Breathing dissipative solitons in three-component reaction-diffusion system

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We investigate the stability of the localized stationary solutions of a three-component reaction-diffusion system with one activator and two inhibitors. A change of the time constants of the inhibitors can lead to a destabilization of the stationary solution. The special case we are interested in is that the breathing mode becomes unstable first and the stationary dissipative soliton undergoes a bifurcation from a stationary to a "breathing" state. This situation is analyzed performing a two-time-scale expansion in the vicinity of the bifurcation point thereby obtaining the corresponding amplitude equation. Also numerical simulations are carried out showing good agreement with the analytical predictions.

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I. INTRODUCTION

The formation and dynamics of localized solitary patterns in dissipative systems have been the subject of substantial interest over the last decade. They can be found in different chemical systems [1–3], in biological systems as nerve pulses [4,5], as solitary states in optics [6–8], and as current filaments in semiconductor devices [9,10] or gas-discharge systems [11–13]. In the case of dissipative systems we refer to these objects as dissipative solitons (DSs), following Ref. [14–16]. Other nomenclature can also be found in the literature, e.g., autosolitons [17], oscillons in granular media [18], as well as spots [19] and pulses [20] in chemical systems.

The modeling of dissipative solitons often is carried out via reaction-diffusion (RD) equations [9,13,15,17,21,22]. In this context a DS is a self-organized well-localized solution of a continuous homogeneous system having in addition a stable stationary homogeneous solution. Recent experiments and numerical simulations for RD systems have shown that pattern formation in the form of localized structures can lead to a rich variety of behavior. We only mention stationary, traveling with a constant velocity [23,24], breathing [25], and oscillating DSs [3], as well as interacting DSs, among which scattering, formation of bound states, generation, and annihilation phenomena are observed [1,12,13,26,27].

In particular, breathing solutions in dissipative systems have attracted a great deal of attention in recent years. They have been found, e.g., as soliton pulsation in fiber laser [28,29], as rocking localized current filaments in *p-n-p-n* devices [30,31], and as breathing spots in chemical reactors [25]. Analytical investigations of breathing localized structures in one spatial dimension have been carried out for two-component reaction-diffusion systems with piecewise linear activator nullcline [32,33]. Numerical studies of breathing domains in an infinite medium in a two-component system with a cubic nonlinearity also in one spatial dimension have been performed in [34]. In the latter study a single stationary domain loses stability via a Hopf bifurcation, and a breathinglike oscillatory motion sets in; as the control parameter is changed further the amplitude of the oscillation grows, thereby leading to the collapse of the domain. Breathing and wiggling motion of layers in reaction-diffusion systems with multiple components in one dimension was investigated in [35]. Quasi-two-dimensional breathing spots have been found experimentally in a disk-shaped chemical reactor [25]. In these experiments similar to the theoretical investigation [34], a circular spot bifurcates to an oscillatory spot when the control parameter is increased beyond some critical value. Further increase of the latter leads to the collapse of the spot. However, these oscillations have been interpreted as an interaction of a front with the system's boundary and not as oscillations of the radius of the spot. Recently, localized breathers have been found in one- and two-dimensional neural media [36,37], described by a two-component system with an integral term. In particular, in two-dimensional excitable neural media the nonlocal inhibition leads to a symmetry breaking instability of the stationary pulse, resulting in a formation of a nonradially symmetric breather. In addition, the number of breathing lobes corresponds to the dominant unstable Fourier mode associated with perturbations of the stationary solution.

The name breather as such arose from studies of the sine-Gordone equation and can be considered as a bound state of its kink and antikink solutions, which oscillate with respect to each other. An important class of breathers is the so-called discrete breathers (or localized modes) [38]. These can be considered as solutions of a nonlinear lattice equation which are periodic in time and localized in space. Localized breathers are found both theoretically and experimentally in various physical systems where the space discreetness arises in a natural way, e.g., photonic crystals [39], Josephson junction [40], or Bose-Einstein condensates in periodic optical traps [41]. Discrete breathers are observed in conservative and dissipative systems in one and two spatial dimensions.

Another interesting example of a localized oscillating structure is oscillon, i.e., a stable localized excitation in a vibrating layer of sand [42]. Typically these localized circular regions oscillate between conical peaks and craters with a period of half of the external driving frequency and exist in a narrow region between the stability regions of spatially extended patterns and the ground state. Oscillons were also found in dissipative fluids and colloidal suspensions [18] and in autonomous chemical systems [3]. However, these localized objects result from interaction of subcritical Turing and Hopf instabilities. In contrast to breathers, an external forcing is needed to produce an oscillon (in the case of autonomous systems, a subcritical Hopf bifurcation plays such a role, possessing simultaneously a stationary steady state and stable limit cycle.)

In this paper we report on the transition from stationary to breathing DSs in three-component reaction-diffusion system with one activator and two inhibitors. In contrast to the reaction-diffusion models, mentioned above, where either piecewise linear nullclines or one-dimensional domains were studied, we consider the case of two spatial dimensions and a nonlinear reaction term. Notice that transition from the *homogeneous* stationary state to an oscillating solution in a spatially extended system was considered by many authors (see, e.g., [43,44]). In this case all wave vectors in some continuous interval become unstable and the instability is described in terms of the Ginzburg-Landau equation (see, e.g., [43], and references cited therein). In our case, however, the bifurcating stationary state is *inhomogeneous* making mathematical treatment much more complicated. On the other hand, as we will see, the solitary solution in question has both a continuous and a discrete spectrum and the bifurcation occurs when at least one discrete mode becomes unstable. Therefore one can develop a reduced description where only the slowly varying amplitudes of the unstable modes are taken into account near the bifurcation point. The amplitudes are subject to a system of ordinary differential equations that can be transformed to a Hopf's normal form.

The system we are interested in was first introduced in [45,46] as an extension of a phenomenological model for a planar dc gas-discharge system with semiconductor electrode:

$$\partial_t u = D_u \Delta u + f(u) - \kappa_3 v - \kappa_4 w + \kappa_1,$$

$$\tau \partial_t v = D_v \Delta v + u - v,$$

$$\theta \partial_t w = D_w \Delta w + u - w. \quad (1)$$

Here $u=u(\mathbf{r},t)$ denotes the activating component, whereas $v=v(\mathbf{r},t)$ and $w=w(\mathbf{r},t)$ denote the inhibiting components (this can be clarified similar to the two-component RD system [13,21]) and $\mathbf{r}\subset\mathbb{R}^2$. In the polynomial function $f(u) = \lambda u - u^3$ the coefficient λ is positive. D_u, D_v, D_w denote the (positive) diffusion coefficients of the components, whereas the positive parameters τ and θ represent dimensionless constants, being the ratios of the characteristic times of both inhibitors with respect to the that of the activator. The coefficient κ_1 violates the inversion symmetry $(u \mapsto -u)$ and has arbitrary sign. Finally, the constants κ_3 and κ_4 staying in the reaction term are also positive, indicating the inhibiting nature of v and w.

The system (1) can be considered as an extension of the FitzHugh-Nagumo equations for nerve pulse transmission [4,5]. Reaction-diffusion systems with more than two components have practical applications in various fields, e.g., blood clotting [47], population dynamics [48], ecology [49], etc. In the gas-discharge context, the model (1) is an extension of a two-component reaction-diffusion system, first derived in [50]. This model allows for the qualitative understanding of many stationary DSs and their bound states [51] in more than one spatial dimension. Moreover, such a model is also used to investigate single moving DSs, which can be stabilized by a global feedback term [23]. However, the latter cannot suppress the growth of antisymmetric combinations of the unstable modes of two or more DSs, so that solutions with several moving and interacting pulses do not exist [52]. This difficulty can be overcome by introducing a second inhibiting component in a phenomenological manner [45,46], which acts as a local feedback. A more detailed description explaining the physical meaning of the parameters can be found in [13,22].

Here we focus on the situation that the system (1) admits a nontrivial stationary solution, which is stable in a certain parameter region. In the simplest case it is a stationary localized structure with rotational symmetry. This stationary DS can lose its stability with the change of one or more control parameters. In what follows, we use the time constants τ and θ as control parameters, as the stationary solution does not depend on them.

Figure 1 shows examples of the behavior of a stationary DS for different control parameters values, as observed in numerical simulation of the system (1). In Fig. 1(a) a stationary DS is stable for given τ and θ, i.e., after some transients the appropriate initial distribution converges to a stationary DS, which is numerically stable on a long time scale. Figure 1(b) shows the behavior of the solution after the bifurcation: a stationary DS bifurcates to an oscillatory DS as a control parameter θ exceeds some critical value and τ is kept fixed. The amplitude of oscillations is built up and achieves some constant value, so that the DS oscillates with a constant amplitude on a long time scale. We refer to this oscillating with constant amplitude soliton as a breathing DS. Figure 1(c) shows another possible instability scenario. The stationary DS bifurcates to an oscillatory DS with increasing amplitude, but in this case it eventually leads to a collapse of the solution. Our goal now is to understand the instability scenario leading to breathing DSs.

II. AMPLITUDE EQUATION IN GENERAL FORM

We start from a reaction-diffusion system in the general form:

$$\partial_t \mathbf{u} = \mathfrak{L}\mathbf{u}. \quad (2)$$

Here $\mathbf{u}=\mathbf{u}(\mathbf{r},t)=(u_1,\ldots,u_n)^T$ is a vector function, $\mathbf{r}\subset\mathbb{R}^2$, $\mathfrak{L}$ is a real-valued nonlinear operator, that depends on some control parameter p,

$$\mathfrak{L} = D\Delta + \mathfrak{f}. \quad (3)$$

Here Δ denotes the Laplace operator, the diagonal matrix D contains the diffusion constants of u_i on the principial diagonal, and vector $\mathfrak{f}(\mathbf{u})$ stands for the nonlinear reaction term. Assume that the system (2) has a stationary solution $\mathbf{u_0}$, which is stable in a certain parameter region. If we consider now a small perturbation $\tilde{\mathbf{u}}$, so that $\mathbf{u}=\mathbf{u_0}+\tilde{\mathbf{u}}$, the corresponding equation takes the form

$$\partial_t \tilde{\mathbf{u}} = \mathfrak{L}'(\mathbf{u_0})\tilde{\mathbf{u}} + \frac{1}{2!}\mathfrak{L}''(\mathbf{u_0})\tilde{\mathbf{u}}\tilde{\mathbf{u}} + \frac{1}{3!}\mathfrak{L}'''(\mathbf{u_0})\tilde{\mathbf{u}}\tilde{\mathbf{u}}\tilde{\mathbf{u}} + \cdots, \quad (4)$$

where we assume that Taylor expansion of $\mathfrak{L}(\mathbf{u_0}+\tilde{\mathbf{u}})$ is possible. Here $\mathfrak{L}^{(n)}(\mathbf{u_0})$ denotes nth Fréchet derivative with re-

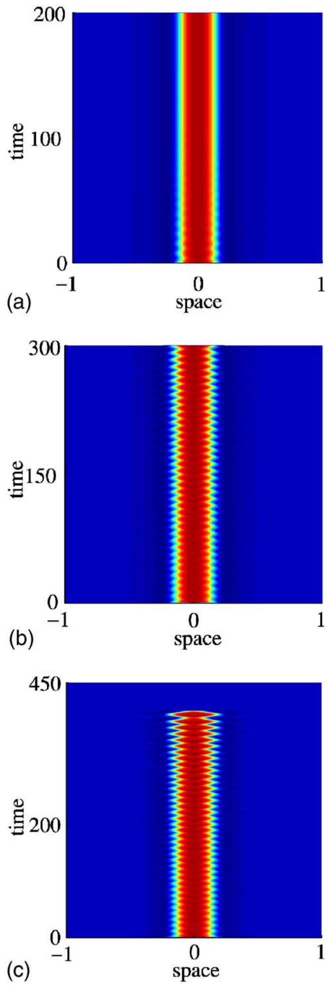

FIG. 1. (Color online) Space-time plots for different control parameters values obtained from numerical solution of Eq. (1). Time evolution of the cross section of the initial pulse close to the stationary solution is shown; (a) τ=0.5, θ=0.63, DS is stable; (b) τ=0.5, θ=0.64, amplitude of oscillations reaches a constant value; (c) τ=0.01, θ=1.22, increase of the control parameter beyond the critical value leads to the collapse of the soliton. Other parameters: $D_u=4.7\times10^{-3}$, $D_v=0$, $D_w=0.01$, $\lambda=5.67$, $\kappa_1=-1.04$, $\kappa_3=1.0$, $\kappa_4=3.33$. The calculations were performed on the rectangular domain $\Omega=[-1,1]\times[-1,1]$ on nonuniform triangular grid with maximal element size 0.1 with Neumann boundary conditions.

spect to $\mathbf{u}$ calculated at $\mathbf{u}=\mathbf{u_0}$. The first coefficient of the expansion (4) is the linearization of the operator $\mathfrak{L}$ around the stationary solution $\mathbf{u_0}$, the second term corresponds to the bilinear operator, acting on two perturbation vectors $\tilde{\mathbf{u}}$, etc. In the following we denote the set of eigenvalues of the linear operator $\mathfrak{L}'(\mathbf{u_0})$ as λ and corresponding eigenfunctions as $\mathcal{F}$, i.e.,

$$\mathfrak{L}'(\mathbf{u_0})\mathcal{F}=\lambda\mathcal{F}. \tag{5}$$

Notice that since the operator $\mathfrak{L}'(\mathbf{u_0})$ generally is not self-adjoint, its eigenvalues and eigenfunctions are usually complex. The stability of $\mathbf{u_0}$ implies that all eigenvalues but $\lambda=0$ [such a critical eigenvalue exists if, e.g., the system (2) features translational invariance] have negative real parts. Note also, that for reaction-diffusion systems with the operators of kind (3) the continuous spectrum of $\mathfrak{L}'(\mathbf{u_0})$ is separated from zero. That is, only a finite number of modes, situated near zero and belonging to the discrete spectrum, can become unstable by the change of the control parameter. The destabilization scenario we are interested in is that a pair of complex-conjugated eigenvalues passes through the imaginary axis as one gradually changes the control parameter p. That is, for some critical value $p=p_c$ the corresponding eigenvalues are purely imaginary, i.e., $\lambda_c=\pm i\omega$. In this case the real-valued vector function $\tilde{\mathbf{u}}$ can be represented as

$$\tilde{\mathbf{u}}=Ae^{i\omega t}\mathcal{F}_c+\text{c.c.}+r, \tag{6}$$

where $\mathcal{F}_c$ is the eigenfunction of the operator $\mathfrak{L}'(\mathbf{u_0})|_{p=p_c}:=\mathfrak{L}'_c$, corresponding to the λ_c. A is a constant oscillation amplitude and r represents the sum of the other decaying modes. Now if we increase the control parameter p, $p=p_c+\varepsilon$, where ε is positive and $\varepsilon\ll1$, the corresponding change of the factor $e^{\lambda t}$ can be "included" in A and the amplitude A becomes a slow function of time, i.e.,

$$\tilde{\mathbf{u}}=A(t)e^{i\omega t}(\mathcal{F}_c+\varepsilon\mathcal{F}_\varepsilon)+\text{c.c.}+r, \tag{7}$$

where $\partial_tA\sim\varepsilon A$. Here $\varepsilon\mathcal{F}_\varepsilon$ represents a deviation of the eigenfunction $\mathcal{F}$ from $\mathcal{F}_c$. Our goal now is to write down the ordinary differential equation for the complex amplitude $A(t)$. For this purpose we substitute Eq. (7) into Eq. (4), equalize the terms with the same frequency, and obtain

$$\varepsilon A(\mathfrak{L}'_c(\mathbf{u_0})-i\omega)\mathcal{F}_\varepsilon=\partial_tA\mathcal{F}_c-\varepsilon A\mathfrak{L}'_\varepsilon(\mathbf{u_0})\mathcal{F}_c -\frac{1}{2}A^2\bar{A}\mathfrak{L}'''_c(\mathbf{u_0})\mathcal{F}_c\mathcal{F}_c\bar{\mathcal{F}}_c, \tag{8}$$

where $\mathfrak{L}'_\varepsilon(\mathbf{u_0})=\frac{\partial\mathfrak{L}'(\mathbf{u_0})}{\partial p}|_{p=p_c}$ represents the deviation of the operator $\mathfrak{L}'(\mathbf{u_0})$ from the $\mathfrak{L}'_c$, so that

$$\mathfrak{L}'\,(\mathbf{u_0})|_{p=p_c+\varepsilon}=\mathfrak{L}'_c+\varepsilon\mathfrak{L}'_\varepsilon.$$

Equation (8) can be interpreted as the equation with respect to the unknown vector function $\mathcal{F}_\varepsilon$. In accordance with Fredholm alternative this equation is solvable if and only if the right-hand side of Eq. (8) is orthogonal to the kernel of the operator, adjoined to the $\mathfrak{L}'_c(\mathbf{u_0})-i\omega$. This kernel can be easily found and is represented by the eigenfunction $\mathcal{F}^*_c$ of the adjoint operator $\mathfrak{L}'^{\dagger}_c$, corresponding to the eigenvalue $-i\omega$. After projection Eq. (8) onto $\mathcal{F}^*_c$ and performing several transformations one obtains the following equation for the complex amplitude $A(t)$:

$$\partial_tA=\varepsilon a_1A+a_2A|A|^2, \tag{9}$$

which can be recognized as a normal form of a Hopf bifurcation [43]. The coefficients a_1 and a_2 are complex and can be expressed as

$$a_1=\frac{\langle\mathfrak{L}'_\varepsilon(\mathbf{u_0})\mathcal{F}_c|\mathcal{F}^*_c\rangle}{\langle\mathcal{F}_c|\mathcal{F}^*_c\rangle},\quad a_2=\frac{\langle\mathfrak{L}'''_c(\mathbf{u_0})\mathcal{F}_c\mathcal{F}_c\bar{\mathcal{F}}_c|\mathcal{F}^*_c\rangle}{2\langle\mathcal{F}_c|\mathcal{F}^*_c\rangle}, \tag{10}$$

where $\langle\cdot|\cdot\rangle$ denotes the scalar product defined in terms of full spatial integration over the considered domain, like, e.g., in [22,24,53].

Equation (9) has a trivial solution $A=0$, which for $\mathrm{Re}(a_1)<0$ is a stable focus. If now $\mathrm{Re}(a_1)$ increases and passes through zero the trivial solution becomes unstable. For $\mathrm{Re}(a_1)>0$ one can also find the nontrivial periodic solution $A_0(t)=\mathrm{R}e^{i\nu t}$, where

$$R=\sqrt{-\varepsilon\frac{\mathrm{Re}(a_1)}{\mathrm{Re}(a_2)}},\quad \nu=\varepsilon\,\mathrm{Im}(a_1)+R^2\,\mathrm{Im}(a_2).$$

Linearization of Eq. (9) about A_0 shows that this solution is stable for $\mathrm{Re}(a_2)<0$ and unstable, if $\mathrm{Re}(a_2)>0$. In the former case the instability is stabilized by a limit cycle (nonlinear stabilization) that corresponds to supercritical bifurcation. In the latter case $[\mathrm{Re}(a_2)>0]$ no stabilization takes place and bifurcation is subcritical ([43,54]).

An additional point to emphasize is that the amplitude equation, derived in a similar situation for the steady state solution of a set of ordinary differential equations, e.g., in [43], contains in its nonlinear part contributions from the second harmonics, which are absent in Eq. (10). In order to find this correction in our case one needs to invert the differential operator $\mathfrak{L}'(\mathbf{u_0})$ explicitly, that usually is not possible. Therefore we estimated this correction for Eqs. (1). Direct calculation for different parameter sets shows that the correction is much smaller than the coefficient a_2 and thus has no significant contributions. This is also confirmed by numerical results, presented hereafter.

III. AMPLITUDE EQUATION FOR THREE-COMPONENT REACTION-DIFFUSION SYSTEM

Let us now apply the amplitude equation (9) to the three-component system (1). First of all we can solve the eigenvalue problem (5) numerically using the implicitly restarted Arnoldi method [55,56] for different values of the control parameters τ and θ, keeping all other parameters fixed. For the sake of simplicity we decompose the small perturbation $\tilde{\mathbf{u}}$ of the radially symmetric stationary solution $\mathbf{u_0}$ into a Fourier series and rewrite the eigenvalue problem (5) in terms of the amplitude $\mathbf{u}_n$ of the perturbation of the stationary solution with angular dependence $e^{in\varphi}$. The influence of the modes with different n on the radial-symmetrical DS can be understood as follows: the mode with $n=0$ (breathing mode) results in the change of the size of the DS, i.e., in the change of the size of the threshold boundary of the radially symmetrical DS. The threshold boundary can be defined as, e.g., a cross section of the DS at the height of a half soliton's amplitude. The mode with $n=1$ describes the shift of the solution and $n\geqslant 2$ leads to different deformations of the DS. The stability diagram and the schematic illustration of the influence of the modes $n=0$ and $n=1$ on the radially symmetrical object can be seen in Fig. 2. Here under influence we understand small perturbations in terms of Fourier modes (dotted line) associated with general perturbations of the threshold boundary of, e.g., the activating component of the stationary DS (solid line).

One can easily see that for small values of τ and θ the stationary solution is stable. An increase of the constants leads to the excitation of either the $n=0$ or $n=1$ mode (see the lines in Fig. 2, which separate different stability regions and indicate the excitation threshold of the mode in question). Notice that for the chosen parameter set the breathing mode $n=0$ becomes unstable first for all θ but $\theta=0$. The latter situation was investigated in, e.g., [24] and leads to the drift bifurcation of DSs. The modes $n\geqslant 2$ become unstable for larger values of τ and θ and cannot be responsible for the primary destabilizaion. So, we consider the destabilization caused by the breathing mode $n=0$. Figure 3 shows an example of this mode calculated as a solution of Eq. (5) for given values of τ and θ. Both real [Fig. 3(a)] and imaginary [Fig. 3(b)] parts of the first component of $\mathcal{F}=(\mathcal{F}_u,\mathcal{F}_v,\mathcal{F}_w)^T$ are shown. As the eigenvalues corresponding to the breathing mode are complex, the linearization operator $\mathfrak{L}'(\mathbf{u_0})$ is not self-adjoint. Nevertheless, it can be represented as a product [27,53],

$$\mathfrak{L}'(\mathbf{u_0})=ML(\mathbf{u_0}),\tag{11}$$

of an invertible matrix

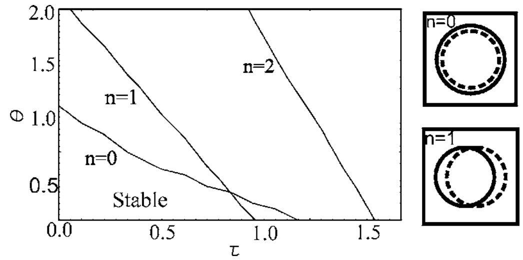

FIG. 2. Stability diagram in (τ,θ) plane, calculated for a solitary solution of the system (1). Lines separate stability regions, corresponding to different modes $n=0,1,2$. Parameters are the same as in Fig. 1. On the right-hand side the dotted line shows the influence of a breathing mode $n=0$ and a drift mode $n=1$ on a radially symmetrical stationary DS (e.g., its activating component), depicted by the solid line.

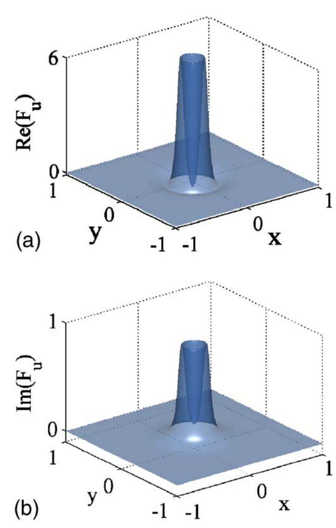

FIG. 3. (Color online) A numerical solution of the eigenvalue problem (5) for $n=0$ mode. Real (a) and imaginary (b) parts of the activator component of the breathing mode are shown. The corresponding eigenvalues are $\lambda=-0.04\pm 0.65i$. Control parameters are $\tau=0.8$, $\theta=0.35$. The other parameters are the same as in Fig. 1.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/\kappa_3\tau & 0 \\ 0 & 0 & -1/\kappa_4\theta \end{pmatrix}$$

and a self-adjoint operator

$$L = \begin{pmatrix} D_u\Delta + \lambda - 3u_0^2 & -\kappa_3 & -\kappa_4 \\ -\kappa_3 & -\kappa_3 D_v\Delta + \kappa_3 & 0 \\ -\kappa_4 & 0 & -\kappa_4 D_w\Delta + \kappa_4 \end{pmatrix}.$$

Because of Eq. (11) the eigenfunctions $\mathcal{F}^*$ of the adjoint operator $\mathcal{L}'^{\dagger}(\mathbf{u_0})$ can be calculated as

$$\mathcal{F}^* = M^{-1}\overline{\mathcal{F}}, \tag{12}$$

where the overline stands for complex conjugate. That is, using the relation (11) and expression (12) one can calculate all scalar products in Eq. (10) alone in terms of the critical eigenfunction $\mathcal{F}_c := (\mathcal{F}_{cu}, \mathcal{F}_{cv}, \mathcal{F}_{cw})^T$ of the linearization operator $\mathcal{L}'(\mathbf{u_0})$. For example, the scalar product, standing in the numerator of the coefficient a_1, can be presented as

$$\begin{aligned}\langle \mathfrak{L}'_{\varepsilon}(\mathbf{u_0})\mathcal{F}_c | \mathcal{F}_c^* \rangle &= \langle M_\theta L'(\mathbf{u_0})\mathcal{F}_c | M_c^{-1}\overline{\mathcal{F}}_c \rangle \\ &= i\omega \langle M_\theta M_c^{-1}\mathcal{F}_c | M_c^{-1}\overline{\mathcal{F}}_c \rangle = i\omega\kappa_4 \langle \mathcal{F}_{cw}^2 \rangle,\end{aligned}$$

where M_c stands for the matrix M, calculated for the critical value of the control parameter $\theta = \theta_c$, $M_\theta = \frac{\partial M}{\partial \theta}|_{\theta=\theta_c}$, and $\langle \mathcal{F}_c^2 \rangle = \langle \mathcal{F}_c | \bar{\mathcal{F}}_c \rangle$. The other scalar products in Eq. (10) can be obtained in a similar way. The relations for the coefficients a_1 and a_2 for the system (1) take the form

$$a_1 = \frac{i\omega\kappa_4 \langle \mathcal{F}_{cw}^2 \rangle}{\langle \mathcal{F}_{cu}^2 \rangle - \kappa_3\tau \langle \mathcal{F}_{cv}^2 \rangle - \kappa_4\theta_c \langle \mathcal{F}_{cw}^2 \rangle},$$

$$a_2 = -\frac{3\langle \mathcal{F}_{cw}^2 | \mathcal{F}_{cu}|^2 \rangle}{\langle \mathcal{F}_{cu}^2 \rangle - \kappa_3\tau \langle \mathcal{F}_{cv}^2 \rangle - \kappa_4\theta_c \langle \mathcal{F}_{cw}^2 \rangle}.$$

IV. NUMERICAL RESULTS

As has already been indicated, the special form of the linearization operator $\mathfrak{L}'(\mathbf{u_0})$ permits us to express the coefficients a_1 and a_2 in terms of $\mathcal{F}_c$. As this critical eigenfunction is not known analytically we have found it together with the corresponding eigenvalue $i\omega$ and critical value of the control parameter θ_c by solving the eigenvalue problem (5) numerically for different values τ and θ. Based on this, the instability increment $\varepsilon \mathrm{Re}(a_1)$ can be calculated. The increment can be also obtained directly from Eq. (1); to this end a time evolution of radius of the breathing DS was calculated. The increment was then found from the corresponding envelope. Numerical results for the increment are presented in Fig. 4; open squares are from the direct simulation and solid triangles are from the amplitude equation (9).

In addition, we have performed direct numerical simulations of Eq. (9) for different values of τ and θ. The results are shown in Fig. 5. Figure 5(a) demonstrates a typical solution of Eq. (9) in the case of nonlinear stabilization, i.e., for

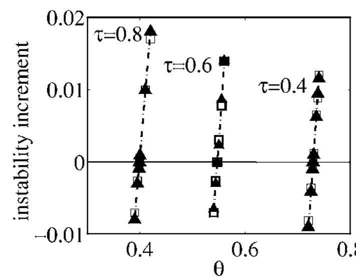

FIG. 4. Dependence of the instability increment $\varepsilon \mathrm{Re}(a_1)$ on the control parameter θ for three different τ as results from the reduced model (9) (▲) and direct simulation of the system (1) (□). Other parameters are $D_u = 4.7 \times 10^{-3}$, $D_v = 0$, $D_w = 0.01$, $\lambda = 5.67$, $\kappa_1 = -1.04$, $\kappa_3 = 1.0$, $\kappa_4 = 3.33$.

$\mathrm{Re}(a_1) > 0$ and $\mathrm{Re}(a_2) < 0$. In this case the solution on the complex plane is represented by an unstable focus surrounded by a stable limit cycle of the radius $R = \sqrt{-\varepsilon \frac{\mathrm{Re}(a_1)}{\mathrm{Re}(a_2)}}$. It is necessary to stress that a direct calculation of the complex amplitude $A(t)$ for all t from the system (1) involves some difficulties, while the derivation of the limit cycle radius is relatively simple. The latter can be seen in Fig. 5(a) as a dotted line.

In Fig. 5(b) a typical solution, corresponding to the subcritical regime with $\mathrm{Re}(a_1) > 0$, $\mathrm{Re}(a_2) > 0$, is shown. In this case the solution on the phase plane corresponds to the unstable focus. In order to compare this result with the full model, direct simulations of the system (1) for the same parameter set have been performed; the obtained solution

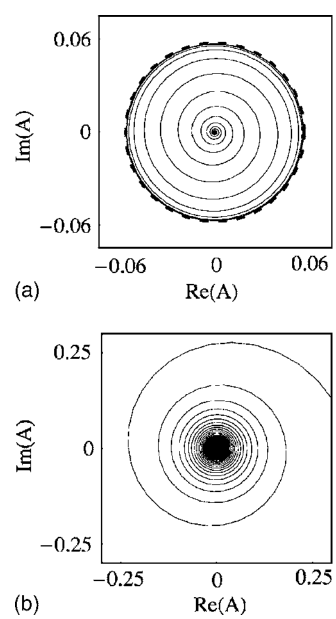

FIG. 5. Numerical solutions of Eq. (9) on the phase plane $[\mathrm{Re}(A), \mathrm{Im}(A)]$ in super- and subcritical regimes. (a) The typical solution of the amplitude equation for $\mathrm{Re}(a_2) < 0$, calculated for $\tau = 0.5$, $\theta = 0.65$. The dotted line corresponds to the limit circle, obtained as the solution of the full three-component system. (b) The typical solution of the reduced system for $\mathrm{Re}(a_2) > 0$. Control parameters are $\tau = 0.7$ and $\theta = 0.55$. Other parameters are the same as in Fig. 4.

clearly shows the absence of the nonlinear stabilization, i.e., the stationary solution becomes unstable and starts to oscillate with an increasing amplitude which finally leads to the destruction of the solution by the oscillations.

It should be mentioned that similar to [25], further increase of the control parameter in the supercritical regime can also lead to the destruction of the limit cycle and the collapse of the soliton. The latter transition cannot be described by the use of Eq. (9). Indeed, the amplitude equation (9) is derived in the vicinity of the bifurcation point θ_c, which is to say that the sign of the coefficients a_1 and a_2 is calculated in this critical point. On the other hand, numerical simulations show that the destruction of the limit cycle usually takes place far beyond the vicinity of the bifurcation point. However, this is out of scope of the amplitude equation (9), presented in this paper.

V. CONCLUSION

In this paper we have presented analytical and numerical investigations of breathing DSs in a three-component reaction-diffusion system. These breathing solitons can be considered as a result of a Hopf bifurcation of a single stationary DS if one gradually changes the control parameter, e.g., the time constant of the second inhibitor. In this case the stationary DS bifurcates to the oscillatory one either with a constant or increasing amplitude, in the latter case the soliton collapses to a homogeneous stable state. This situation was analyzed performing two-time-scale expansion in the vicinity of the bifurcation point and the corresponding amplitude equation, being a normal form of the Hopf bifurcation, is derived. The information about the system behavior in the vicinity of the bifurcation point is now contained in the complex coefficients of this equation. The latter are the functions of the stationary solution and breathing eigenfunction and can be calculated from the original system. Depending on the sign of the coefficients this equation shows the two instability scenarios. We also have calculated the full system as well as the reduced amplitude equation numerically. The results show that both approaches are in good agreement.

Notice that the amplitude equation (9) is a well-known normal form of Hopf bifurcation, studied in detail for the steady-state solution of a set of ordinary differential equations [43]. In our case, in contrast, the instability of nonhomogeneous stationary solution in two spatial dimensions is discussed, which makes the considered problem more complicated. In particular, the calculation of the correction, connected with a second harmonic, was complicated because of difficulties with the inversion of the operator $\mathfrak{L}'(\overline{\mathbf{u}})$, whereas for the steady state solution it is rather simple because in this case one needs to invert just a matrix.

To conclude, let us consider if a breathing DS can be found in other systems of reaction-diffusion type. The key assumption underlying the very derivation of the amplitude equation (9) is that the stationary DS is destabilized via the supercritical Hopf mechanism leading to the primary excitation of the breathing mode. To our knowledge the only general recipe here is to solve the corresponding eigenvalue problem for the linearization operator. If this property is established, our results for $a_{1,2}$ in Eq. (9) have a very general nature and the coefficients can be calculated for any RD system. The existence of the breathing DS is then guaranteed if the nonlinear term in Eq. (9) stabilizes the instability.

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