder has the option to enter a swap in which he receives at a fixed rate R_S and pays at a floating rate, is given by the formula for the call option for a coupon bond. Suppose the swaption C_{II} matures at time T_0; the payoff function on a principal amount V is given by

$$C_{II}(T_0,R_S) = V\left(B(T_0,T_0+N\ell) + \ell R_S\right.$$

$$\left. \times \sum_{n=1}^{N} B(T_0,T_0+n\ell) - 1\right)_+. \quad (25)$$

Comparing the payoff for C_{II} given above with the payoff for the coupon bond call option given in Eq. (11), one obtains the following for the swaption coefficients

$$c_n = \ell R_S, \quad n = 1,2,\ldots,(N-1),$$

$$\text{payment at time } T_0 + n\ell,$$

$$c_N = 1 + \ell R_S, \quad \text{payment at time } T_0 + N\ell,$$

$$K = 1. \quad (26)$$

The price of C_I is given from C_{II} by using the put-call relation given in Eq. (24).

There are swaptions traded in the market in which the floating rate is paid at $\ell = 90$ days intervals, and with the fixed rate payments being paid at intervals of $2\ell = 180$ days. For a swaption with fixed rate payments at 90 days intervals—at times $T_0+n\ell$, with $n=1,2,\ldots,N$—there are N payments. For payments made at 180 days intervals, there are only $N/2$ payments[5] made at times $T_0+2n\ell$, $n=1,2,\ldots,N/2$, and of amount $2R_S$. Hence the payoff function for the swaption is given by

$$C_I(T_0;R_S) = V\left(1 - B(T_0,T_0+N\ell) - 2\ell R_S\right.$$

$$\left. \times \sum_{n=1}^{N/2} B(T_0,T_0+2n\ell)\right)_+$$

$$= V\left(1 - \sum_{n=1}^{N/2} \tilde{c}_n B(T_0,T_0+2n\ell)\right)_+. \quad (27)$$

The par value at time t_0 is fixed by the forward swap contract, and from Eq. (22) is given by

$$2\ell R_P(t_0) = \frac{B(t_0,T_0) - B(t_0,T_0+N\ell)}{\sum_{n=0}^{N/2} B(t_0,T_0+2n\ell)} \quad (28)$$

and reduces at $t_0=T_0$ to the par value of the fixed interest rate payments being given by

$$2\ell R_P = \frac{1 - B(T_0,T_0+N\ell)}{\sum_{n=1}^{N/2} B(T_0,T_0+2n\ell)}.$$

The equivalent coupon bond put option payoff function is given by

$$S_{Put}(t_*) = \left(K - \sum_{n=1}^{N/2} \tilde{c}_n B(t_*,T_0+2n\ell)\right)_+ \quad (29)$$

and from Eq. (27), has the coefficients and strike price given by

$$\tilde{c}_n = 2\ell R_S, \quad n = 1,2,\ldots,(N-1)/2,$$

$$\text{payment at time } T_0 + 2n\ell,$$

$$\tilde{c}_{N/2} = 1 + 2\ell R_S, \quad \text{payment at time } T_0 + N\ell,$$

$$K = 1.$$

The price of C_I for the 180 days fixed interest payment case is given from C_{II} by using the put-call relation similar to that given in Eq. (24).

Note that it is only due to asymmetric nature of the last coefficient, namely c_N and $\tilde{c}_{N/2}$ for the two cases discussed

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[5]Suppose the swaption has a duration such that N is even. Note that $N=4$ for a year long swaption.

above, that the swap interest rate R_S does not completely factor out (up to a rescaling of the strike price) from the swaption price.

Options on S_I and S_{II}, namely C_I and C_{II}, are both call options since it gives the holder the option to either receive fixed or receive floating payments, respectively. When expressed in terms of coupon bond options, it can be seen from Eqs. (23) and (25) that the swaption for receiving fixed payments is equivalent to a coupon bond put option, whereas the option to receive floating payments is equivalent to a coupon bond call option.

V. FORWARD MEASURE FOR EUROPEAN COUPON BOND OPTIONS

Any numeraire can be used for discounting the payoff function for options for a financial instrument as long as the numeraire yields a martingale evolution for the financial instrument. The choice of the numeraire that yields a martingale measure also fixes the drift $\alpha(t,x)$ [3].

Recall from Eq. (1) that $F_i \equiv F(t_0, t_*, T_i)$ is the forward price, at time t_0, of the zero coupon Treasury Bond $B(t_*, T)$ that is to be issued at time $t_* > t_0$ in the future. A choice of numeraire renders the forward bond price F_i into a martingale [1], and is called the forward measure. The forward measure is more convenient for the option pricing problem since one can dispense discounting with the stochastic (money market) numeraire, namely by $\exp[\int_{t_0}^{t_*} r(t)dt]$, and instead discount using the nonstochastic (present value of a) zero coupon bond $B(t_0, t_*)$.

Call and put options for the coupon bonds using the forward measure are given by

$$C(t_0, t_*, K) = B(t_0, t_*) E_F\left[\left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K\right)_+\right]$$

$$= B(t_0, t_*)\langle S(t_*)\rangle,$$

$$P(t_0, t_*, K) = B(t_0, t_*) E_F\left[\left(K - \sum_{i=1}^{N} c_i B(t_*, T_i)\right)_+\right], \quad (30)$$

where the subscript F in $E_F[\cdots]$ refers to the expectation being evaluated using the forward measure.

The corresponding drift for the forward measure is given by

$$\alpha_F(t,x) = \int_{t_*}^{x} dx' M(x,x';t): \text{forward drift.} \quad (31)$$

Writing the bonds in terms of the forward interest rates yields, from Eq. (3),

$$B(t_*, T_i) = \exp\left(-\int_{t_*}^{T_i} dx f(t_*, x)\right) = e^{-\alpha_i - Q_i} F(t_0, t_*, T_i),$$

where

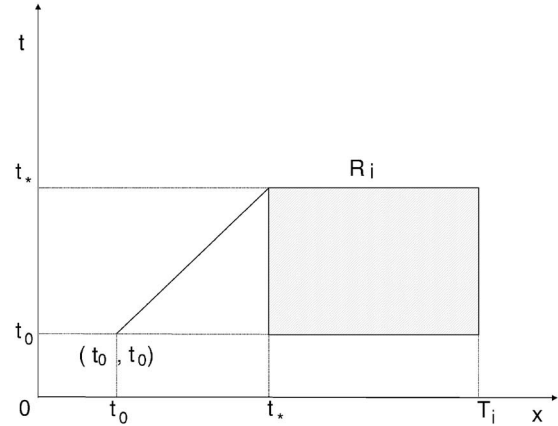$$F(t_0, t_*, T_i) = \left(-\int_{t_*}^{T_i} dx f(t_0, x)\right), \quad \alpha_i = \int_{R_i} \alpha(t,x),$$

FIG. 2. The shaded area is the domain of integration R_i.

$$Q_i = \int_{R_i} \sigma(t,x) \mathcal{A}(t,x) \equiv \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \sigma(t,x) \mathcal{A}(t,x).$$

The domain of integration R_i is given in Fig. 2.

The values of the forward bond prices are plotted in Fig. 3; it can be seen that the forward price falls rapidly as is expected given the exponential discounting of the bond prices.

The coefficient α_i, the integrated form of the forward measure drift, is given by

$$\alpha_i = \int_{R_i} \alpha_F(t,x) \equiv \frac{1}{2} \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx dx' M(x,x';t).$$
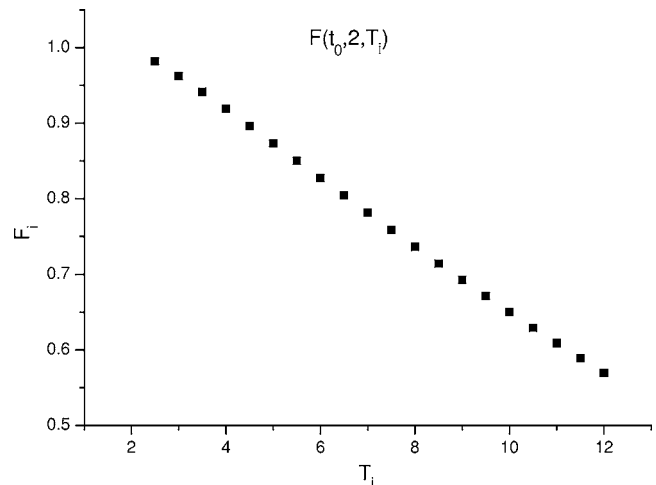
The price of the coupon bond is rewritten as

FIG. 3. Market forward bond prices $F_i = F(t_0, t_*, T_i)$ for Treasury Bonds maturing at times different times T_i, with $t_* = 2$ years in the future. The forward interest rates $f(t_0, x)$ were obtained from the US$ zero coupon yield curve for $t_0 = $ 29 January, 2003.

$$\sum_{i=1}^{N} c_i B(t_*, T_i) = \sum_{i=1}^{N} c_i e^{-\alpha_i - Q_i} F(t_0, t_*, T_i)$$

$$= \sum_{i=1}^{N} c_i F_i + \sum_{i=1}^{N} c_i [B(t_*, T_i) - F_i]$$

$$\equiv F + V \qquad (32)$$

with definitions

$$J_i \equiv c_i F_i, \quad F_i = \exp\left(-\int_{t_*}^{T_i} dx f(t_0, x)\right), \qquad (33)$$

$$F \equiv \sum_{i=1}^{N} c_i F_i = \sum_{i=1}^{N} J_i, \qquad (34)$$

$$V \equiv \sum_{i=1}^{N} c_i [B(t_*, T_i) - F_i] = \sum_{i=1}^{N} J_i (e^{-\alpha_i - Q_i} - 1). \qquad (35)$$

The payoff function is rewritten using the properties of the Dirac delta function. Since

$$\delta(W) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\eta \, e^{i\eta W} \qquad (36)$$

it follows from Eq. (32) that

$$\left(\sum_{i=1}^{N} c_i B(t_*, T_i) - K\right)_+ = (F + V - K)_+$$

$$= \int_{-\infty}^{+\infty} dW \delta(V - W)(F + W - K)_+$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dW d\eta \, e^{i\eta(V-W)} (F + W - K)_+.$$

Hence the price of the call option, from Eq. (30), can be written as

$$C(t_0, t_*, K) = B(t_0, t_*) \frac{1}{2\pi} \int_{-\infty}^{+\infty} dW d\eta (F + W - K)_+ e^{-i\eta W} Z(\eta) \qquad (37)$$

with the partition function given by

$$Z(\eta) = \langle e^{i\eta V} \rangle_F = \frac{1}{Z} \int D\mathcal{A} e^S e^{i\eta V}, \quad Z = \int D\mathcal{A} e^S. \qquad (38)$$

A perturbation expansion is developed that evaluates the partition function $Z(\eta)$ as a series in the volatility function $\sigma(t, x)$.

To see how put-call parity is expressed in terms of the partition function $Z(\eta)$, note from Eqs. (30) and (37),

$$C(t_0, t_*, K) - P(t_0, t_*, K) = B(t_0, t_*) \frac{1}{2\pi} \int_{-\infty}^{+\infty} dW d\eta (F + W - K)$$

$$\times e^{-i\eta W} Z(\eta). \qquad (39)$$

The integration over the variable W in the equation above

can be performed exactly, and from the definition of the Dirac delta function given in Eq. (36), yields a delta function and its derivative in the η variable; hence the integration over η can also be performed exactly, and one obtains from Eq. (39) that

$$C(t_0, t_*, K) - P(t_0, t_*, K)$$

$$= B(t_0, t_*) \int_{-\infty}^{+\infty} d\eta \left((F - K)\delta(\eta) + i\frac{\partial}{\partial\eta}\delta(\eta)\right) Z(\eta)$$

$$= B(t_0, t_*) \left((F - K)Z(0) - i\frac{\partial}{\partial\eta} Z(\eta)\Big|_{\eta=0}\right). \qquad (40)$$

To satisfy put-call parity, any approximation scheme for evaluating the partition function $Z(\eta)$ must satisfy, comparing Eqs. (19) and (40), the following two conditions:

$$Z(0) = 1, \quad \frac{\partial}{\partial\eta} Z(\eta)\Big|_{\eta=0} = 0. \qquad (41)$$

VI. FEYNMAN PERTURBATION EXPANSION FOR THE COUPON BOND OPTION

In general, computing the price of a European option for a coupon bond is a nonlinear problem and is usually studied numerically. In this section an analytic expression for the approximate price of the coupon bond option is derived.

It is well known [1,8] that the volatility of the forward interest rates is a small quantity, of the order of $\sigma(t, x) \simeq 10^{-2}/\text{year}$. The volatility function $\sigma(t, x)$ can hence be used as an expansion parameter; the approximation can be systematically improved by going to higher and higher orders in powers of $\sigma(t, x)$. Hence it is logical to find a mathematical expansion that yields the price of the coupon bond option as a power series in $\sigma(t, x)$, and in effect provides a perturbation expansion of the partition function $Z[\eta]$.

One of the advantages of the quantum finance formulation of coupon bond options is that one has a representation of the problem that yields itself to approximations schemes that in the conventional formulation of finance, relying heavily as it does on stochastic calculus, are far from obvious.

The partition function of the coupon bond option price can be written more explicitly.

From the expression for the partition function given in Eq. (38), the effective action for the pricing of the coupon bond option, from Eq. (32), is given by

$$S_{\text{Effective}} \equiv S[\mathcal{A}] + i\eta V \qquad (42)$$

$$= S[A] + i\eta \sum_{i=1}^{N} J_i (e^{-\alpha_i - Q_i} - 1)$$

$$= S[\mathcal{A}] + i\eta \sum_{i=1}^{N} J_i (e^{-\alpha_i} e^{-\int_{R_i} \sigma\mathcal{A}} - 1). \qquad (43)$$

Equation (43) yields a highly nonlinear and nonlocal two-dimensional quantum field theory, with the coupon bond op-

tion payoff function providing an effective nonlocal exponential potential for the quantum field $\mathcal{A}(t,x)$.

Nonlinear quantum field theories are usually intractable, and the best that one can do is to develop a consistent perturbation expansion for the partition function $Z(\eta)$. Feynman diagrams provide the standard technique in quantum field theory for studying nonlinear field theories [2]. The field theory formulation of the forward interest rates provides a natural perturbation expansion of the partition function using the well-known technique of Feynman diagrams.

A cumulant expansion of the partition function in a power series in η yields

$$Z(\eta) = e^{i\eta D - (1/2)\eta^2 A - i(1/3!)\eta^3 B + (1/4!)\eta^4 C + \cdots}. \tag{44}$$

The coefficients A, B, C, ... are evaluated using Feynman diagrams.

From the put-call parity constraint given in Eq. (41), the first condition $Z(0)=1$ is satisfied automatically, and the second condition implies that $D=0$. Hence any approximate scheme for $Z(\eta)$ that fulfills the put-call parity relation must yield

$$Z(\eta) = e^{-(1/2)\eta^2 A - i(1/3!)\eta^3 B + (1/4!)\eta^4 C + \cdots}. \tag{45}$$

Expanding the partition function in a power series to fourth order in η yields

$$Z(\eta) = \frac{1}{Z} \int D\mathcal{A} e^{i\eta V} e^{S[\mathcal{A}]}$$

$$= \frac{1}{Z} \int D\mathcal{A} e^{S[\mathcal{A}]} \left(1 + i\eta V + \frac{1}{2!}(i\eta)^2 V^2 + \frac{1}{3!}(i\eta)^3 V^3 \right.$$

$$\left. + \frac{1}{4!}(i\eta)^4 V^4 + \cdots \right). \tag{46}$$

Comparing Eqs. (44) and (46), we have to fourth order in η the following:

$$D = \langle V \rangle, \tag{47}$$

$$A = \langle V^2 \rangle - D^2, \tag{48}$$

$$B = \langle V^3 \rangle - D^3, \tag{49}$$

$$C = \langle V^4 \rangle - 3A^2 - D^4. \tag{50}$$

The coefficient D is exactly zero since the martingale condition for the forward measure yields

$$D = \langle V \rangle_F = \sum_{i=1}^{N} J_i [E_F(e^{-\alpha_i - Q_i}) - 1] = 0. \tag{51}$$

Put-call parity is satisfied by the approximation scheme since $D=0$; one can see that the martingale condition is essential in the realization of put-call parity.

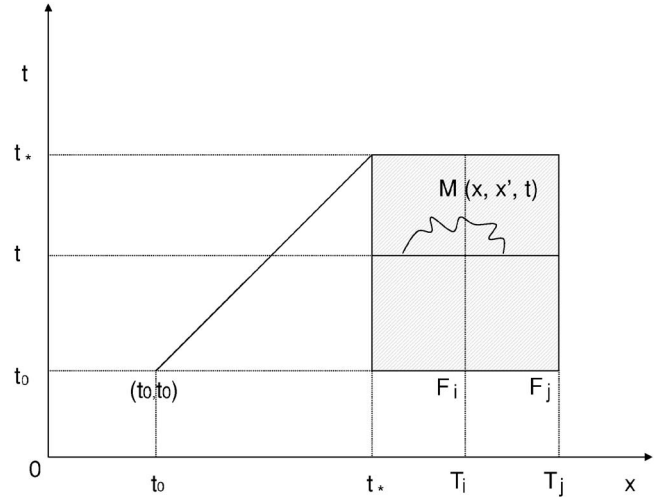Define the dimensionless forward bond price correlator by

FIG. 4. The shaded domain of the forward interest rates contribute to the correlator $G_{ij} = \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \int_{t_*}^{T_j} dx' M(x,x';t)$. For a typical point t in the time integration the figure shows the correlation function $M(x,x';t)$ connecting two different values of the forward interest rates at future time x and x'.

$$G_{ij} \equiv G_{ij}(t_0, t_*, T_i, T_j; \sigma)$$

$$= \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \int_{t_*}^{T_j} dx' M(x,x';t)$$

$$= G_{ji} : \text{real and symmetric.} \tag{52}$$

The evaluation of G_{ij} is illustrated in Fig. 4, and Fig. 5 shows it's dependence on T_i and T_j. G_{ij} is the forward bond propagator that expresses the correlation in the fluctuations of the forward bond prices $F_i = F(t_0, t_*, T_i)$ and $F_j = F(t_0, t_*, T_j)$.

For any application of the coupon bond option price to the financial markets, one must take into account market, or psy-
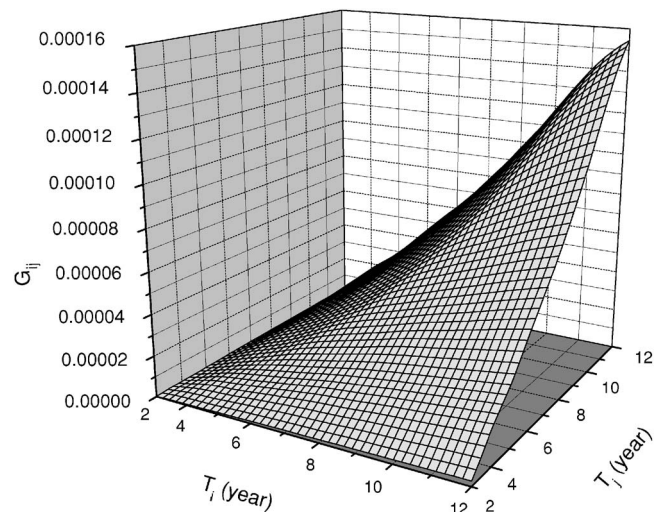

FIG. 5. The forward bond price correlator $G_{ij} = \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \int_{t_*}^{T_j} dx' M(x,x';t) = G_{ij}(t_0, t_*, T_i, T_j)$, is plotted against T_i and T_j with $t_* - t_0 = 2$ years, where $M(x,x';t)$ is taken from swaption data.

chological, future time defined by an equation similar to Eq. (2), and is given by [1]

$$\frac{\partial f(t,\theta)}{\partial t} = \alpha(t,z(\theta)) + \sigma(t,z(\theta))\mathcal{A}(t,z(\theta)),$$

$$\theta = x - t, \quad z = \theta^{\nu}$$

and which yields the following representation for Treasury Bonds:

$$B(t,T) = \exp\left(-\int_0^{T-t} d\theta f(t,\theta)\right).$$

The forward interest rates correlator has the property that, for a given time regime of the market $M(t;x,x')=M(x-t,x'-t)$ [1]. The market correlator for the forward bond prices is then given by

$$G_{ij}^{\text{Market}} \equiv G_{ij}^{\text{Market}}(t_0,t_*;\sigma;\nu)$$

$$= \int_{t_0}^{t_*} dt \int_{t_*-t}^{T_i-t} d\theta \int_{t_*-t}^{T_j-t} d\theta' M(z(\theta),z(\theta')).$$

A field theory computation for the coefficients is carried out in Appendix A, and yields the following result:

$$A = \sum_{ij=1}^N J_i J_j (e^{G_{ij}} - 1), \tag{53}$$

$$B = \sum_{ijk=1}^N J_i J_j J_k (e^{G_{ij}+G_{jk}+G_{ki}} - e^{G_{ij}} - e^{G_{jk}} - e^{G_{ki}} + 2), \tag{54}$$

$$C = \sum_{ijkl=1}^N J_i J_j J_k J_l [e^{G_{ij}+G_{ik}+G_{il}+G_{jk}+G_{jl}+G_{kl}} - e^{G_{ij}+G_{jk}+G_{ki}}$$

$$- e^{G_{ij}+G_{jl}+G_{li}} - e^{G_{ik}+G_{kl}+G_{li}} - e^{G_{jk}+G_{kl}+G_{lj}} - e^{G_{ij}+G_{kl}} - e^{G_{jk}+G_{il}}$$

$$- e^{G_{ik}+G_{jl}} + 2(e^{G_{ij}} + e^{G_{ik}} + e^{G_{il}} + e^{G_{jk}} + e^{G_{jl}} + e^{G_{kl}}) - 6] \tag{55}$$

As one can see, the terms required to determine the coefficients rapidly proliferate.

As things stand, all coefficients A, B, C, ... seem to be of equal magnitude. A consistent expansion is obtained if one assumes that $\sigma(t,x)$ is small for all values of its argument. For Libor, data indicates that $\sigma(t,x) \simeq 10^{-2}$/year; furthermore, normalizing the propagator such that $M(x,x;t) = \sigma(t,x)^2$ yields $N(x,x';t) \simeq 1$ for all x,x'. G_{ij} is dimensionless and is of order of magnitude of σ^2, and which yields that $G_{ij} \simeq 10^{-4}$.

Hence G_{ij} can be taken to be a small expansion parameter, with all the coefficients $A, B, C, ...$ being expressed as power series in G_{ij}. Expanding the exponential functions in Eqs. (53)–(55) yields the following result:

$$A = \sum_{ij=1}^N J_i J_j \left(G_{ij} + \frac{1}{2}G_{ij}^2\right) + O(G_{ij}^3), \tag{56}$$

$$B = 3\sum_{ijk=1}^N J_i J_j J_k G_{ij} G_{jk} + O(G_{ij}^3),$$

$$C = 16\sum_{ijkl=1}^N J_i J_j J_k J_l G_{ij} G_{jk} G_{kl} + O(G_{ij}^4). \tag{57}$$

Denote the magnitude of the matrix elements G_{ij} by G; using the fact that $G \simeq \sigma^2$, the partition function Z has an order of magnitude expansion given by

$$Z(\eta) \simeq e^{-\sum_{l=2}^{\infty} a_l \eta^l G^{l-1}} \simeq e^{-\sum_{l=2}^{\infty} a_l \eta^l \sigma^{2l-2}}$$

$$= e^{-a_2\zeta^2 - a_3\zeta^3\sigma - a_4\zeta^4\sigma^2 + \cdots}, \quad \zeta = \sigma\eta, \tag{58}$$

where all the coefficients $a_l \simeq O(1)$.

The quadratic term in the exponential for $Z(\eta)$ fixes the magnitude of the fluctuations of the $\zeta = \sigma\eta$ variable to be of $O(1)$; hence, the remaining terms are of order σ, σ^2 and so on to higher and higher order. One can consequently go to any order of accuracy required in the expansion parameter G, or equivalently in σ, and self-consistently terminate the expansion at any order. Hence the perturbation expansion for the partition function $Z(\eta)$ is consistent, with the higher order terms in η being smaller than the lower order ones.

The perturbation expansion for the partition function $Z(\eta)$ has an intuitive representation using Feynman diagrams. The forward bond propagator G_{ij} that represents the correlation between forward bond price F_i and F_j is indicated with a wavy line in Fig. 6(a). Note that all the diagrams for the partition function $Z(\eta)$ are connected in that none of the forward bond prices are decoupled from the forward bond propagator G_{ij}. In contrast Figs. 7(a)–7(c) are examples of disconnected Feynman diagrams; for example, in Fig. 7(a) the forward bond prices in the first line, denoted by F_j, do not couple to the propagator.

VII. APPROXIMATE PRICE OF THE COUPON BOND OPTION

From Eqs. (37) and (45), one can do an expansion for the partition function of the cubic and quartic terms in η, and then perform the Gaussian integrations over η; this yields

$$C(t_0,t_*,K) = B(t_0,t_*)\frac{1}{2\pi}\int_{-\infty}^{+\infty} dW d\eta (F + W - K)_+$$

$$\times e^{-i\eta W} e^{-(1/2)\eta^2 A - i(1/3!)\eta^3 B + (1/4!)\eta^4 C + \cdots}$$

$$= B(t_0,t_*)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dW(F + W\sqrt{A} - K)_+$$

$$\times f(\partial_w)e^{-(1/2)W^2} + O(\sigma^4), \tag{59}$$

where, for $\partial_w \equiv \partial/\partial W$, one has the following:

$$f(\partial_w) \equiv 1 - \left(\frac{B}{6A^{3/2}}\right)\partial_w^3 + \left(\frac{C}{24A^2}\right)\partial_w^4 + \frac{1}{2}\left(\frac{B}{6A^{3/2}}\right)^2\partial_w^6 + O(\sigma^4),$$
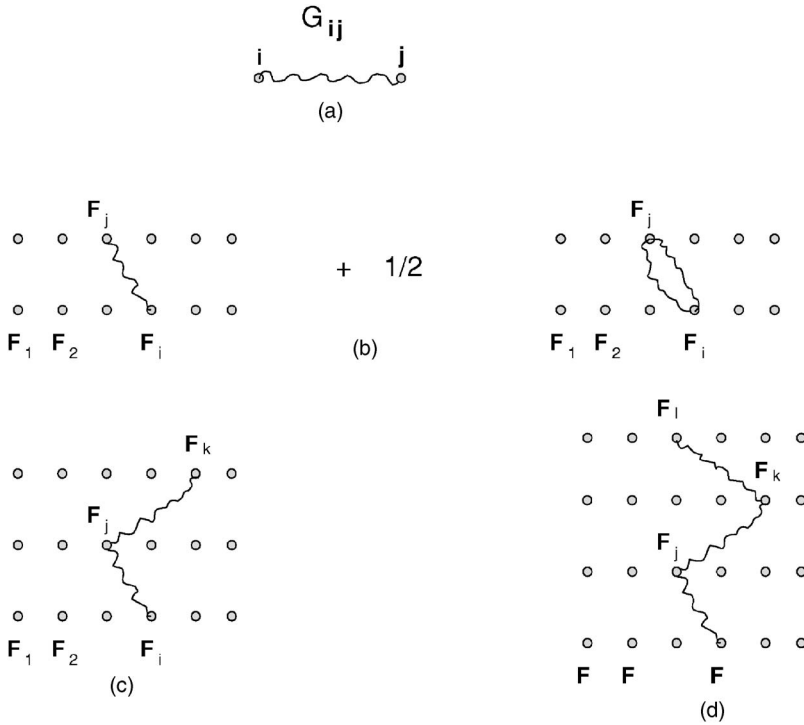
FIG. 6. Feynman diagrams for the partition function $Z(\eta)$. In diagram (a) the dots represent the forward bond prices F_i and F_j and the wavy line is their correlator G_{ij}. Coefficient A is evaluated by diagrams given in (b); coefficient B is evaluated by the diagram given in (c) and coefficient C is evaluated by diagram given in (d).

$$A \simeq O(\sigma^2), \quad \frac{B}{A^{3/2}} \simeq O(\sigma), \quad \frac{C}{A^2} \simeq O(\sigma^2). \quad (60)$$

Due to the properties of $\Theta(x)$, the Heaviside theta function, the second derivative of the payoff is equal to the Dirac delta function, namely

$$\partial_w^2(F + W\sqrt{A} - K)_+ = \sqrt{A}\,\delta(W - X), \quad (61)$$

(a)

(b)

(c)

FIG. 7. Disconnected diagrams are ones in which some of the forward bond prices, indicated by the top line, do not couple to the other forward bond prices. (a), (b), and (c) are second, third, and fourth order disconnected diagrams, do not contribute to $\ln[Z(\eta)]$ and hence do not contribute to any of the coefficients.

$$X = \frac{K - F}{\sqrt{A}}, \quad \text{dimensionless.} \quad (62)$$

Using the equations above and Eqs. (59) and (60) yields, after integration by parts, the following:
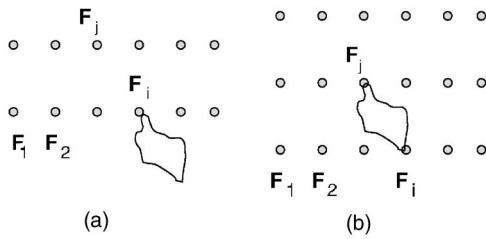
$$C(t_0, t_*, K) = B(t_0, t_*)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dW\Bigg\{ (F + W\sqrt{A} - K)_+$$

$$+ \sqrt{A}\,\delta(W - X)\Bigg[-\frac{B}{6A^{3/2}}\partial_w + \frac{C}{24A^2}\partial_w^2$$

$$+ \frac{1}{2}\bigg(\frac{B}{6A^{3/2}}\bigg)^2\partial_w^4 \Bigg] \Bigg\}e^{-(1/2)W^2} + O(\sigma^4). \quad (63)$$

Note

$$I(X) = \int_{-\infty}^{+\infty} dW(W - X)_+ e^{-(1/2)W^2}$$

$$= e^{-(1/2)X^2} - \sqrt{\frac{\pi}{2}}X\Bigg[1 - \Phi\bigg(\frac{X}{\sqrt{2}}\bigg) \Bigg], \quad (64)$$

where the error function is given by

$$\Phi(u) = \frac{2}{\sqrt{\pi}}\int_0^u dW e^{-w^2}.$$
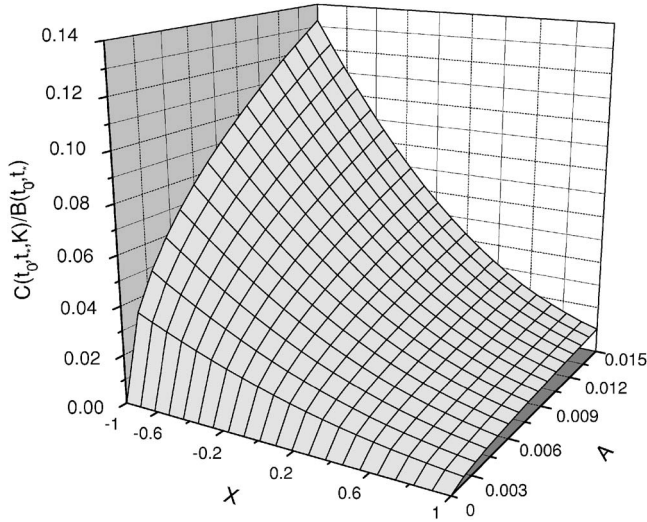
Hence the price of the coupon bond is given by

FIG. 8. The value of the swaption $C(t_0,t_*,K)/B(t_0,t_*)$ $=\sqrt{A/2\pi}I(X)+O(\sigma^2)$, plotted as a function of A and X.

$$C(t_0,t_*,K) = B(t_0,t_*)\sqrt{\frac{A}{2\pi}}\left(\frac{B}{6A^{3/2}}X + \frac{C}{24A^2}(X^2-1)\right.$$

$$\left. + \frac{1}{72}\frac{B^2}{A^3}(X^4 - 6X^2 + 3)\right)e^{-(1/2)X^2} + B(t_0,t_*)$$

$$\times \sqrt{\frac{A}{2\pi}}I(X) + O(\sigma^4) \tag{65}$$

and is graphed in Fig. 8; the reason the surface is smooth is because the variables X and A are varied continuously.

The asymptotic behavior of the error function $\Phi(u)$ yields the following limits:

$$I(X) = \begin{cases} 1 - \sqrt{\dfrac{\pi}{2}}X + O(X^2), & X \approx 0, \\[2ex] \dfrac{e^{-(1/2)X^2}}{X^2}\left[1 + O\left(\dfrac{1}{X^2}\right)\right], & X \gg 0. \end{cases}$$

For the coupon bond and swaption at the money $F=K$; hence the option's price close to the money has $X\approx 0$ and to leading order yields the price to be

$$C(t_0,t_*,K) \approx B(t_0,t_*)\sqrt{\frac{A}{2\pi}} - \frac{1}{2}B(t_0,t_*)(K-F) + O(X^2).$$

$$\tag{66}$$

Put-call parity for approximate option price. The approximate price of the coupon bond call option in Eq. (65) expressed in terms of the expansion coefficients can be written symbolically as $C(t_0,t_*,K) \equiv C_{[t_0,t_*,K]}(A,B,C,X)$. Recall that the put option is given by an expression similar to that of the call option given in Eq. (59), namely
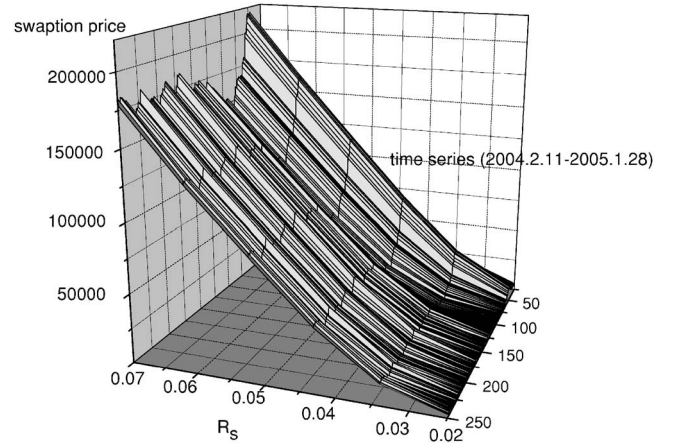

FIG. 9. The price of a 1×3 swaption. The swaption has a 3 year duration that matures by 1 year in the future, with floating payments made at 90 days intervals and fixed payments being made at 180 days intervals. The x axis is the value of R_S and time is plotted along the y axis.

$$P(t_0,t_*,K) = B(t_0,t_*)\frac{1}{2\pi}\int_{-\infty}^{+\infty} dW d\eta (K - F - W)_+$$

$$\times e^{-i\eta W}e^{-(1/2)\eta^2 A - i(1/3!)\eta^3 B + (1/4!)\eta^4 C + \cdots}$$

$$= C_{[t_0,t_*,K]}(A, -B, C, -X).$$

For the approximate price of the put and call options, since $X=(K-F)/\sqrt{A}$, put-call parity yields

$$C(t_0,t_*,K) - P(t_0,t_*,K)$$

$$= C_{[t_0,t_*,K]}(A,B,C,X) - C_{[t_0,t_*,K]}(A, -B, C, -X)$$

$$= B(t_0,t_*)\sqrt{\frac{A}{2\pi}}\left(-2\sqrt{\frac{\pi}{2}}X\right) + (\sigma^4)$$

$$= B(t_0,t_*)(F - K)$$

$$= \sum_i c_i B(t_0,T_i) - KB(t_0,t_*).$$

Hence, as expected from Eq. (19), the expansion in the volatility parameter σ yields an approximate price for the call and put option that obeys put-call parity order by order.

Numerical price of swaptions. The swaption price is given by Eq. (65), with the factor of $B(t_0,t_*)$ for the case of coupon bond replaced for the case of the swaption by $VB(t_0,T_0)$. Putting c_n to be equal to it's values for the swaption coefficients as given in Eq. (26), and setting strike price of $K=1$ in Eq. (65) yields the price of the swaption $C_{\text{II}}(t_0,T_0,R_S)$ at time t_0.

Figure 9 shows the time series for the price of a swaption on Libor with values for the fixed interest rate R_S being equal to the par value for the different times. The reason the surface of the market swaption price shown in Fig. 9 is rough compared to Fig. 8 is because the par interest rate for the market varies discontinuously, leading to sharp changes in the price of the swaption.
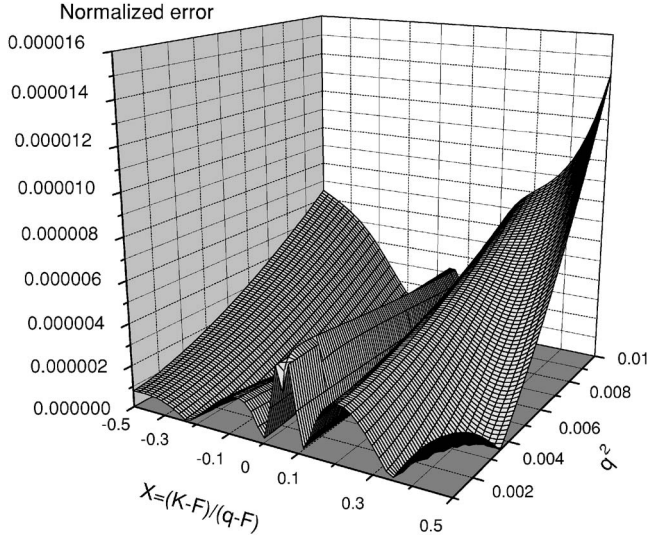
FIG. 10. The normalized difference of the exact and approximate option prices for the zero coupon bond, namely $(C_{zcb}^{Exact} - C_{zcb}^{Approx})/C_{zcb}^{Exact}$, for various values of q^2 and strike price $X = (K - F)/qF$.

A. Zero coupon bond option price

An exact quantum finance result for the option price of a zero coupon bond is given in Ref. [1]. For the approximate price of the coupon bond option to be consistent it must reproduce the price of the zero coupon bond option as one of it's limiting values. In this section the general result obtained for the price of a coupon bond option is shown to correctly reproduce order, to $O(q^4)$, the known exact result of the zero coupon bond option.

The zero coupon bond is a special case of the coupon bond when only one of the coefficient functions c_i is nonzero. Let $c_1 = 1$ and $T_1 = T$, with the rest of the coupon payments being zero, that is, $c_i = 0$, $i = 2, 3, \ldots, N$. From Eq. (11) the payoff function, the forward price for the zero coupon bond and the propagator is as follows:

$$S(t_*) = (B(t_*, T) - K)_+,$$

$$F = c_1 F_1 = \exp\left(-\int_{t_*}^{T} dx f(t_0, x)\right),$$

$$G_{11} \equiv q^2 = \int_{t_0}^{t_*} dt \int_{t_*}^{T} dx \int_{t_*}^{T} dx' M(x, x'; t). \quad (67)$$

The coefficients in the expansion for the price of the coupon bond option yield

$$A = F_1^2\left(G_{11} + \tfrac{1}{2}G_{11}^2\right) = F^2\left(q^2 + \tfrac{1}{2}q^4\right) + O(q^6),$$

$$\frac{B}{A^{3/2}} = \frac{3F_1^3 G_{11}^2}{A^{3/2}} = 3q + O(q^3),$$

$$\frac{C}{A^2} = \frac{16F_1^4 G_{11}^3}{A^2} = 16q^2 + O(q^4),$$

$$X = \frac{K - F}{qF} + O(q). \quad (68)$$

Note that the expansion for coefficient A must to be kept to $O(q^4)$ since it appears in the payoff function and yields the next leading order term for the payoff function which is a term of $O(q^3)$.

The price of the coupon bond call option, from Eqs. (63) and (68), simplifies in the case of the zero coupon option to

$$C_{zcb}(t_0, t_*, K) = B(t_0, t_*)\frac{qF}{\sqrt{2\pi}}\left(\frac{1}{2}qX + \frac{2}{3}q^2(X^2 - 1)\right.$$
$$\left. + \frac{1}{8}q^2(X^4 - 6X^2 + 3)\right)e^{-(1/2)X^2} + B(t_0, t_*)\frac{1}{\sqrt{2\pi}}$$
$$\times \int_{-\infty}^{+\infty} dW\left[F + qF\left(1 + \frac{1}{4}q^2\right)W - K\right]_+$$
$$\times e^{-(1/2)W^2} + O(q^4). \quad (69)$$

The payoff function is expressed to $O(q^3)$ using Taylors expansion; using $We^{-W^2/2} = -\partial_w e^{-W^2/2}$ for doing an integration by parts, and from Eq. (61), the last term in Eq. (69) yields

$$B(t_0, t_*)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dW e^{-(1/2)W^2}\left(1 + \frac{1}{4}q^2 W\partial_w\right)(F + qFW - K)_+$$
$$= B(t_0, t_*)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dW\left((F + qFW - K)_+\right.$$
$$\left. + \frac{1}{4}q^3 F\delta(W - X)\right)e^{-(1/2)W^2}$$
$$= B(t_0, t_*)\frac{qF}{\sqrt{2\pi}}\left(I(X) + \frac{1}{4}q^2 e^{-(1/2)X^2}\right), \quad (70)$$

where $I(X)$ is given in Eq. (64). Hence from Eqs. (69) and (70), the price of the zero coupon bond option, after some simplifications, is given by

$$C_{zcb}(t_0, t_*, K) = B(t_0, t_*)\frac{qF}{\sqrt{2\pi}}\left(\frac{1}{2}qX + \frac{1}{6}q^2(X^2 - 1)\right.$$
$$\left. + \frac{1}{8}q^2(X^2 - 1)^2\right)e^{-(1/2)X^2} + B(t_0, t_*)\frac{qF}{\sqrt{2\pi}}I(X)$$
$$+ O(q^4). \quad (71)$$

It is shown in Appendix B that the exact result for the zero coupon bond option price, when expanded in a power series in q^2, yields the same result as the one obtained in Eq. (71).

B. Numerical estimates of zero coupon bond option

The accuracy of the volatility expansion of the zero coupon bond option price is studied by comparing the approximate expression for the call option price given in Eq. (71) with the exact expression for zero coupon bond option given in Eq. (B1).

The normalized difference of the exact and approximate option prices, namely $(C_{zcb}^{Exact} - C_{zcb}^{Approx})/C_{zcb}^{Exact}$ is plotted in
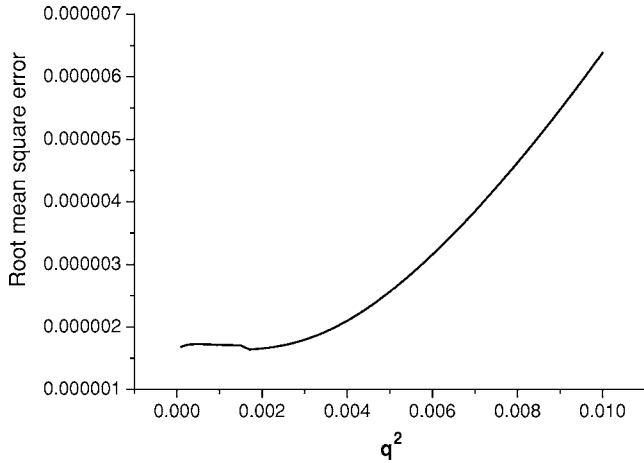
FIG. 11. The normalized root mean square error $\sqrt{\mathrm{var}[(C_{\mathrm{zcb}}^{\mathrm{Exact}} - C_{\mathrm{zcb}}^{\mathrm{Approx}})/C_{\mathrm{zcb}}^{\mathrm{Exact}}]}$ of the fit for the entire zero coupon bond option price, as a function of effective volatility q^2.

Fig. 10 for various values of $X = (K - F)/qF$ and for different values of q^2; the result shows that the approximate value of the option price has a negligible normalized error of about 10^{-5} for $0 \leq q^2 \leq 0.01$. The (normalized) root mean square is computed for the entire fit, and is given in Fig. 11; the normalized error is again about 10^{-5} over the same range of q^2.

For the coupon bond with N coupons, the effective expansion is approximately Nq^2; from the results of the zero coupon bond, one can estimate that as long as $Nq^2 \leq 0.01$ the approximation has an accuracy of about 10^{-3}; since a typical value of $q^2 \simeq 10^{-3}$, one can conservatively conclude that for the coupon bond option the perturbation expansion is valid for $N \simeq 100$.

VIII. HJM LIMIT OF THE APPROXIMATE COUPON BOND OPTION PRICE

The coupon bond option price in the one factor HJM model with exponential volatility [7,9] is compared with the HJM limit of the approximate field theory coupon bond option price.

The generic HJM limit for field theory models is given by forward interest rates that are exactly correlated; this limit corresponds to correlator being given by $D(x, x'; t) \rightarrow 1$ [1], and hence $M(x, x'; t) \rightarrow \sigma(x - t)\sigma(x' - t)$. The case of $D(x, x'; t) \rightarrow 1$ is studied to ascertain the importance of having a nontrivial correlation for the forward interest rates.

Taking $\sigma(x - t)$ equal to the market volatility of the forward rates, the percentage difference between the daily field theory price of a 2×10 swaption at the money and it's limit of $D(x, x'; t) \rightarrow 1$ is plotted in Fig. 12. In the HJM limit of $D(x, x'; t) \rightarrow 1$ the daily option price is seen to be overpriced by 4% to 9% in comparison with the correlated field theory option price. This result shows the important role of the nontrivial correlations in pricing coupon bond options.

The HJM limit of the field theory swaption price is now studied by considering the volatility function to have the very special form of $\sigma(t, x) = \sigma_0 e^{-\lambda(x - t)}$. Jarrow and Turnbul
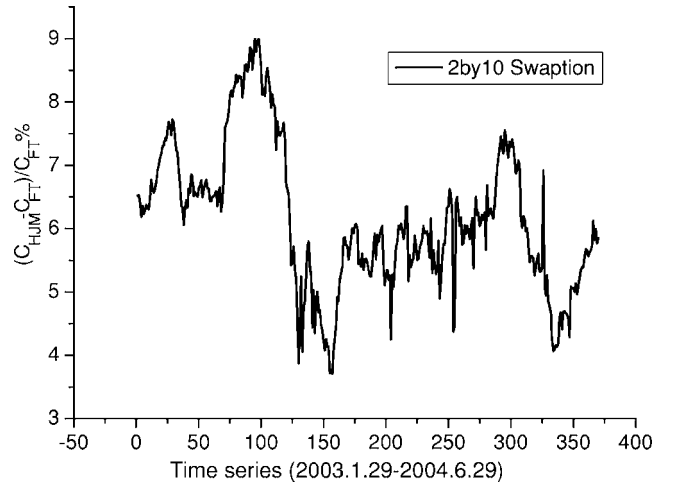

FIG. 12. The percentage difference of a 2×10 swaption daily prices, at the money, for the field theory model and of it's generic HJM limit given by taking $D(x, x'; t) \rightarrow 1$. The volatility function $\sigma(t, x)$ is taken from the Libor market as are the daily forward bond prices F_i. Note the HJM model systematically overprices the swaption by 4%–9%.

[7,9] obtained the following explicit expression for the coupon bond option:

$$C_{\mathrm{HJM}}(t_0, t_*, K) = \sum_{i=1}^{N} c_i B(t_0, T_i)N(d_i) - KB(t_0, t_*)N(d),$$

$$d_i \equiv \frac{r'}{\sigma_R} + Y(t_*, T_i)\sigma_R, \quad d = \frac{r'}{\sigma_R},$$

$$Y(t_*, T_i) \equiv \frac{1}{\lambda}(1 - e^{-\lambda(T_i - t_*)}), \quad \sigma_R^2 = \frac{\sigma_0^2}{2\lambda}(1 - e^{-2\lambda(t_* - t_0)}).$$
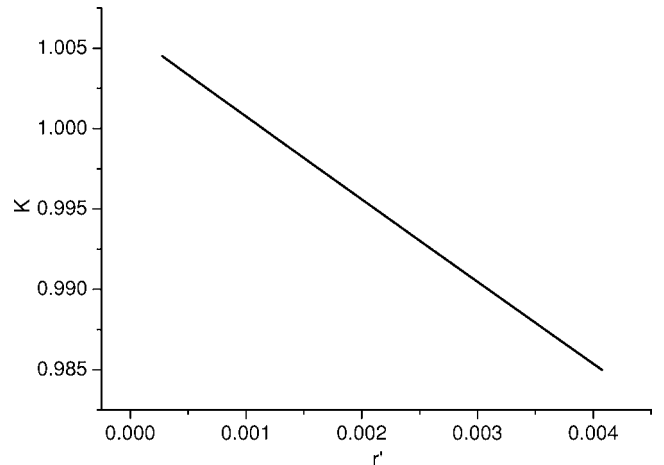
$$(72)$$

FIG. 13. The nonlinear relation of K and r'.

The quantity r' is related to the strike price K by a nonlinear transformation that depends on the initial coupon bond price [7]; Fig. 13 shows a typical dependence of K on r'. $N(d)$ is the probability integral for the normal distribution.

The field theory option price has been derived as a perturbation expansion in the volatility taken as a small quantity; hence for making a comparison the HJM result must be expanded as a power series in the volatility constant σ_0,

which is taken to be small. The value of r' is taken to be such that r'/σ_0 is small, which in turn yields that all the d_i, d are small. Hence

$$N(d) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{d} dz e^{-(1/2)z^2} \simeq \frac{1}{2} + \sqrt{\frac{1}{2\pi}} d + O(d^2).$$

Hence from above and Eq. (72) the HJM-option price yields

$$
\begin{aligned}
C_{\mathrm{HJM}}(t_0,t_*,K) &\simeq \sum_{i=1}^{N} c_i B(t_0,T_i)\left(\frac{1}{2} + \sqrt{\frac{1}{2\pi}}d_i\right) - KB(t_0,t_*)\left(\frac{1}{2} + \sqrt{\frac{1}{2\pi}}d\right) + O(d_i^2,d^2) = \frac{1}{2}\left(\sum_{i=1}^{N} c_i B(t_0,T_i) - KB(t_0,t_*)\right) \\
&+ \sqrt{\frac{1}{2\pi}}\sum_{i=1}^{N} c_i B(t_0,T_i) Y(t_*,T_i)\sigma_R + \sqrt{\frac{1}{2\pi}}\frac{r'}{\sigma_R}\left(\sum_{i=1}^{N} c_i B(t_0,T_i) - KB(t_0,t_*)\right) + O(d_i^2,d^2) = \frac{1}{2}B(t_0,t_*)(F-K) \\
&+ \sqrt{\frac{1}{2\pi}}B(t_0,t_*)\sum_{i=1}^{N} J_i Y(t_*,T_i)\sigma_R + O\left(\frac{r'}{\sigma_0}(F-K),(F-K)^2\right),
\end{aligned}
\tag{73}
$$

where recall that $J_i = c_i B(t_0,T_i)/B(t_0,t_*)$ and $F = \Sigma_{i=1}^{N} J_i$.

From Eq. (66), to leading order, the field theory option price is given by

$$C(t_0,t_*,K) = B(t_0,t_*)\sqrt{\frac{A}{2\pi}} - \frac{1}{2}B(t_0,t_*)(K-F) + O(X^2,\sigma_0^2).$$

To lowest order, from Eqs. (16), (52), and (56),

$$A = \sum_{ij=1}^{N} J_i J_j G_{ij},$$

$$G_{ij} = \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \int_{t_*}^{T_j} dx' \sigma(t,x) D(x,x';t)\sigma(t,x').$$

Taking the HJM limit of $D(x,x';t) \to 1$ and taking volatility $\sigma(t,x) = \sigma_0 e^{-\lambda(x-t)}$ yields

$$
\begin{aligned}
G_{ij} &\to \sigma_0^2 \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx e^{-\lambda(x-t)} \int_{t_*}^{T_j} dx' e^{-\lambda(x'-t)} \\
&= \sigma_R^2 Y(t_*,T_i) Y(t_*,T_j) \\
\Rightarrow \sqrt{A} &= \sqrt{\sigma_R^2 \sum_{ij=1}^{N} J_i J_j Y(t_*,T_i) Y(t_*,T_j)} \\
&= \sigma_R \sum_{i=1}^{N} J_i Y(t_*,T_i).
\end{aligned}
\tag{74}
$$

Hence, the limit of the field theory option price is given by

$$
\begin{aligned}
C(t_0,t_*,K) \to{}& \sqrt{\frac{1}{2\pi}}B(t_0,t_*)\sum_{i=1}^{N} J_i Y(t_*,T_i)\sigma_R \\
&- \frac{1}{2}B(t_0,t_*)(K-F) + O((K-F)^2,\sigma_0^2)
\end{aligned}
$$

and from Eq. (73), to leading order in σ_0, is seen to be equal to $C_{\mathrm{HJM}}(t_0,t_*,K)$.

The HJM limit of the field theory result shows a number of special features of the HJM-option price. First, the one-factor HJM coupon bond option price for exponential volatility is given by a single sum over expressions that are similar to the zero coupon bond options; this result is due to the specific exponential form chosen for the volatility. In general the limit of $D(x,x';t) \to 1$ for a general volatility function would not remove the square root on A. Second, the field theory and HJM-option prices are seen to be exactly equal to lowest order in the volatility, and at the money.

Figure 14 shows the result of a numerical evaluation of both the HJM and the HJM limit of the field theory option price—at the money $K=F$ and for fixed forward bond prices F_i—with exponential volatility $\sigma_0 e^{-\lambda(x-t)}$; only σ_0 is allowed to vary. Figure 14 shows that the HJM limit of the approximate field theory coupon bond price starts to deviate from the HJM price for $\sigma_0 \gtrsim 0.1$; this is to be expected since the field theory approximation in the first place is expected to hold for only small values of $\sigma_0^2 \simeq 10^{-2}$.

The result of Fig. 14 does not mean that the HJM-model price is as good as the field theory model for small σ_0; rather the result shows that the HJM approximation of the field theory model agrees with the exact result of the one factor HJM model for small σ_0. In fact, as shown in Fig. 12, the
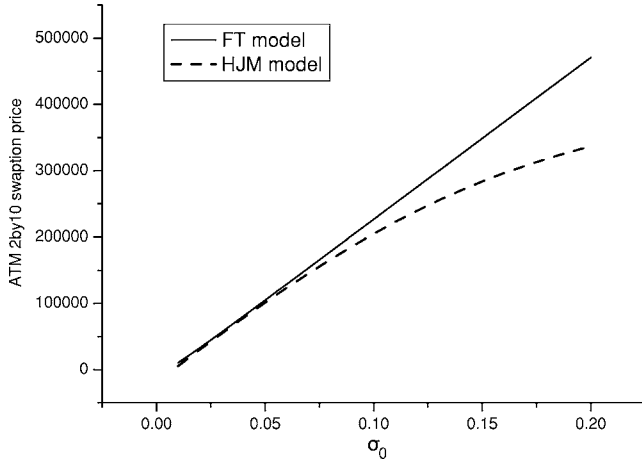
FIG. 14. The difference of the swaption price, at the money, as a function of σ_0, of the one factor HJM model and the field theory model with $D(x,x';t) \to 1$; the volatility function is $\sigma_0 e^{-\lambda(x-t)}$ with $\lambda = 0.1$.

best version of the HJM model, which uses the market volatility and forward interest rates that are taken to be exactly correlated, systematically overprices the actual market price of the swaption, with the field theory approximation being much more accurate.

IX. CONCLUSIONS

Swaptions are amongst the most complex of financial instruments, and the pricing and hedging of swaptions is of major interest in the debt market. It is seen that the formalism of quantum finance can solve problems that otherwise would be analytically intractable and only amenable to numerical analysis.

The empirical value of the forward interest rates' volatility is a small parameter and hence can be used for developing a power series expansion for the price of the coupon bond options. The quantum finance formalism of forward interest rates provides a scheme for approximately evaluating the price of coupon bond options and swaptions.

The perturbation expansion using Feynman diagrams is realized by expanding the nonlinear terms in the partition function, and performing the path integral order by order using Gaussian path integrations. The approximate coupon bond option price shows that the correlation between the forward prices of bonds of different maturities plays a crucial role in yielding an accurate price for the swaptions. This result agrees with our intuition since it is the interaction between the various forward bond prices that should determine the price of a coupon bond option.

The case of the one-factor HJM swaption price with exponential volatility is seen to be a particular limit of the field theory formula. The HJM limit of an exactly correlated forward interest rates is seen to be inaccurate, leading to a systematic overpricing of the swaptions.

The pricing of swaptions and coupon bond options using the formalism of quantum finance yields a result that is empirically better than the industry standard HJM models as demonstrated in Ref. [6]. One may tentatively conclude that the formalism of quantum finance is a flexible and transparent theoretical tool that yields accurate results for interest rate derivatives.

APPENDIX A: COUPON BOND OPTION PRICE

A detailed field theory derivation is given of the coefficients A, B, C. Since $D=0$, Eqs. (48)–(50) yield the following:

$$A = \langle V^2 \rangle, \tag{A1}$$

$$B = \langle V^3 \rangle, \tag{A2}$$

$$C = \langle V^4 \rangle - 3A^2. \tag{A3}$$

The computation for the coefficient A is carried in complete detail. Writing out the expression for A yields

$$A = \sum_{i,j=1}^{N} J_i J_j \int D\mathcal{A}(e^{-\alpha_i - Q_i - \alpha_j - Q_j} - e^{-\alpha_i - Q_i} - e^{-\alpha_j - Q_j} + 1)$$

$$\times e^{S[\mathcal{A}]}/Z = \sum_{i,j=1}^{N} J_i J_j \int D\mathcal{A}(e^{-\alpha_i - Q_i - \alpha_j - Q_j} - 1)e^{S[\mathcal{A}]}/Z$$

since $\langle e^{-\alpha_i - Q_i} \rangle = 1$ due to the martingale condition for the forward measure. Note from the definition of Q_i given in Eq. (32),

$$Q_i + Q_j = \int_{R_i} \sigma\mathcal{A} + \int_{R_j} \sigma\mathcal{A}$$

$$= \int_{t_0}^{t_*} dt \int_{t_*}^{\infty} (h_i + h_j)(t,x)\mathcal{A}(t,x) \equiv \int_{R_{ij}} (h_i + h_j)\mathcal{A}. \tag{A4}$$

On performing the Gaussian integration over the quantum field $\mathcal{A}(t,x)$ using Eq. (8) one obtains, in abbreviated notation

$$A = \sum_{i,j=1}^{N} J_i J_j (e^{-\alpha_i - \alpha_j + (1/2) \int_{R_{ij}} (h_i+h_j)(t,x) D(x,x';t)(h_i+h_j)(t,x')} - 1)$$

$$= \sum_{i,j=1}^{N} J_i J_j (e^{G_{ij}} - 1) \tag{A5}$$

$$\Rightarrow G_{ij} = \int_{R_{ij}} h_i(t,x)D(x,x';t)h_j(t,x')$$

$$= \int_{t_0}^{t_*} dt \int_{t_*}^{T_i} dx \int_{t_*}^{T_j} dx' M(x,x';t). \tag{A6}$$

Note the diagonal terms involving $\int_{R_i} h_i(t,x)D(x,x';t)h_i(t,x')$, $\int_{R_j} h_j(t,x)D(x,x';t)h_j(t,x')$ in Eq. (A5) cancel against the drift terms α_i, α_j terms, respectively. The cross term $G_{ij} = \int_{R_{ij}} h_i(t,x)D(x,x';t)h_j(t,x')$ yields the final result for the coefficient A.[6]

A similar calculation yields the coefficient B given in Eq. (49). To evaluate coefficient C note from Eq. (A3), and writing out the coefficient A^2 using Eq. (A1) in a symmetric form, one obtains

$$C = \langle V^4 \rangle - 3A^2 = \left\langle \sum_{ijkl=1}^{N} J_i J_j J_k J_l (e^{-\alpha_i - Q_i} - 1)(e^{-\alpha_j - Q_j} - 1) \right.$$

$$\left. \times (e^{-\alpha_k - Q_k} - 1)(e^{-\alpha_l - Q_l} - 1) \right\rangle$$

$$- \sum_{ijkl=1}^{N} J_i J_j J_k J_l (e^{G_{ij} + G_{kl}} + e^{G_{jk} + G_{li}} + e^{G_{ik} + G_{jl}} - 3). \tag{A7}$$

Doing a calculation similar to the one carried out for the A coefficient, and using the martingale condition for the forward measure yields

$$C = \sum_{ijkl=1}^{N} J_i J_j J_k J_l [e^{G_{ij} + G_{ik} + G_{il} + G_{jk} + G_{jl} + G_{kl}} - e^{G_{ij} + G_{jk} + G_{ki}}$$

$$- e^{G_{ij} + G_{jl} + G_{li}} - e^{G_{ik} + G_{kl} + G_{li}} - e^{G_{jk} + G_{kl} + G_{lj}} - e^{G_{ij} + G_{kl}} - e^{G_{jk} + G_{il}}$$

$$- e^{G_{ik} + G_{jl}} + 2(e^{G_{ij}} + e^{G_{ik}} + e^{G_{il}} + e^{G_{jk}} + e^{G_{jl}} + e^{G_{kl}}) - 6]. \tag{A8}$$

To understand the significance of the various terms for coefficient C in Eq. (A8) consider the case of the forward bond propagator G_{ij} being a small parameter; an expansion of the coefficient C as a power series in G_{ij} yields

$$C = 16 \sum_{ijkl=1}^{N} J_i J_j J_k J_l G_{ij} G_{jk} G_{kl} + O(G_{ij}^4).$$

One sees that the terms in Eq. (A8) combine to cancel terms that are of lower order than the cubic term in the propagator, yielding the leading term to be of $O(G_{ij}^3)$. Furthermore, all the disconnected, generically represented by the Feynman diagram given in Fig. 7, are canceled out by the terms appearing after the leading term, and yield the final result that leading order quartic term consists of only the connected Feynman diagrams given in Fig. 6(d).

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[6]Field theorists will recognize that $e^{-\alpha_i - Q_i}$ is equal to the normal ordered expression: e^{-Q_i}:.

In general, for all field theories the partition function $Z(\eta)$ is given by the sum of all Feynman diagrams, both connected and disconnected, whereas the log of the partition function $\ln[Z(\eta)]$ is given by the sum of only the connected Feynman diagrams; for this reason all the coefficients $A, B, C, \ldots$ are given by only the connected Feynman diagrams [2].

APPENDIX B: ZERO COUPON BOND OPTION PRICE

The exact zero coupon bond option price for the field theory model is given by [1]

$$C_{zcb}(t_0, t_*, K) = B(t_0, t_*) \frac{1}{\sqrt{2\pi q^2}}$$

$$\times \int_{-\infty}^{+\infty} dW e^{-(1/2q^2)[W + \int_{t_*}^T dx f(t_0,x) + (q^2/2)]^2}$$

$$\times (e^W - K)_+. \tag{B1}$$

Making a change of variable yields

$$C_{zcb}(t_0, t_*, K) = B(t_0, t_*) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dW e^{-(1/2)W^2}$$

$$\times (F e^{qW - (q^2/2)} - K)_+, \tag{B2}$$

where $F \equiv \exp[-\int_{t_*}^T dx f(t_0,x)]$. A Taylors expansion in the W variable for the payoff function to $O(q^4)$, for $X = (K-F)/qF$, yields

$$(F e^{qW - (q^2/2)} - K)_+ = (F + qFW - K)_+ + qF \left[\left(\frac{q}{2}(W^2 - 1) \right. \right.$$

$$\left. + \frac{q^2}{6}(W^3 - 3W) \right) \partial_w (W - X)_+$$

$$\left. + \frac{q^2}{8}(W^2 - 1)^2 \partial_w^2 (W - X)_+ \right]. \tag{B3}$$

Using $(W^2 - 1)e^{-(1/2)W^2} = \partial_w^2 e^{-(1/2)W^2}$ and $(W^3 - 3W)e^{-(1/2)W^2} = -\partial_w^3 e^{-(1/2)W^2}$, doing an integration by parts, and using Eqs. (B3) and (61), yields

$$C_{zcb}(t_0, t_*, K) = \frac{B(t_0, t_*)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dW \left[qF\delta(W - X) \left(-\frac{q}{2} \partial_w e^{-(1/2)W^2} \right. \right.$$

$$\left. + \frac{q^2}{6} \partial_w^2 e^{-(1/2)W^2} + \frac{q^2}{8}(W^2 - 1)^2 e^{-(1/2)W^2} \right)$$

$$\left. + (F + qFW - K)_+ e^{-(1/2)W^2} \right] + O(q^4)$$

$$= B(t_0, t_*) \frac{qF}{\sqrt{2\pi}} \left(\frac{1}{2} qX + \frac{1}{6} q^2(X^2 - 1) + \frac{1}{8} q^2(X^2 \right.$$

$$\left. - 1)^2 \right) e^{-(1/2)X^2} + B(t_0, t_*) \frac{qF}{\sqrt{2\pi}} I(X) + O(q^4), \tag{B4}$$

where $I(X)$ is given in Eq. (64). The result obtained above agrees, as expected, with the zero coupon bond limit of the coupon bond option price given in Eq. (71).

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