Resonant symmetry lifting in a parametrically modulated oscillator

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We study a parametrically modulated oscillator that has two stable states of vibration at half the modulation frequency ω_F. Fluctuations of the oscillator lead to interstate switching. A comparatively weak additional field can strongly affect the switching rates because it changes the switching activation energies. The change is linear in the field amplitude. When the additional field frequency ω_d is $\omega_F/2$, the field makes the populations of the vibrational states different, thus lifting the states symmetry. If ω_d differs from $\omega_F/2$, the field modulates the state populations at the difference frequency, leading to fluctuation-mediated wave mixing. For an underdamped oscillator, the change of the activation energy displays characteristic resonant peaks as a function of frequency.

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I. INTRODUCTION

A parametrically modulated oscillator is one of the simplest physical systems that display spontaneous breaking of time-translation symmetry. When the modulation is sufficiently strong, the oscillator has two states of vibration at half the modulation frequency, the period-2 states [1]. They are identical except for the phase shift by π, but for each of them the symmetry with respect to time translation by the modulation period is broken. Fluctuations of the oscillator lead to switchings between the states. Switching rates are equal by symmetry, for stationary fluctuations. The switchings ultimately make the state populations equal, thus restoring the full time-translation symmetry. Experimental studies of fluctuation-induced switching in classical parametrically modulated systems were done for trapped electrons [2], optically trapped atoms [3,4], and microelectromechanical systems [5]. The obtained switching rates are in good agreement with the theory [6].

The degeneracy of period-2 states can be lifted if, in addition to parametric modulation at frequency ω_F, a system is driven at frequency $\omega_F/2$. In the frame oscillating at frequency $\omega_F/2$, the period-2 states and the additional field look static. The system reminds an Ising ferromagnet, with the period-2 states and the additional field playing the roles of spin orientations and an external magnetic field, respectively. One can expect that the role of the direction of the magnetic field is played by the phase of the additional field counted off from the phase of one of the period-2 states. Depending on the field phase, one or the other state should be predominantly occupied.

In this paper, we study resonant symmetry lifting in a parametric oscillator, which occurs where the frequency of the additional field is close to the oscillator eigenfrequency. The field makes the rates of switching between the period-2 states, W_{12} and W_{21}, different from each other. In turn, this leads to a difference of the stationary state populations. This difference may become large even for a comparatively weak field, as can be surmised from the analogy with the problem of a ferromagnet. There the change of the state populations becomes large when the energy difference of the states due to an external magnetic field exceeds kT, which happens already for weak fields where this difference itself is small compared to the internal energy.

In contrast to the case of a ferromagnet, the energy of a parametrically modulated oscillator is not conserved and its stationary distribution is not of the Boltzmann form. However if the fluctuation intensity is small, the dependence of the switching rates on this intensity is often of the activation type, see Refs. [7,8] and papers cited therein. This applies not only to a classical, but also to a quantum oscillator, where switching is due to quantum fluctuations [9]. An additional field changes the effective switching activation energies, and when this change exceeds the fluctuation intensity, the overall change of the switching rates becomes large. The effect is particularly strong if the field is resonant.

In what follows, we develop a theory of the switching rates W_{nm} for a classical oscillator. We find the dependence of the switching activation energies on the amplitude, phase, and frequency of the additional field and on the oscillator parameters. We first study the symmetry lifting by the field at frequency $\omega_d=\omega_F/2$. Of particular interest here is the vicinity of the bifurcation point where the period-2 states merge because, in this range, the rates W_{nm} are comparatively large and easy to control.

We are also interested in the situation where the additional field frequency ω_d is close but not equal to $\omega_F/2$. Here, the field-induced modulation of the switching probabilities causes oscillations of the state populations at frequency $|\omega_d-\omega_F/2|$. Such oscillations, superimposed on the oscillator vibrations at frequency $\omega_F/2$ in period-2 states, lead to vibrations at frequency $|\omega_F-\omega_d|$, i.e., to a strong effective fluctuation-induced three-wave mixing.

The amplitude of the population oscillations becomes small when $|\omega_d-\omega_F/2|$ largely exceeds the switching rates. Yet, the rates themselves may be significantly changed by the additional field. Of primary interest in this case are the rates W_{nm} averaged over the period $4\pi/|2\omega_d-\omega_F|$ and their dependence on the additional field amplitude A_d and frequency ω_d. One may expect that, as in the case of equilibrium systems [10–12], the rate change is quadratic in A_d for small A_d and corresponds to effective heating of the system by the field. For underdamped equilibrium systems, this heating can be resonant, as seen for modulated Josephson junctions [13,14]. However, for somewhat stronger fields the change of the logarithm of the switching rate should become linear in A_d [15,16]. This happens when the properly scaled field amplitude exceeds temperature.

We show that, for a parametric oscillator, the activation energies are indeed linear in the additional field amplitude A_d, when it is not too small (but is also not too large). When the oscillator is underdamped, the factor multiplying A_d displays a characteristic, very different from equilibrium systems resonant frequency dependence. We develop a technique that allows us to find this dependence in an explicit form. The asymptotic analytical results are compared to the results of numerical calculations of the activation energies.

In Sec. II, we discuss the Langevin equation for a parametrically modulated nonlinear oscillator in the rotating frame and give the general expression for the probability of switching between coexisting stable vibrational states. In Sec. III, we obtain a general expression for the correction to the activation energy of switching and show that it is linear in A_d. We study resonant symmetry lifting of the switching rates. In Sec. IV, we investigate low-frequency oscillations of the state populations and fluctuations-mediated resonant wave mixing. In Sec. V, an expression for the period-averaged switching rate is given. In Sec. VI, we consider symmetry lifting and the frequency dependence of the activation energy close to the bifurcation point where the period-2 states merge. In Sec. VII, we study the case of weak damping. We show that the change of the activation energy may display characteristic asymmetric resonant peaks as a function of the additional field frequency. Section VIII contains concluding remarks.

II. LANGEVIN EQUATION AND SWITCHING RATES

We will study switching between period-2 states of a nonlinear oscillator, which is parametrically modulated by a force $F\cos(\omega_F t)$ and additionally driven by a comparatively weak field $A_d\cos(\omega_d t+\phi_d)$ at frequency $\omega_d\approx\omega_F/2$. The Hamiltonian of the oscillator is a sum of the term that describes the motion without the extra field and the term proportional to the field, $H_{\rm osc}=H_{\rm osc}^{(0)}+H_{\rm osc}^{(d)}$,

$$H_{\rm osc}^{(0)}=\frac{1}{2}p_0^2+\frac{1}{2}q_0^2[\omega_0^2+F\cos(\omega_F t)]+\frac{1}{4}\gamma q_0^4,$$

$$H_{\rm osc}^{(d)}=-q_0A_d\cos(\omega_d t+\phi_d) \tag{1}$$

(q_0 and p_0 are the coordinate and momentum of the oscillator). We will assume that the modulation frequency ω_F is close to twice the frequency of small amplitude vibrations ω_0, and that the driving force F is not too large so that the oscillator nonlinearity remains small,

$$|\omega_F-2\omega_0|,\quad |\omega_d-\omega_0|\ll\omega_0,$$

$$F\ll\omega_0^2,\quad |\gamma|\langle q^2\rangle\ll\omega_0^2. \tag{2}$$

In what follows, for concreteness we set $\gamma>0$.

Following the standard procedure [1], we change to the rotating frame and introduce the dimensionless canonical coordinate Q and momentum P,

$$q_0(t)=C\left[P\cos\left(\frac{\omega_F t}{2}\right)-Q\sin\left(\frac{\omega_F t}{2}\right)\right],$$

$$p_0(t)=-C\frac{\omega_F}{2}\left[P\sin\left(\frac{\omega_F t}{2}\right)+Q\cos\left(\frac{\omega_F t}{2}\right)\right], \tag{3}$$

where $C=(2F/3\gamma)^{1/2}$. In these variables, the Hamiltonian becomes equal to $\tilde{H}_{\rm osc}=(F^2/6\gamma)g(Q,P)$, with $g=g^{(0)}+g^{(d)}(\tau)$,

$$g^{(0)}=\frac{1}{4}(P^2+Q^2)^2+\frac{1}{2}(1-\mu)P^2-\frac{1}{2}(1+\mu)Q^2,$$

$$g^{(d)}(\tau)=-a_d[P\cos(\nu_d\tau+\phi_d)+Q\sin(\nu_d\tau+\phi_d)]. \tag{4}$$

Here, we introduced dimensionless time τ and dimensionless parameters μ and ν_d. These parameters characterize, respectively, the detuning of the modulation frequency from twice the oscillator eigenfrequency and the detuning of the weak-field frequency from $\omega_F/2$, i.e., an effective "beat frequency" with the subharmonic of the strong field,

$$\mu=\frac{\omega_F(\omega_F-2\omega_0)}{F},\quad \nu_d=\frac{\omega_F(2\omega_d-\omega_F)}{F},$$

$$\tau=\frac{tF}{2\omega_F}. \tag{5}$$

The parameter $a_d=A_d(6\gamma/F^3)^{1/2}$ is the dimensionless amplitude of the additional driving field. In obtaining Eq. (4), we used the rotating wave approximation and disregarded fast oscillating terms $\propto\exp(\pm in\omega_F t), n\geqslant 1$. In the quantum formulation, the eigenvalues of $g^{(0)}(Q,P)$ give the scaled quasienergy of the system [9], and in what follows for brevity we call g quasienergy.

We will assume that the interaction with a bath that leads to dissipation of the oscillator is sufficiently weak, so that the oscillator is underdamped. Then, under fairly general assumptions [17], in the rotating frame dissipation is described by an instantaneous friction force (no retardation). Also the noise spectrum is, generally, practically flat in a comparatively narrow frequency range of width $\sim F/\omega_F$ centered at ω_0; this is the most interesting range, since the oscillator filters out noise at frequencies far from this range. Therefore, with respect to the slow time τ, the noise can be assumed white. The oscillator motion is described by the Langevin equation, which can be conveniently written in a vector form as

$$\dot{\mathbf{q}}\equiv\frac{d\mathbf{q}}{d\tau}=\mathbf{K}+\mathbf{f}(\tau),\quad \mathbf{K}=\mathbf{K}^{(0)}+\mathbf{K}^{(d)}, \tag{6}$$

with

$$\mathbf{K}^{(0)}=-\zeta^{-1}\mathbf{q}+\hat{\epsilon}\,\boldsymbol{\nabla}g^{(0)},\quad \mathbf{K}^{(d)}(\tau)=\hat{\epsilon}\,\boldsymbol{\nabla}g^{(d)}(\tau). \tag{7}$$

Here, all vectors have two components, $\mathbf{q}\equiv(Q,P)$, $\mathbf{K}\equiv(K_Q,K_P)$, and $\boldsymbol{\nabla}\equiv(\partial_Q,\partial_P)$, while $\hat{\epsilon}$ is the permutation tensor, $\epsilon_{QQ}=\epsilon_{PP}=0$, $\epsilon_{QP}=-\epsilon_{PQ}=1$. The parameter ζ^{-1} in Eq. (6) gives the oscillator friction coefficient in the units of $F/2\omega_F$. We use for ζ and μ the same notations as in Refs. [6,9].

Because of the extra field $\propto a_d$, the function $\mathbf{K}^{(d)}$ explicitly depends on time.

The function $\mathbf{f}(\tau)$ is a random force. Its two components are independent white Gaussian noises with the same intensity [17],

$$\langle f_Q(\tau) f_Q(0)\rangle = \langle f_P(\tau) f_P(0)\rangle = 2D\delta(\tau).$$

The noise intensity D is the smallest parameter of the theory. If both the friction force and the noise come from coupling to a thermal reservoir at temperature T, then we have $D = 6\gamma kT/F\zeta\omega_F^2$ (the parameter D corresponds to $D\zeta^{-2}/2$ in Ref. [6]).

A. Oscillator dynamics in the absence of noise

In the absence of noise and the symmetry-breaking field $\propto A_d$, in the range

$$\mu_B^{(1)} < \mu < \mu_B^{(2)}, \quad \mu_B^{(1,2)} = \mp (1-\zeta^{-2})^{1/2} \tag{8}$$

the parametrically modulated oscillator has two stable period-two states $\mathbf{q}_{1,2}^{(0)}$ and an unstable state $\mathbf{q}_b^{(0)}$. These states are the stationary solutions of equation $\mathbf{K}^{(0)}=0$. They merge for $\mu=\mu_B^{(1)}$. The stable states 1, 2 are inversely symmetrical, $\mathbf{q}_2^{(0)} = -\mathbf{q}_1^{(0)}$. For concreteness, we choose

$$Q_1^{(0)} = -Q_2^{(0)} > 0.$$

The vibration amplitude in the unstable state is zero, $\mathbf{q}_b^{(0)} = 0$. For $\mu > \mu_B^{(2)}$, the state $\mathbf{q}=0$ becomes stable and there additionally emerge two unstable period-2 states [1].

We will be interested in a comparatively weak symmetry-breaking field. Respectively, we will assume that the reduced field amplitude a_d is small, so that the field does not lead to new stable states. It just makes the stationary states periodic, for $\nu_d \neq 0$, or shifts them, for $\nu_d=0$. The correspondingly modified states are given by the periodic solutions of equation $\dot{\mathbf{q}}=\mathbf{K}$ or by equation $\mathbf{K}=\mathbf{0}$. In the laboratory frame, the periodic stable states $\mathbf{q}_{1,2}(\tau)$ correspond to oscillator vibrations at frequency $\omega_F/2$ weakly modulated at frequency $|\omega_F - 2\omega_d|/2$. They have spectral components at $\omega_F/2, \omega_d$, and the "mirror" frequency $|\omega_F - \omega_d|$.

We will limit ourselves to the analysis of switching in the parameter range (8). In this range escape from a period-2 state leads to switching to a different period-2 state. For $\mu > \mu_B^{(2)}$, escape may result in a transition to the zero-amplitude state (from which the system may also escape to one of the period-2 states). The results of the paper immediately extend to this range, but this extension will not be discussed. Therefore we use the terms "escape" and "switching" intermittently.

B. Switching rates: General formulation

The noise $\mathbf{f}(t)$ leads to fluctuations about the stable states and to interstate transitions. When the noise is weak, fluctuations have small amplitude on average. Interstate transitions require large outbursts of noise and therefore occur infrequently. For Gaussian δ-correlated noise the probability of a transition from nth to mth stable period-2 state has activation dependence on the noise intensity D and is given by the expression [7,8]

$$W_{nm} = C_W \exp\left(-\frac{R_n}{D}\right),$$

$$R_n = \min \int_{-\infty}^{\infty} d\tau\, L(\dot{\mathbf{q}}, \mathbf{q}; \tau), \quad L = \frac{1}{4}(\dot{\mathbf{q}} - \mathbf{K})^2. \tag{9}$$

The quantity R_n is the activation energy of a transition. It is given by the solution of a variational problem. The minimum in Eq. (9) for R_n is taken with respect to trajectories $\mathbf{q}(\tau)$ that start for $\tau \to -\infty$ at the initially occupied stable state $\mathbf{q}_n(\tau)$ and asymptotically approach $\mathbf{q}_b(\tau)$ for $\tau \to \infty$. The optimal trajectory $\mathbf{q}_{n\,\mathrm{opt}}(\tau)$ that minimizes R_n is called the most probable escape path (MPEP). The system is most likely to follow this trajectory in a large fluctuation that leads to escape.

The prefactor in the transition probability is $C_W \propto F/\omega_F$ in unscaled time t. It weakly depends on the noise intensity and will not be discussed in what follows.

The variational problem (9) for R_n can be associated with the problem of dynamics of an auxiliary system with Lagrangian L and coordinates $\mathbf{q}=(Q,P)$, and with Hamiltonian

$$H = H^{(0)} + H^{(d)}, \quad H^{(0)} = \mathbf{p}^2 + \mathbf{p}\mathbf{K}^{(0)},$$

$$H^{(d)} = \mathbf{p}\mathbf{K}^{(d)}. \tag{10}$$

In terms of this auxiliary Hamiltonian system, the MPEP of the original dissipative system corresponds to a heteroclinic trajectory that goes from the periodic (or stationary, for $\nu_d = 0$) state $[\mathbf{q}_n(\tau), \mathbf{p}=\mathbf{0}]$ to the periodic state $[\mathbf{q}_b(\tau), \mathbf{p}=\mathbf{0}]$.

III. PERTURBATION THEORY FOR ACTIVATION ENERGY

When the additional field $\propto a_d$ is weak, the activation energy R_n is close to its value $R^{(0)}$ for $a_d=0$, which is the same for the both states $n=1,2$. Even though the field-induced correction to $R^{(0)}$ is small compared to $R^{(0)}$, it may significantly exceed the noise intensity D. Then the overall change of the transition probability W_{nm} will be exponentially large.

The correction to the activation energy was studied earlier for thermal equilibrium systems additionally modulated by a comparatively weak periodic field [15,16]. As mentioned in the Introduction, it was found that the correction to R_n is linear in the field amplitude. The factor multiplying the amplitude gives the logarithm of the transition probability and therefore was called the logarithmic susceptibility (LS). We show now that the correction to the activation energy for a parametrically modulated oscillator is also linear in the amplitude of the additional field, $\delta R_n = R_n - R^{(0)} \propto a_d$, and find the proportionality coefficient, that is the LS.

Because of the modulation at frequency ω_F, the oscillator is far away from thermal equilibrium even without the field $\propto a_d$. This leads to a significant difference of the problem of switching from that for equilibrium systems. Its physical origin is the lack of time reversibility in nonequilibrium systems. As a result, the auxiliary Hamiltonian system described by the Hamiltonian $H^{(0)}$ (10), which gives optimal fluctuational trajectories and the MPEP of the original dissipative system, is nonintegrable [18].

The Hamiltonian dynamics described by the full Hamiltonian H (10), which includes the additional time-dependent field $\propto a_d$, is also nonintegrable. However, for small a_d the heteroclinic Hamiltonian trajectory $\mathbf{q}_{n\ \mathrm{opt}}(\tau)$ that gives the MPEP remains close to the unperturbed heteroclinic trajectory $\mathbf{q}^{(0)}_{n\ \mathrm{opt}}(\tau)$ for $a_d=0$. Then, as in the case of modulated equilibrium systems [15,16], the lowest-order correction in a_d to the variational functional R_n (9) can be found by calculating the perturbing term in the Lagrangian along $\mathbf{q}^{(0)}_{n\ \mathrm{opt}}(\tau)$.

The trajectory $\mathbf{q}^{(0)}_{n\ \mathrm{opt}}(\tau)$ is a real-time counterpart of an instanton [19] and is often called a real-time instanton. It goes from $\tau\to-\infty$ to $\tau\to\infty$. As in the case of standard instantons, $\dot{\mathbf{q}}^{(0)}_{n\ \mathrm{opt}}(\tau)$ looks like a pulse. It is large only for a time of the order of the relaxation time of the system in the absence of noise. The position of the center of the pulse on the time axis τ_c is arbitrary.

Periodic modulation lifts the degeneracy with respect to τ_c. It synchronizes switching events. It is this synchronization that leads to the correction to R_n being linear in a_d for a nonzero frequency detuning $|\nu_d|$. The synchronization corresponds to calculating the field-induced correction δR_n using as a zeroth-order approximation the unperturbed MPEP for the nth state $\mathbf{q}^{(0)}_{n\ \mathrm{opt}}(\tau-\tau_c)$ and adjusting τ_c in such a way that the overall functional $R_n=R^{(0)}+\delta R_n$ be minimal. That is, τ_c is found from the condition of the minimum of the function $\delta R_n(\tau_c)$,

$$\delta R_n = \min_{\tau_c} \delta R_n(\tau_c),$$

$$\delta R_n(\tau_c) = -\int_{-\infty}^{\infty} d\tau\, \boldsymbol{\chi}_n(\tau-\tau_c)\mathbf{K}^{(d)}(\tau),$$

$$\boldsymbol{\chi}_n(\tau) = \frac{1}{2}[\dot{\mathbf{q}}^{(0)}_{n\ \mathrm{opt}}(\tau) - \mathbf{K}^{(0)}(\mathbf{q}^{(0)}_{n\ \mathrm{opt}}(\tau))]. \qquad (11)$$

Clearly, τ_c is determined modulo the dimensionless modulation period $2\pi/|\nu_d|$. Generically, there is one MPEP per period that provides the absolute minimum to R_n.

Equation (11) shows that the correction to the activation energy of escape is indeed linear in a_d. The coefficient at a_d is determined by the Fourier components $\tilde{\boldsymbol{\chi}}_n(\pm\nu_d)$ of the function $\boldsymbol{\chi}(\tau)$,

$$\tilde{\boldsymbol{\chi}}_n(\nu) = \int_{-\infty}^{\infty} d\tau\, e^{i\nu\tau}\boldsymbol{\chi}_n(\tau). \qquad (12)$$

The function $\boldsymbol{\chi}_n(\tau)$ and its Fourier transform $\tilde{\boldsymbol{\chi}}_n(\nu)$ give the LS in the time and frequency representation. They are of central interest for the studies of symmetry lifting and switching-rate modulation. Interestingly, the functions $\boldsymbol{\chi}_n(\tau),\tilde{\boldsymbol{\chi}}_n(\nu)$ have two components each, even though there is only one additional force that drives the oscillator. This leads to important consequences discussed below and is in contrast to the case of a modulated equilibrium system.

We note that the condition $d[\delta R_n(\tau_c)]/d\tau_c=0$, Eq. (11), corresponds to the condition that the Mel'nikov function $\mathcal{M}(\tau_c)$ defined for our Hamiltonian system with two degrees of freedom as

$$\mathcal{M}(\tau_c) = \int_{-\infty}^{\infty} d\tau \{H^{(0)}(\tau-\tau_c), H^{(d)}(\tau-\tau_c;\tau)\}, \qquad (13)$$

be equal to zero. Here, $\{A,B\}$ is the Poisson bracket with respect to the dynamical variables of the auxiliary system $\mathbf{q},\mathbf{p}$. The Poisson bracket is evaluated along the unperturbed trajectory $\mathbf{q}^{(0)}_{n\ \mathrm{opt}}(\tau-\tau_c),\mathbf{p}^{(0)}_{n\ \mathrm{opt}}(\tau-\tau_c)$. Respectively, the functions $\mathbf{q},\mathbf{p}$ in $H^{(0)},H^{(d)}$ are evaluated at time $\tau-\tau_c$, which is indicated by the first argument in $H^{(0)},H^{(d)}$; the second argument in $H^{(d)}$ indicates the explicit time dependence of $H^{(d)}$, Eq. (10), due to the field $\propto a_d$. For systems with one degree of freedom, the condition $\mathcal{M}(\tau_c)=0$ shows that τ_c for the unperturbed trajectory is chosen in such a way that this trajectory is close to the trajectory $\mathbf{q}_{n\ \mathrm{opt}}(\tau-\tau_c),\mathbf{p}_{n\ \mathrm{opt}}(\tau-\tau_c)$ [20]. For our system, the condition $\mathcal{M}(\tau_c)=0$ is necessary for the applicability of perturbation theory. The corresponding analysis will be provided elsewhere.

A. Symmetry lifting by a field at subharmonic frequency

The analysis of the switching rates should be done somewhat differently in the case where the frequencies of the additional field and the parametrically modulating field satisfy the condition $\omega_d=\omega_F/2$. In this case $\nu_d=0$ and the force $\mathbf{K}^{(d)}$ is independent of time. Therefore the function $\delta R_n(\tau_c)$ is independent of τ_c and minimizing it over τ_c is irrelevant. However, a perturbation theory in a_d can still be developed. The first-order correction to the activation energy $\delta R_n \equiv \delta R^{\mathrm{res}}_n(\phi_d)$ can be obtained by evaluating the term $\propto a_d$ in the Lagrangian L (9) along the unperturbed path $\mathbf{q}^{(0)}_{n\ \mathrm{opt}}(\tau)$. It has a simple explicit form

$$\delta R^{\mathrm{res}}_n(\phi_d) = a_d[\tilde{\chi}_{nQ}(0)\cos\phi_d - \tilde{\chi}_{nP}(0)\sin\phi_d], \qquad (14)$$

where the subscripts Q,P enumerate the components of $\tilde{\boldsymbol{\chi}}$ [we have also used Eqs. (4) and (7) for $\mathbf{K}^{(d)}$]. Both components of $\tilde{\boldsymbol{\chi}}_n(0)$ contribute to the change of the activation energy for subharmonic driving.

An important feature to emphasize is that the function $\delta R^{\mathrm{res}}_n(\phi_d)$ is *not* the limit of $\min \delta R_n(\tau_c)$ for $\nu_d\to 0$. Physically this is because Eq. (11) gives the change of the logarithm of the escape rate W_{nm} averaged over the period $\pi/|2\omega_d-\omega_F|$ (or $2\pi/|\nu_d|$, in dimensionless time τ). Such averaging is meaningful as long as the frequency detuning $|2\omega_d-\omega_F|\gg W_{nm}$. In the opposite limit, the occupation of the state n changes significantly over the period $\pi/|2\omega_d-\omega_F|$; this change is not characterized by the period-averaged W_{nm}.

An important property of the static LS $\tilde{\boldsymbol{\chi}}_n(0)$ is that it has opposite signs for the states 1 and 2. Indeed, these states are inversely symmetric, $\mathbf{q}^{(0)}_1=-\mathbf{q}^{(0)}_2$, and since the vector $\mathbf{K}^{(0)}$ is also antisymmetric, $\mathbf{K}^{(0)}\to-\mathbf{K}^{(0)}$ for $\mathbf{q}\to-\mathbf{q}$, it is clear that $\mathbf{q}^{(0)}_{1\ \mathrm{opt}}(\tau)=-\mathbf{q}^{(0)}_{2\ \mathrm{opt}}(\tau)$, and therefore $\tilde{\boldsymbol{\chi}}_1(0)=-\tilde{\boldsymbol{\chi}}_2(0)$. Then from Eqs. (9) and (14), we have for the switching rates W_{12} and W_{21}

$$\frac{W_{12}}{W^{(0)}} = \frac{W^{(0)}}{W_{21}} = \exp\left[-\frac{\delta R_1^{\text{res}}(\phi_d)}{D}\right], \tag{15}$$

where $W^{(0)}=W_{12}^{(0)}=W_{21}^{(0)}$ is the switching rate for $a_d=0$, $W^{(0)} \propto \exp(-R^{(0)}/D)$. Equation (15) applies for arbitrary a_d/D. The corrections to the prefactor $\propto a_d$ (but not a_d/D) have been disregarded.

In the parameter range $\mu_B^{(1)}<\mu<\mu_B^{(2)}$, where the only stable states of the system are the period-2 states [cf. Eq. (8)], the ratio of the stationary populations of the states w_1 and w_2 is the inverse of the escape probabilities ratio. Therefore, from Eq. (15),

$$\frac{w_1}{w_2} = \frac{W_{21}}{W_{12}} = \exp\left[\frac{2\delta R_1^{\text{res}}(\phi_d)}{D}\right].$$

It is seen from this expression that even a comparatively weak symmetry-lifting field, where $|\delta R_1^{\text{res}}| \ll R^{(0)}$, can lead to a significant change of the state populations. This happens when $|\delta R_1^{\text{res}}| \gg D$. The ratio of the state populations is determined by the phase of the field ϕ_d. As mentioned in the Introduction, there is a similarity with magnetic-field-induced symmetry breaking for an Ising spin, with ϕ_d playing the role of the orientation of the magnetic field. We note that both the absolute value and the *sign* of the coefficients $\tilde{\boldsymbol{\chi}}_n(0)$ depend on the oscillator parameters for $a_d=0$. Therefore, one can change the population ratio not only by varying the phase and amplitude of the additional field, but also by varying these parameters, for example the amplitude F of the parametrically modulating field.

IV. LOW-FREQUENCY MODULATION OF STATE POPULATIONS

The LS $\tilde{\boldsymbol{\chi}}_n(\nu)$ displays frequency dispersion for ν of the order of the inverse dimensionless relaxation time of the oscillator or higher. If the additional field is so closely tuned to the subharmonic frequency that $|2\omega_d-\omega_F| \lesssim W^{(0)}$, the period averaging of W_{nm} implied in Eq. (11) becomes inapplicable, as explained above. One should rather think of the instantaneous values of the switching rates $W_{nm}(\tau)$, which are given by Eq. (15) with $\phi_d \to \nu_d\tau + \phi_d(0)$. Slow time-dependent modulation of the switching rates leads to modulation of the state populations,

$$\frac{dw_1}{d\tau} = -W_{12}(\tau)w_1 + W_{21}(\tau)w_2, \quad w_1+w_2=1. \tag{16}$$

Such modulation has attracted much attention in the context of stochastic resonance [21,22]. In contrast to a particle in a slowly modulated double-well static potential that has been most frequently studied in stochastic resonance, here the stable states are fast oscillating, and strong modulation of their populations is induced by a high-frequency driving field. In this sense, there is similarity with stochastic resonance in a resonantly driven oscillator with coexisting period-1 states (see Ref. [21]) that was recently observed in experiment [23]. In contrast to a resonantly driven oscillator, for a parametrically modulated oscillator the populations of period-2 states are equal in the absence of extra field for any parameter values.

Oscillations of the state populations at frequency $|\nu_d|$ ($|\omega_d-\omega_F/2|$ in the laboratory frame) have a comparatively large amplitude $\propto a_d/D$, for small noise intensity. They lead to vibrations of the oscillator in the laboratory frame at frequencies ω_d and $|\omega_F-\omega_d|$, with the average coordinate being

$$\langle q_0(t)\rangle \approx \left|\frac{2F}{3\gamma}\right|^{1/2}[w_1(t)-w_2(t)] \times\left[P_1^{(0)}\cos\left(\frac{\omega_F t}{2}\right) - Q_1^{(0)}\sin\frac{\omega_F t}{2}\right].$$

The vibration amplitude is much larger than the amplitude of noise-free vibrations about attractors. This indicates strong fluctuation-induced effective three-wave mixing. Moreover, because oscillations of $w_{1,2}(\tau)$ are nonsinusoidal for large a_d/D, there also occurs multiple-wave mixing. Further analysis of this effect is beyond the scope of the present paper.

V. PERIOD-AVERAGED SWITCHING RATE

For the difference frequency $|2\omega_d-\omega_F| \gg W_{nm}^{(0)}$, the major effect of the additional field is the change of the switching rates averaged over the dimensionless period $2\pi/|\nu_d|$. It is given by the change of the activation energies δR_n. From Eq. (11), it follows that δR_n is independent of time. From the symmetry relations $\mathbf{q}_{1\ \text{opt}}^{(0)}(\tau)=-\mathbf{q}_{2\ \text{opt}}^{(0)}(\tau)$ and $\mathbf{K}^{(d)}(\tau)=-\mathbf{K}^{(d)}(\tau+\pi\nu_d^{-1})$, it follows that the minimization over τ_c in Eq. (11) leads to $\delta R_1=\delta R_2$. It is straightforward to show that

$$\delta R_1 = \delta R_2 = -a_d\tilde{\chi}_{1c}(\nu_d),$$

$$\tilde{\chi}_{1c}(\nu) = [|\tilde{\boldsymbol{\chi}}_1(\nu)|^2 - i\,\tilde{\boldsymbol{\chi}}_1(\nu)\,\hat{\epsilon}\,\tilde{\boldsymbol{\chi}}_1^*(\nu)]^{1/2}. \tag{17}$$

From Eq. (17), the change of the activation energy is fully determined by the LS at the scaled frequency difference ν_d. The function $\tilde{\chi}_{1c}$ is nonnegative. It displays a characteristic dependence on the oscillator parameters. As we show, it may have resonant peaks in the regime of small damping, $\zeta \gg 1$.

VI. SCALING BEHAVIOR NEAR THE BIFURCATION POINT

A. Symmetry lifting

The dynamics of the oscillator is simplified near the bifurcation point $\mu_B^{(1)}$ where the period-2 states merge together (a supercritical pitchfork bifurcation [20]). Here, motion is controlled by one slow variable (soft mode) $Q'=Q\cos\beta + P\sin\beta$, where $\beta=\frac{1}{2}(\pi-\arcsin\zeta^{-1})$. From Eq. (6), to the lowest order in the distance to the bifurcation point $\eta=\mu-\mu_B^{(1)}$ the Langevin equation for this variable has the form

$$\dot{Q}' = -\partial_{Q'}U + f'(\tau), \quad U = \frac{1}{2}\mu_B^{(1)}\eta\zeta Q'^2 - \frac{1}{4}\mu_B^{(1)}\zeta^3 Q'^4 + Q' a_d\cos(\nu_d\tau + \phi_d + \beta), \tag{18}$$

where $f'(\tau)$ is white noise of intensity D.

The potential U in the absence of additional field has a familiar form of a quartic parabola (note that $\mu_B^{(1)}<0$). The values of the slow variable at the period-2 states correspond to the minima of U, $Q'^{(0)}_{1,2}=\pm\eta^{1/2}\zeta^{-1}$, whereas the unstable zero-amplitude state is at the local maximum of U at $Q'^{(0)}=0$. The activation energy $R^{(0)}=|\mu_B^{(1)}|\eta^2/4\zeta$ is just the height of the potential barrier ΔU [6]; the probability distribution near the pitchfork bifurcation point was discussed in Refs. [24,25].

For an additional field at exact subharmonic frequency, the change of the activation energy is simply the change of the barrier height. With the chosen convention that $Q_1^{(0)}>0$, we obtain from Eq. (18)

$$\delta R_1^{\rm res}=-\delta R_2^{\rm res}=-\eta^{1/2}\zeta^{-1}a_d\cos(\phi_d+\beta). \tag{19}$$

This expression shows that the correction to the activation energy decreases as the system approaches the bifurcation point, i.e., $\eta=\mu-\mu_B^{(1)}$ decreases. However, the decrease of $\delta R_1^{\rm res}$ is much slower than the decrease of $R^{(0)}$. Therefore, the relative correction to the activation energy sharply increases as $\mu\to\mu_B^{(1)}$. It is important that the sign of $\delta R_1^{\rm res}$, which shows which of the states 1 and 2 is predominantly occupied, depends on $\phi_d+\beta$, that is, not only on the relative phase of the additional field, but also on the scaled relaxation parameter ζ that determines the value of β. Therefore, by varying ζ, one can control which of the states is predominantly occupied.

B. High-frequency modulation

We now consider the case where the additional field is detuned from the subharmonic frequency, $|2\omega_d-\omega_F|\gg W_{12}^{(0)}$. Since the motion near the bifurcation point is controlled by one variable Q', there is no phase shift between the components $\chi_{1Q}(\nu_d)$ and $\chi_{1P}(\nu_d)$. The change of the activation energy is $\delta R_1=\delta R_2=-a_d\,\tilde{\chi}_{1c}(\nu_d)=-a_d|\tilde{\boldsymbol{\chi}}_1(\nu_d)|$. The problem as a whole coincides with that for a periodically modulated overdamped equilibrium particle. The LS for an overdamped particle in a quartic potential is already known [8]. In the present case, we have

$$\tilde{\chi}_{1c}(\nu)=\pi^{-1/2}\zeta^{-1}\eta^{1/2}\times\left|\Gamma\left(\frac{1-i\nu'}{2}\right)\Gamma\left(1+\frac{i\nu'}{2}\right)\right|, \tag{20}$$

where $\nu'=\nu/|\mu_B^{(1)}|\eta\zeta$ and $\Gamma(z)$ is the Gamma function. The function $\tilde{\chi}_{1c}(\nu)$ is proportional to a smaller power of the distance to the bifurcation point η than $R^{(0)}$. This shows that the correction to the period-averaged switching rate becomes relatively larger as the system approaches the bifurcation point. As seen from Eq. (20), for small η the LS $\tilde{\chi}_{1c}(\nu)$ has a peak at the frequency detuning $\nu=0$ and monotonically decreases with increasing $|\nu|$. The typical width of the peak of $\tilde{\chi}_{1c}(\nu)$ is $\zeta|\mu_B^{(1)}|\eta$. It decreases as μ approaches $\mu_B^{(1)}$.

VII. WEAK DAMPING LIMIT

The LS can be analyzed also in the limit of weak damping, $\zeta\gg 1$. Here we consider damping that is weak in the rotating frame. This means that not only is the oscillator decay slow compared to frequency ω_0, but also compared to the much smaller frequency F/ω_F. If there were no damping and noise, the motion of the oscillator in the rotating frame would be vibrations with given quasienergy g, which are described by equation $\dot{\mathbf{q}}=\hat{\epsilon}\nabla g$. Damping causes the quasienergy to decrease toward its value in one of the stable states whereas noise leads to quasienergy diffusion away from these states. On the MPEP $\mathbf{q}_{n\,\rm opt}(\tau)$, the quasienergy increases from its value g_n in the stable state n to its value at the saddle point g_b [6] [the quasienergy G in Ref. [6] differs from $g^{(0)}(Q,P)$ in sign]. As a result, $\mathbf{q}_{n\,\rm opt}(t)$ is a spiral.

A. Symmetry lifting

We will start the analysis with the case of the additional field at exact subharmonic frequency, $\omega_d=\omega_F/2$. In this case, $g(Q,P)$ is independent of time. The general expression for the activation energy of switching in the limit of weak damping, which is not limited to small a_d, has the form [6,9,17]

$$R_n^{\rm res}=\zeta^{-1}\int_{g_n}^{g_b}dg\,\frac{M_n(g)}{N_n(g)},\quad M_n(g)=\int\int_{A_n(g)}dQdP,$$

$$N_n(g)=\frac{1}{2}\int\int_{A_n(g)}\boldsymbol{\nabla}^2 g\,dQdP. \tag{21}$$

Integration with respect to Q,P in functions $M_n(g),N_n(g)$ for given g is done over the area $A_n(g)$ of phase plane (Q,P), which is limited by the phase trajectory $g(Q,P)=g$ that lies within the basin of attraction to the stable state $\mathbf{q}_n$.

For comparatively small a_d, the integrals in Eq. (21) can be calculated by perturbation theory. To first order in a_d, we obtain the following expression for the correction $\delta R_n^{\rm res}\equiv\delta R_n^{\rm res}(\phi_d)$ to the activation energy $R^{(0)}$ for $\mu_B^{(1)}<\mu<\mu_B^{(2)}$,

$$\delta R_1^{\rm res}(\phi_d)=-\delta R_2^{\rm res}(\phi_d)=\mathcal{X}_1\zeta^{-1}a_d\sin\phi_d, \tag{22}$$

where

$$\mathcal{X}_1=\int_{g_{\rm min}^{(0)}}^{0}dg\,\frac{1}{N^{(0)}}\left[\delta M_1-\frac{M^{(0)}}{N^{(0)}}\delta N_1\right]+\frac{(\mu+1)^{1/2}}{2+\mu},$$

$$\delta M_1=2\int_{Q_{1\,\rm min}}^{Q_{1\,\rm max}}dQ\,\frac{Q}{|\partial_P g^{(0)}|},$$

$$\delta N_1=\int_{Q_{1\,\rm min}}^{Q_{1\,\rm max}}dQ\,\frac{Q\boldsymbol{\nabla}^2 g^{(0)}}{|\partial_P g^{(0)}|}. \tag{23}$$

Here $M^{(0)}$ and $N^{(0)}$ are the values of $M_{1,2}(g)$ and $N_{1,2}(g)$ for $a_d=0$, whereas δM_1 and δN_1 are the field-induced corrections to M_1 and N_1 divided by $a_d\sin\phi_d$; $g_{\rm min}^{(0)}=-(\mu+1)^2/4$. The limits $Q_{1\,\rm min}(g),Q_{1\,\rm max}(g)$ of the integrals over Q are given by equation $g^{(0)}(Q,0)=g$, with $0<Q_{1\,\rm min}<Q_{1\,\rm max}$. The arguments of the integrals over Q in the expressions for $\delta M_1,\delta N_1$ are calculated for P given by equation $g^{(0)}(Q,P)=g$ with $P>0$.

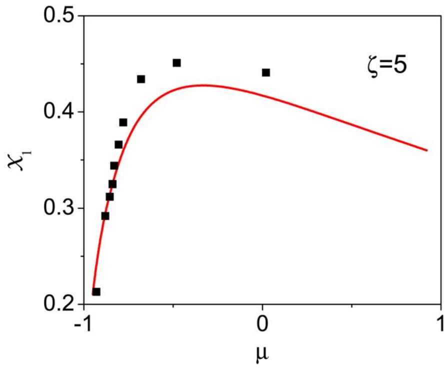

FIG. 1. (Color online) Symmetry lifting in the weak-damping limit, $\zeta^{-1} \ll 1$. Solid line: the scaling factor $\mathcal{X}_1$, Eq. (23), in the correction to the activation energy $\delta R_1^{\rm res} = \mathcal{X}_1 \zeta^{-1} a_d \sin\phi_d$. The dots show the scaled LS $-\zeta \tilde{\chi}_{1P}(0)$. It determines the correction $\propto \sin\phi_d$ to $\delta R_1^{\rm res}$, see Eq. (14), and is obtained by calculating the MPEPs for $\zeta=5$.

It follows from Eq. (22) that, for small damping, the dependence of the correction to the activation energy on the phase of the additional field has a simple form $\delta R_1^{\rm res} \propto \sin\phi_d$. Both $\delta R_1^{\rm res}$ and $R^{(0)}$ are $\propto \zeta^{-1}$, they decrease with the decreasing scaled friction coefficient ζ^{-1}. The function $\mathcal{X}_1$ depends on one parameter, the relative frequency detuning of the strong field μ. This dependence is shown in Fig. 1.

It is seen from Fig. 1 that the change of the activation energy is nonmonotonic as a function of μ. For μ close to the bifurcation value $\mu_B^{(1)} = -1$, we have $\mathcal{X}_1 \approx (\mu+1)^{1/2}$. This shows that the weak-damping expression for $\delta R_1^{\rm res}$, Eqs. (22) and (23), smoothly goes over into expression (19) obtained near the bifurcation point in the opposite limit of overdamped motion; note that in Eq. (19) $\beta \approx \pi/2$ for $\zeta^{-1} \ll 1$.

The analytical results are compared in Fig. 1 with the numerical results obtained by directly solving the Hamiltonian equations of motion for the MPEP and calculating the LS from Eqs. (11) and (12). Since in the limit of small damping $\delta R_1^{\rm res} \propto \sin\phi_d$, one expects from Eq. (14) that $\tilde{\chi}_{1Q}(0)$ should be small and $\mathcal{X}_1 \approx -\zeta \tilde{\chi}_{1P}(0)$. Close to the bifurcation point $\mu_B^{(1)}$ the numerical results agree with the asymptotic theory already for moderately small damping, $\zeta^{-1} = 0.2$.

We found numerically that the component $\tilde{\chi}_{1Q}(0)$ increases with μ. Close to the second bifurcation point $\mu_B^{(2)}$, it becomes of the same order of magnitude as $\tilde{\chi}_{1P}(0)$, for the chosen ζ. We note that the MPEP is a fast oscillating function of time, for weak damping. When $\tilde{\boldsymbol{\chi}}_1(0)$ is calculated, the oscillations largely compensate each other. The numerical value of $\tilde{\boldsymbol{\chi}}_1(0)$ is therefore very sensitive to numerical errors in the MPEP. With increasing μ, the frequency of oscillations of the MPEP increases and so does the sensitivity. Therefore, we present numerical results only for a limited range of μ.

B. Resonant peaks of the logarithmic susceptibility

Oscillations of the MPEP may be expected to lead to peaks of the LS as function of frequency for the dimensionless frequency detuning $|\nu| \gg \zeta^{-1}$. Since the period-averaged corrections to the activation energy for the states 1 and 2, δR_1 and δR_2, are equal, we will consider the MPEP for the state 1 and will drop the subscript 1. The analysis of the LS peaks can be done by extending to the MPEP of a parametrically modulated oscillator the approach developed in Ref. [17].

As a first step, it is convenient to change variables from Q,P to quasienergy-angle variables g,ψ. This is accomplished by seeking the optimal path of the unperturbed system, $a_d=0$, in the form

$$\mathbf{q}_{1\,{\rm opt}}^{(0)}(\tau) = (x(g_{\rm opt},\psi_{\rm opt}), y(g_{\rm opt},\psi_{\rm opt})).$$

Here, x and $y$are the coordinate and momentum of vibrations with given unperturbed quasienergy, $g^{(0)}(x,y)=g$; ψ is the vibration phase: x and y are 2π-periodic in ψ. We denote the vibration frequency by $\omega(g)$. The equations for x,y are of the form

$$\omega(g)\partial_\psi x = \partial_y g^{(0)}, \quad \omega(g)\partial_\psi y = -\partial_x g^{(0)}. \qquad (24)$$

Functions $x(g,\psi)$ and $y(g,\psi)$ can be expressed in terms of the Jacobi elliptic functions [6].

On the optimal path, the quasienergy $g \equiv g_{\rm opt}(\tau)$ is a function of time, with $\dot{g}_{\rm opt}, |\dot{\psi}_{\rm opt} - \omega(g_{\rm opt})| \propto \zeta^{-1}$. Using the explicit form of equations of motion for the Lagrangian (9), one can show that, to first order in ζ^{-1}, on the optimal path

$$\left(\frac{dx}{d\tau}\right)_{\rm opt} \approx K_x^{(0)}(x,y) + \zeta^{-1}F(g)\partial_\psi y,$$

$$\left(\frac{dy}{d\tau}\right)_{\rm opt} \approx K_y^{(0)}(x,y) - \zeta^{-1}F(g)\partial_\psi x. \qquad (25)$$

Here, the force $\mathbf{K}^{(0)}$, with components $K_x^{(0)}, K_y^{(0)}$, is defined by Eq. (7) in which Q and P are replaced by x and y, respectively.

The function $F(g)$ in Eq. (25) remains arbitrary, to first order in ζ^{-1}. It can be found from the analysis of the terms $\propto \zeta^{-2}$ in the Lagrange equation for the optimal path [17], which is cumbersome. The calculation can be simplified by noting that $F(g)$ determines the change of the quasienergy $\bar{g}$ averaged over vibration period $2\pi/\omega(g)$. From Eq. (25), taking into account the explicit form of $\mathbf{K}^{(0)}$ and expression (21), we have

$$\begin{aligned}\left(\frac{d\bar{g}}{d\tau}\right)_{\rm opt} &= -\zeta^{-1}\{\overline{x\partial_x g^{(0)}} + \overline{y\partial_y g^{(0)}} \\ &\quad + F(\bar{g})\omega^{-1}(\bar{g})[\overline{(\partial_x g^{(0)})^2} + \overline{(\partial_y g^{(0)})^2}]\} \\ &= -(\pi\zeta)^{-1}[\omega(\bar{g})M^{(0)}(\bar{g}) + F(\bar{g})N^{(0)}(\bar{g})], \qquad (26)\end{aligned}$$

where overline denotes period averaging.

Alternatively the evolution of quasienergy on the optimal path can be found using the Langevin equation for $d\bar{g}/d\tau$, which can be obtained by averaging Eq. (6) over the period $2\pi/\omega(\bar{g})$. The resulting equation describes drift and diffusion of $\bar{g}$. The optimal path for $\bar{g}$ can be obtained using the variational formulation similar to Eq. (9). Since the corresponding variational problem is one-dimensional, it is easy to find the optimal path. As is often the case for one-dimensional systems driven by white noise, the optimal path is the time-reversed path in the absence of noise,

$$\left(\frac{d\bar{g}}{d\tau}\right)_{\rm opt} = (\pi\zeta)^{-1}\omega(\bar{g})M^{(0)}(\bar{g}).$$

Comparing this expression with Eq. (26), we find

$$F(g) = -\frac{2\omega(g)M^{(0)}(g)}{N^{(0)}(g)}. \quad (27)$$

To lowest order in ζ^{-1}, we can replace $\bar{g}$ with g on the optimal path.

Equations (11), (12), (25), and (27) describe the LS in a simple form of the Fourier transform of the functions $F(g)\partial_\psi x$, $F(g)\partial_\psi y$ on the optimal path. The general expression is further simplified near the peaks of $\tilde{\boldsymbol{\chi}}_1(\nu)$. They occur where ν is close to the frequency $\omega(g)$ or its overtones for certain values of g. It is convenient to write x and y as Fourier series,

$$x(g,\psi) = \sum_m x(m;g)\exp(im\psi)$$

and similarly for y. Then calculating the LS is reduced to taking the Fourier transform of the oscillating factors $\exp[im\psi_{\rm opt}(\tau)]$ weighted with functions of $g_{\rm opt}(\tau)$ that smoothly depend on time.

On the MPEP the leading term in the phase $\psi_{\rm opt}(\tau)$ is $\psi_{\rm opt}(\tau) \approx \int^\tau d\tau' \omega(g_{\rm opt}(\tau'))$. The function $\omega(g)$ monotonically decreases with increasing g. Since $g_{\rm opt}(\tau)$ is monotonic as function of time, $m\omega(g_{\rm opt})$ on the optimal path can be in resonance with ν only at a certain time. This allows us to single out resonant contributions $\tilde{\boldsymbol{\chi}}_1(m;\nu)$ to $\tilde{\boldsymbol{\chi}}_1(\nu)$ from the corresponding mth overtones of $x(g,\psi), y(g,\psi)$. Near resonance integration over τ in Eq. (12) can be done by the steepest descent method, for slowly varying $g(\tau)$. It gives

$$\tilde{\boldsymbol{\chi}}_1(m;\nu) = C_m \pi\zeta^{-1/2}\left[\frac{2\nu M^{(0)}}{|d\omega/dg|N^{(0)2}}\right]^{1/2}_{g_{\nu m}} \times (y(-m;g_{\nu m}), -x(-m;g_{\nu m}))$$

Here, C_m is a phase factor, $|C_m|=1$. The subscript $g_{\nu m}$ indicates that the expression in the brackets should be calculated for $g=g_{\nu m}$, with $g_{\nu m}$ given by the condition $m\omega(g_{\nu m})=\nu$.

The change of the activation energy δR_1 is determined by the LS $\tilde{\chi}_{1c}(\nu)$ defined in Eq. (17). From the explicit form of the matrix elements $x(m;g), y(m;g)$ found in Ref. [9] it follows that $\arg[y(-m;g)x^*(-m;g)] = -\pi/2$ for $m>0$, in the considered range of g and μ. Then from the expression for $\tilde{\boldsymbol{\chi}}_1(m;\nu)$, we obtain that the spectral peak of $\tilde{\chi}_{1c}$ near an mth overtone, $\tilde{\chi}_{1c}(m;\nu)$, is given by the expression

$$\tilde{\chi}_{1c}(m;\nu) = \pi\zeta^{-1/2}\left[\frac{2\nu M^{(0)}}{|d\omega/dg|N^{(0)2}}\right]^{1/2}_{g_{\nu m}} \times [|x(-m;g_{\nu m})| + |y(-m;g_{\nu m})|] \quad (m>0). \quad (28)$$

The Fourier components $|x(m;g)|, |y(m;g)|$ rapidly decrease with increasing $|m|$ (exponentially, for large $|m|$ [9]). Therefore, the major peak of $\tilde{\chi}_{1c}(\nu)$ is the peak of $\tilde{\chi}_{1c}(1;\nu)$. From Eq. (28), near the maximum it has the form

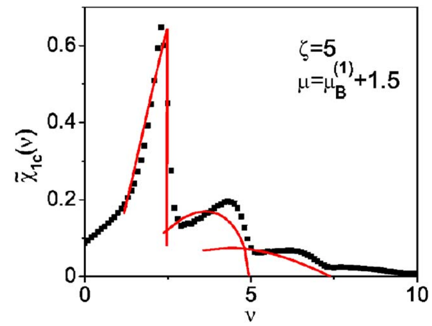

FIG. 2. (Color online) The multiple-peak LS $\tilde{\chi}_{1c}(\nu)$ for an underdamped system. Solid lines show the overtones $\tilde{\chi}_{1c}(m;\nu)$, Eq. (28), with $m=1,2,3$. Squares show the LS calculated by numerically finding the MPEP. The data refer to $\zeta=5$, $\mu-\mu_B^{(1)}=1.5$.

$$\tilde{\chi}_{1c}(1;\nu) \propto (1-az)\theta(z), \quad z=\omega(g_{\rm min})-\nu, \quad (29)$$

where $a\sim 1$ is a numerical factor and $\theta(x)$ is the step function $[\omega(g_{\rm min})=2(\mu+1)^{1/2}]$. In obtaining this expression, we used that, for small $g-g_{\rm min}$, vibrations with given g are nearly sinusoidal, with $x(1;g), y(1;g) \propto (g-g_{\rm min})^{1/2}$ and $M^{(0)}, N^{(0)} \propto g-g_{\rm min}$. The sharp asymmetry of the peak of $\tilde{\chi}_{1c}(1;\nu)$ is due to the fact that the eigenfrequencies $\omega(g)$ have a cutoff at $\omega(g_{\rm min})$.

A similar calculation shows that the peak from the second overtone is smoother, with $\tilde{\chi}_{1c}(2;\nu) \propto [2\omega(g_{\rm min})-\nu]^{1/2}$ for small $2\omega(g_{\rm min})-\nu$. The maximum of $\tilde{\chi}_{1c}(2;\nu)$ is shifted from $2\omega(g_{\rm min})$ to lower frequencies and has a smaller height than $\tilde{\chi}_{1c}(1;\nu)$. Higher-order peaks have still smaller heights.

The multi-peak structure of the LS for small damping is clearly seen in Fig. 2. The lines in this figure show asymptotic expressions (28) for the overtones $\tilde{\chi}_{1c}(m;\nu)$. The squares are obtained by numerically finding the optimal path $\mathbf{q}^{(0)}_{\rm opt}(t)$ and then calculating $\tilde{\chi}_{1c}(\nu)$ from Eqs. (11), (12), and (17). As expected from Eq. (29), the major peak of the numerically calculated LS is at frequency $\omega(g_{\rm min})$. Other peaks are located near overtones of $\omega(g_{\rm min})$, on the low-frequency side. All peaks have characteristic strongly asymmetric shapes.

Equation (28) for the major peak, $m=1$, is in a good agreement with the numerical calculations. The agreement for the overtones is worse because the peaks are much broader and the contributions to the LS from different overtones overlap. The full LS is not given just by a sum over m of $\tilde{\chi}_{1c}(m;\nu)$. It is necessary to take into account interference of the contributions from vibrations with different quasienergies $g_{\nu m}$ but close $m\omega(g_{\nu m})$ and, of course, the effect of relaxation. We note that for $\nu \propto 2\omega_d-\omega_F<0$ the peaks of the LS have much smaller amplitudes. In contrast to equilibrium systems, $\tilde{\chi}_{1c}(-\nu) \neq \tilde{\chi}_{1c}(\nu)$.

The above results demonstrate that, for an underdamped oscillator, the rate of switching between period two states can be resonantly increased by applying an extra field with frequency $\omega_d \approx \omega_F/2$. The amplitude of the LS peaks is $\propto \zeta^{-1/2}$, it is parametrically larger than the LS at zero frequency, which determines the symmetry lifting and is $\propto \zeta^{-1}$.

We note that, for extremely weak damping, a change of the switching rate with the field amplitude A_d may be due to the field-induced mixing of the attraction basins, as in equilibrium systems [26]. We do not discuss this mechanism here.

VIII. CONCLUSIONS

In this paper, we considered an oscillator parametrically modulated by a comparatively strong field at nearly twice its eigenfrequency and additionally driven by a comparatively weak nearly resonant field. Because of the parametric modulation, the oscillator displays period doubling. It has two vibrational states that differ only by phase in the absence of the additional field. Even a comparatively weak additional field can strongly affect the oscillator by changing the rates of switching between the period-2 states. We have shown that the rate change depends exponentially on the ratio of the amplitude of this field A_d to the characteristic fluctuation intensity D. For small D, this change becomes large even where the field only weakly perturbs the dynamics of the system.

If the frequency of the additional field ω_d coincides with the frequency of the period-2 states $\omega_F/2$, the switching rates W_{12} and W_{21} between the states become different from each other. As a result, the stationary populations of the states also become different. This is the effect of symmetry lifting. It depends exponentially strongly on the field amplitude.

For small frequency difference, $|\omega_d - \omega_F/2| \lesssim W_{nm}$, the additional field leads to oscillations of the state populations, which, in turn, lead to fluctuation-induced three- and multiple-wave mixing. For a larger detuning, $|\omega_d - \omega_F/2| \gg W_{nm}$, the major effect of the additional field is the increase of the switching rates W_{nm} averaged over the beat period $4\pi/|\omega_F - 2\omega_d|$.

For both small and comparatively large $|\omega_F - 2\omega_d|$, the change of the switching rates is characterized by the LS $\tilde{\boldsymbol{\chi}}_n$. The latter gives the proportionality coefficient between the field-induced change of the activation energy of switching from an nth state δR_n and the field amplitude A_d. We have obtained an explicit general expression for the LS in terms of the path that the system is most likely to follow in switching, the MPEP. For $\omega_d = \omega_F/2$, the two components of the vector $\tilde{\boldsymbol{\chi}}_n$ give the coefficients in δR_n at $\cos \phi_d$ and $\sin \phi_d$, where ϕ_d is the phase of the additional field relative to the phase of the strong field. For $|\omega_d - \omega_F/2| \gg W_{nm}$, the quantity of interest is $\tilde{\chi}_{1c}$ defined by Eq. (17).

The major qualitative features of the LS are (i) scaling behavior near the bifurcation point where the period-2 states merge and (ii) the occurrence of resonant peaks as a function of frequency for weak damping. We have found that $\tilde{\boldsymbol{\chi}}_n$ scales with the distance η to the bifurcation point as $\tilde{\boldsymbol{\chi}}_n \propto \eta^{1/2}$. Thus, the field-induced correction to the activation energy of switching decreases as the system approaches the bifurcation point. However, this decrease is significantly slower than that of the major term in the activation energy, which is $\propto \eta^2$.

Resonant peaks of the LS become pronounced where, in the rotating frame, oscillator vibrations about the period-2 states are underdamped. The vibration frequencies are much less than ω_d, and the condition that these vibrations are underdamped is more restrictive than the requirement that the oscillator be underdamped in the laboratory frame. The LS displays several peaks. The major peak is shifted from $\omega_F/2$ by the frequency of small-amplitude vibrations about the period-2 states. Other peaks are shifted approximately by the overtones of this frequency and have smaller amplitudes. All LS peaks have characteristic strongly asymmetric shapes.

The effects discussed in this paper are not limited to a parametric oscillator, they can be observed in other systems with period-2 states. The results on modulation of switching rates by an additional field can be extended also to other systems with coexisting vibrational states, including resonantly driven nonlinear nano- and micromechanical oscillators and Josephson junctions. Fluctuational interstate transitions in these systems were recently studied experimentally [27–29]. Although the theory of the LS for modulated oscillators was not developed until this paper, a resonant increase of the switching rate by an additional field was expected from qualitative arguments based on the analogy with static systems [30]. The preliminary experimental data indicate that the effect occurs in Josephson junctions when the frequencies of the both strong and weak fields are close to the plasma frequency [31].

In conclusion, we have studied the effect of an additional field on a fluctuating parametrically modulated oscillator. We predict strong change of the populations of the period-2 states by a comparatively weak field. We also predict that the logarithm of the rate of interstate switching is linear in the field amplitude, and the proportionality coefficient may display resonant peaks as a function of the field frequency.

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[1] L. D. Landau and E. M. Lifshitz, *Mechanics*, 3rd ed. (Elsevier, Amsterdam, 2004).

[2] L. J. Lapidus, D. Enzer, and G. Gabrielse, Phys. Rev. Lett. **83**, 899 (1999).

[3] K. Kim, M. S. Heo, K. H. Lee, H. J. Ha, K. Jang, H. R. Noh, and W. Jhe, Phys. Rev. A **72**, 053402 (2005).

[4] K. Kim, M. S. Heo, K. H. Lee, K. Jang, H. R. Noh, D. Kim, and W. Jhe, Phys. Rev. Lett. **96**, 150601 (2006).

[5] H. B. Chan and C. Stambaugh, e-print cond-mat/0609394.
[6] M. I. Dykman, C. M. Maloney, V. N. Smelyanskiy, and M. Silverstein, Phys. Rev. E **57**, 5202 (1998).
[7] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical Systems*, 2nd ed. (Springer-Verlag, New York, 1998).
[8] M. I. Dykman, B. Golding, L. I. McCann, V. N. Smelyanskiy, D. G. Luchinsky, R. Mannella, and P. V. E. McClintock, Chaos **11**, 587 (2001).
[9] M. Marthaler and M. I. Dykman, Phys. Rev. A **73**, 042108 (2006).
[10] A. Larkin and Y. Ovchinnikov, J. Low Temp. Phys. **63**, 317 (1986).
[11] B. Ivlev and V. Mel'nikov, Phys. Lett. A **116**, 427 (1986).
[12] S. Linkwitz and H. Grabert, Phys. Rev. B **44**, 11901 (1991).
[13] M. H. Devoret, D. Esteve, J. M. Martinis, A. Cleland, and J. Clarke, Phys. Rev. B **36**, 58 (1987).
[14] E. Turlot, S. Linkwitz, D. Esteve, C. Urbina, M. H. Devoret, and H. Grabert, Chem. Phys. **235**, 47 (1998).
[15] M. I. Dykman, H. Rabitz, V. N. Smelyanskiy, and B. E. Vugmeister, Phys. Rev. Lett. **79**, 1178 (1997).
[16] V. N. Smelyanskiy, M. I. Dykman, H. Rabitz, and B. E. Vugmeister, Phys. Rev. Lett. **79**, 3113 (1997).
[17] M. I. Dykman and M. A. Krivoglaz, Zh. Eksp. Teor. Fiz. **77**, 60 (1979).
[18] R. Graham and T. Tél, J. Stat. Phys. **35**, 729 (1984).
[19] J. S. Langer, Ann. Phys. **41**, 108 (1967).
[20] J. Guckenheimer and P. Holmes, *Nonlinear Oscillators, Dynamical Systems and Bifurcations of Vector Fields* (Springer-Verlag, Berlin, 1987).
[21] M. I. Dykman, D. G. Luchinsky, R. Mannella, P. V. E. McClintock, N. D. Stein, and N. G. Stocks, Nuovo Cimento D **17**, 661 (1995).
[22] K. Wiesenfeld and F. Jaramillo, Chaos **8**, 539 (1998).
[23] H. B. Chan and C. Stambaugh, Phys. Rev. B **73**, 224301 (2006).
[24] E. Knobloch and K. A. Wiesenfeld, J. Stat. Phys. **33**, 611 (1983).
[25] R. Graham and T. Tél, Phys. Rev. A **35**, 1328 (1987).
[26] S. M. Soskin, R. Mannella, M. Arrayas, and A. N. Silchenko, Phys. Rev. E **63**, 051111 (2001).
[27] I. Siddiqi, R. Vijay, F. Pierre, C. M. Wilson, M. Metcalfe, C. Rigetti, L. Frunzio, and M. H. Devoret, Phys. Rev. Lett. **93**, 207002 (2004).
[28] J. S. Aldridge and A. N. Cleland, Phys. Rev. Lett. **94**, 156403 (2005).
[29] C. Stambaugh and H. B. Chan, Phys. Rev. B **73**, 172302 (2006).
[30] M. I. Dykman (unpublished).
[31] M. Devoret and I. Siddiqi (private communication).