Generalized matrix equivalence theorem for polarization theory

S. N. Savenkov,[1,*] V. V. Marienko,[1] and E. A. Oberemok[1]
[1]*Department of Radiophysics, Kiev Taras Shevchenko University, 01033 Kiev, Ukraine*

O. Sydoruk[2]
[2]*Department of Physics, University of Osnabrück, D-49069 Osnabrück, Germany*

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A generalized equivalence theorem for polarization theory is formulated and proven. It is shown that anisotropic properties of homogeneous nondepolarizing media can be presented as a combination of four basic mechanisms: linear and circular phase and linear and circular amplitude anisotropy. Expressions for the generalized effect operators of algebraic (or operator) optics are obtained and the inverse problem of crystal optics is solved in terms of physically realizable anisotropy parameters.

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I. INTRODUCTION

The history of matrix models of homogeneous anisotropic media goes back to the early works of Jones, who in 1941 introduced the formalism of 2×2 matrix [1] describing the response of anisotropic media to the incident polarized light. Relying on this approach, Hurwitz and Jones proved three so-called equivalence theorems [2] providing the basis of the matrix models for a number of classes of homogeneous anisotropic media.

According to the first theorem, any combination of polarization elements with circular and linear phase anisotropy (i.e., rotators and retardation plates) can be presented by an optical system consisting of only two elements, one with circular and the other with linear phase anisotropy. The second theorem is analogous to the first and deals with combinations of elements with circular phase and amplitude anisotropy (rotators and partial polarizers). Finally, the third theorem states that any combination of elements with amplitude and phase anisotropy is equivalent to an optical system containing only four elements: two with linear phase, one with circular phase, and one with linear amplitude anisotropy.

The development of matrix descriptions of media properties motivated further research in the field based on the methods of linear algebra and matrix analysis. In particular, Barakat [3] gave an analytical proof of the third Jones equivalence theorem in terms of singular decomposition [4].

Another approach to modeling homogeneous anisotropic media, alternative to that of Jones, is based on the polar decomposition theorem [4]. According to this theorem, an arbitrary matrix $\mathbf{T}$ can be presented by a product

$$\mathbf{T} = \mathbf{T}_P \mathbf{T}_R \quad \text{or} \quad \mathbf{T} = \mathbf{T}_R \mathbf{T}'_P, \tag{1}$$

where $\mathbf{T}_P$ and $\mathbf{T}'_P$ are Hermitian matrices and $\mathbf{T}_R$ is a unitary one. The Hermitian matrix is associated with amplitude anisotropy and the unitary matrix, with phase anisotropy [5]. The matrices $\mathbf{T}_P$ and $\mathbf{T}_R$ are called dichroic and phase polar forms [5–7].

Polar decomposition for polarization theory was first employed in Ref. [5] without, however, finding explicit expressions for $\mathbf{T}_P$ and $\mathbf{T}_R$. They were obtained later independently by Gil and Bernabeu [6] and Lu and Chipman [7]. Alternatively, dichroic and phase polar forms may be presented relying on the spectral problem of linear algebra [8].

The models of homogeneous anisotropic media based on polar decomposition contain six independent parameters, three for the phase $\mathbf{T}_R$ and three for the dichroic $\mathbf{T}_P$ polar forms, for four complex elements of the Jones matrix. Two additional degrees of freedom are associated with isotropic changes of the phase and amplitude of the light propagating in the medium.

A direct consequence of the generality of polar and singular decompositions is that they can be employed for representing arbitrary optical systems. One has, however, to pay the price for using these formal mathematical approaches by losing in physical interpretability of the decomposition results [9]. What is needed is a theorem that is general on the one hand and relies on the physically realizable parameters on the other [10]. The most significant issue for any model of media would be its physical validity. In crystal optics, only those matrix models that simultaneously take into account both inertia of medium properties and nonlocality of the medium response on light (i.e., time and spatial dispersion) can be accepted as rigorous and adequate. It is known that these properties of a crystalline medium determine, in the general case, the character of its anisotropy, namely, linear amplitude and phase, and circular amplitude and phase anisotropies (see, e.g., Refs. [11,12]). Formulation and proof of a generalized equivalence theorem is an issue of the present paper. This theorem is a direct generalization of the first and second Jones equivalence theorems, and it determines a matrix model for a homogeneous stationary anisotropic medium.

We start by introducing necessary conventions. Polarization of light changes if either amplitudes or phases of components of electric vector $\mathbf{E}$ change [8,13,14]. It is therefore customary to distinguish between two classes of anisotropic media, dichroic (or possessing amplitude anisotropy), which

*E-mail address: sns@univ.kiev.ua

influence only the amplitudes, and birefringent (or possessing phase anisotropy), influencing only the phases of the components of the electric vector. Among these two classes, four types of anisotropic mechanisms are recognized as basic (or primitive) [1,2]: linear and circular phase and linear and circular amplitude anisotropies. Their Jones matrices are well known [13].

The matrix of linear phase anisotropy is

$$\mathbf{T}^{LP}=\begin{pmatrix}\cos^2\alpha+\exp(-\mathrm{i}\Delta)\sin^2\alpha & [1-\exp(-\mathrm{i}\Delta)]\cos\alpha\sin\alpha\\ [1-\exp(-\mathrm{i}\Delta)]\cos\alpha\sin\alpha & \sin^2\alpha+\exp(-\mathrm{i}\Delta)\cos^2\alpha\end{pmatrix}, \tag{2}$$

where Δ is a value (i.e., phase shift between two orthogonal linear components of the electric vector) and α is an azimuth of the anisotropy.

Linear amplitude anisotropy has the Jones matrix

$$\mathbf{T}^{LA}=\begin{pmatrix}\cos^2\theta+P\sin^2\theta & (1-P)\cos\theta\sin\theta\\ (1-P)\cos\theta\sin\theta & \sin^2\theta+P\cos^2\theta\end{pmatrix}, \tag{3}$$

where P is a value (relative absorption of two linear orthogonal components of the electric vector) and θ is an azimuth of the anisotropy.

Circular phase anisotropy has the Jones matrix

$$\mathbf{T}^{CP}=\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}, \tag{4}$$

where ϕ is a phase shift introduced for two orthogonal circular components of the electric vector.

Finally, the Jones matrix of circular amplitude anisotropy is

$$\mathbf{T}^{CA}=\begin{pmatrix}1 & -\mathrm{i}R\\ \mathrm{i}R & 1\end{pmatrix}, \tag{5}$$

where R is a value of the anisotropy, i.e., relative absorption of two orthogonal circular components of the electric vector.

Six quantities α, Δ, P, θ, ϕ, and R are called anisotropy parameters. The ranges of their physically realizable values are

$$\begin{gathered}0\le P\le 1\\ -1\le R\le 1\\ -\pi/2\le\alpha\le\pi/2\\ -\pi/2\le\theta\le\pi/2\\ 0\le\Delta\le 2\pi\\ 0\le\phi\le 2\pi.\end{gathered} \tag{6}$$

Accepting the style of Jones [2], we formulate the following theorem.

Any combination of elements with linear and circular phase and linear and circular amplitude anisotropy is equivalent to an optical system containing only one element of each kind in the order

$$\mathbf{T}^{\mathrm{Gen}}=\mathbf{T}^{CP}\mathbf{T}^{LP}\mathbf{T}^{CA}\mathbf{T}^{LA}. \tag{7}$$

Formulated in the form of Eq. (7) the generalized equivalence theorem is free from the drawbacks of both polar and singular decompositions. First, the results of the suggested decomposition can be directly used for an analysis of the anisotropy of an optical system, which is based on physically meaningful and realizable parameters. Second, in contrast to singular decomposition (or the third Jones equivalence theorem), Eq. (7) includes circular amplitude anisotropy. Therefore, it provides a more physically adequate and complete description of the anisotropic properties of the general case of media possessing all four kinds of anisotropy mechanisms.

We shall give a proof of this theorem in Sec. II and examples demonstrating its application in Sec. III. Discussion of the results is presented and conclusions are drawn in Sec. IV.

II. PROOF OF THE GENERAL EQUIVALENCE THEOREM

The theorem shall be proven in two stages, which are essentially the same as given in Ref. [2]. First, for an arbitrary Jones matrix we derive the values of the anisotropy parameters and then show that the obtained values are always physically realizable.

A. Derivation of the anisotropy parameters

An explicit form of the elements of the matrix $\mathbf{T}^{\mathrm{Gen}}$ [Eq. (7)] is presented in Appendix A. It can be seen that, as far as the corresponding expressions Eq. (A1) are bulky, it is hardly possible to directly determine the values of the anisotropy parameters, α, Δ, P, θ, ϕ, and R, in Eq. (7). Instead, we are going to use the following method.

We shall analyze the response of the optical system [Eq. (7)] to the incident light with a certain polarization state. Choosing proper polarization states, it is possible to determine the anisotropy parameters one by one in Eq. (7), starting with the linear amplitude and finishing with the circular phase. We assume from now on that $\det\mathbf{T}^{\mathrm{Gen}}\neq 0$.

It can be seen from Eq. (7) that if the incident light is linearly polarized and has a varying azimuth γ, then after passing the first element in the sequence Eq. (7), $\mathbf{T}^{LA}$, it remains linearly polarized, but its intensity changes periodically with γ. The maximums and minimums of the intensity are determined by the azimuth of the linear amplitude anisotropy θ. Moreover, as follows from the form of the matrix [Eq. (5)], the intensity dependence on γ does not change after passing the remaining elements of the optical system $\mathbf{T}^{CP}\mathbf{T}^{LP}\mathbf{T}^{CA}$ either. To prove that, we write the polarization state of the light after passing the element $\mathbf{T}^{LA}$ in the form

$$\mathbf{E}^{\text{in}} = \begin{pmatrix} A\cos\gamma \\ B\sin\gamma \end{pmatrix}, \tag{8}$$

where A and B depend on the parameters P and θ. It has an intensity of $I^{\text{in}}=A^2\cos^2\gamma+B^2\sin^2\gamma$. Since the optical system $\mathbf{T}^{CP}\mathbf{T}^{LP}$ does not apparently influence the intensity of light, it is sufficient to consider the interaction of the polarized light [Eq. (8)] only with the optical system $\mathbf{T}^{CA}$

$$\mathbf{E}^{\text{out}} = \begin{pmatrix} 1 & -\mathrm{i}R \\ \mathrm{i}R & 1 \end{pmatrix}\begin{pmatrix} A\cos\gamma \\ B\sin\gamma \end{pmatrix} = \begin{pmatrix} A\cos\gamma - \mathrm{i}RB\sin\gamma \\ B\sin\gamma + \mathrm{i}RA\cos\gamma \end{pmatrix}, \tag{9}$$

and

$$I^{\text{out}} = (1+R^2)(A^2\cos^2\gamma + B^2\sin^2\gamma) = (1+R^2)I^{\text{in}}. \tag{10}$$

It can, therefore, be seen that the influence of the optical system $\mathbf{T}^{CA}$ on the intensity of the outgoing light, when the system is probed by a linearly polarized light, is described by the scaling factor $1+R^2$, whereas, the exact form of I^{out}, as a function of the parameters P, θ, γ, remains unchanged. Taking into account the fact that the matrix of the optical system $\mathbf{T}^{LA}$ is Hermitian and, consequently, its eigenpolarizations are orthogonal, it is clear that the maximum and minimum of I^{out} would be reached for two values of the azimuth of the linear amplitude anisotropy, $\theta_{\max}$ and $\theta_{\max}+\pi/2$, where $\theta_{\max}$ is the azimuth of maximum transmission of the element $\mathbf{T}^{LA}$. Therefore, determining the values of γ for which the output light has maximum and minimum intensity, it is possible to obtain the anisotropy parameters of linear amplitude anisotropy, θ and P.

For the intensity, we have

$$\begin{aligned} I(\theta) = {} & (\cos\theta\,\mathrm{Re}\,t_{11} + \sin\theta\,\mathrm{Re}\,t_{12})^2 \\ & + (\cos\theta\,\mathrm{Im}\,t_{11} + \sin\theta\,\mathrm{Im}\,t_{12})^2 \\ & + (\cos\theta\,\mathrm{Re}\,t_{21} + \sin\theta\,\mathrm{Re}\,t_{22})^2 \\ & + (\cos\theta\,\mathrm{Im}\,t_{21} + \sin\theta\,\mathrm{Im}\,t_{22})^2, \end{aligned} \tag{11}$$

where t_{ij} $(i,j=1,2)$ are the elements of $\mathbf{T}^{\text{Gen}}$. Finding the maximums and minimums of the intensity

$$\begin{aligned} \frac{dI(\theta)}{d\theta} = {} & \sin 2\theta(|t_{12}|^2 + |t_{22}|^2 - |t_{11}|^2 - |t_{21}|^2) \\ & + 2\,\mathrm{Re}(t_{22}t_{21}^* + t_{11}t_{12}^*)\cos 2\theta = 0, \end{aligned} \tag{12}$$

we determine the value of θ

$$\theta = -\frac{1}{2}\arctan\frac{2\,\mathrm{Re}(t_{22}t_{21}^* + t_{11}t_{12}^*)}{|t_{12}|^2 + |t_{22}|^2 - |t_{11}|^2 - |t_{21}|^2}, \tag{13}$$

where * denotes complex conjugation.

The value of linear amplitude anisotropy P can now be determined by probing the optical system, $\mathbf{T}^{\text{Gen}}$, with the linearly polarized light that has two different azimuthes corresponding to the maximum and minimum transmission, and equal intensities. At the output of the optical system the intensities are

$$\begin{aligned} I_1 = P^2[& (|t_{21}| + |t_{11}|)\cos^2\theta + (|t_{12}| + |t_{22}|)\sin^2\theta \\ & + 2\,\mathrm{Re}(t_{22}t_{21}^* + t_{11}t_{12}^*)\cos\theta\sin\theta \end{aligned} \tag{14}$$

$$\begin{aligned} I_2 = {} & (|t_{21}| + |t_{11}|)\sin^2\theta + (|t_{12}| + |t_{22}|)\cos^2\theta \\ & - 2\,\mathrm{Re}\,(t_{22}t_{21}^* + t_{11}t_{12}^*)\cos\theta\sin\theta, \end{aligned} \tag{15}$$

yielding for P

$$P = \sqrt{\frac{|t_{12}\cos\theta - t_{11}\sin\theta| + |t_{22}\cos\theta - t_{21}\sin\theta|}{|t_{12}\sin\theta + t_{11}\cos\theta| + |t_{22}\sin\theta + t_{21}\cos\theta|}}. \tag{16}$$

Equations (13) and (16) determine the anisotropy parameters of the matrix $\mathbf{T}^{LA}$ in Eq. (7). As far as $\mathbf{T}^{LA}$ is now known, it is possible to exclude it by multiplying $\mathbf{T}^{\text{Gen}}$ by the inverse of $\mathbf{T}^{LA}$ on the right-hand side

$$\mathbf{T}' = \mathbf{T}^{\text{Gen}}(\mathbf{T}^{LA})^{-1} = \mathbf{T}^{CP}\mathbf{T}^{LP}\mathbf{T}^{CA}. \tag{17}$$

The value of the circular amplitude anisotropy R can then be obtained from the ratio of the intensities of the light coming out of the optical system described by $\mathbf{T}'$, which interacts with incident light with right- and left-circular polarizations.

After some algebra we obtain for R

$$R = \frac{-(|t'_{11}|^2 + 0|t'_{12}|^2 + |t'_{21}|^2 + |t'_{22}|^2) \pm \sqrt{(|t'_{11}|^2 + |t'_{12}|^2 + |t'_{21}|^2 + |t'_{22}|^2)^2 - 4[\mathrm{Im}(t'^{\,*}_{11}t'_{12} + t'^{\,*}_{22}t'_{21})]^2}}{-2\mathrm{Im}(t'^{\,*}_{11}t'_{12} + t'^{\,*}_{22}t'_{21})}, \tag{18}$$

which determines the matrix $\mathbf{T}^{CA}$ in Eq. (7). Analogously to the previous case, this matrix can be excluded from consideration multiplying $\mathbf{T}'$ by the inverse of $\mathbf{T}^{CA}$

$$\mathbf{T}'' = \mathbf{T}'(\mathbf{T}^{CA})^{-1} = \mathbf{T}^{CP}\mathbf{T}^{LP}. \tag{19}$$

The matrix $\mathbf{T}''$ describes the optical system possessing only phase anisotropy. This matrix is unitary [1,5,14] and the parameters ϕ, Δ, and α can then be directly determined from Eq. (19)

$$\phi = \arctan \frac{t''_{12} - t''_{21}}{t''_{11} + t''_{22}} \tag{20}$$

$$\alpha = \frac{1}{2}\left(\arctan \frac{t''_{21} + t''_{12}}{t''_{11} - t''_{22}} + \phi \right) \tag{21}$$

$$\Delta = -2 \arctan\left(i \frac{t''_{12} + t''_{21}}{t''_{11} + t''_{22}} \frac{\cos \phi}{\sin(\phi - 2\alpha)} \right). \tag{22}$$

B. Physical realizability of obtained values of the anisotropy parameters

Equations (13), (16), (18), and (20)–(22) give the values of six anisotropy parameters and determine the matrices in Eq. (7). Equation (6) determines the ranges of the anisotropy parameters, which follow from their physical meaning [8]. The issue of the analysis of Eqs. (13), (16), (18), and (20)–(22) is to determine the possibility of obtaining values of the corresponding anisotropy parameters that lie outside the physically realizable ranges, as given by Eq. (6). Therefore, for the proof of the theorem it is necessary, depending on the physical nature of the specific anisotropy parameter and the structure of the expression for its determination, to analyze only those certain particular cases, which lie outside Eq. (6).

As long as any 2×2 matrix with complex elements describes a physically realizable optical system [15], the elements t_{ij} may be arbitrary complex numbers. In order to ensure that the optical system is passive [16], we can consider, without losing in generality, that its matrix is normalized to one of the elements.

We start with the value of the linear amplitude anisotropy P. It follows from Eq. (6) that in Eq. (16) the following cases are subject of interest:

1. P is real. It can be seen that, since the radicand in Eq. (16) is always positive, P is real for arbitrary values of t_{ij}.

2. $P=0$. The radicand in Eq. (16) is a ratio of the intensities of the linear polarized light with the azimuthes corresponding to the maximum and minimum transmissions of $\mathbf{T}^{LA}$. Zero value of P would mean that one of the polarization states is totally absorbed, which is not the case as long as we assume $\det \mathbf{T}^{\text{Gen}} \neq 0$.

3. $P=\infty$. Analogously to the previous case this cannot be realized because $\det \mathbf{T}^{\text{Gen}} \neq 0$.

4. $P=0/0$ can be realized only for the null matrix.

5. $P=1$ corresponds to the case where the intensities of the linear polarized light with the azimuthes corresponding to the maximum and minimum transmissions of $\mathbf{T}^{LA}$ are equal, yielding from Eqs. (14) and (15)

$$\begin{aligned}(|t_{21}| + |t_{11}| - |t_{12}| - |t_{22}|)(\cos^2\theta - \sin^2\theta) \\ + 4\,\mathrm{Re}(t_{22}t_{21}^* + t_{11}t_{12}^*) \cos\theta \sin\theta \\ = (|t_{21}| + |t_{11}| - |t_{12}| - |t_{22}|)\cos 2\theta + 2\,\mathrm{Re}(t_{22}t_{21}^* \\ + t_{11}t_{12}^*) \sin 2\theta = 0,\end{aligned} \tag{23}$$

which holds if

$$\begin{aligned}|t_{21}| + |t_{11}| - |t_{12}| - |t_{22}| = 0, \\ \mathrm{Re}(t_{22}t_{21}^* + t_{11}t_{12}^*) = 0.\end{aligned} \tag{24}$$

If both expressions in Eq. (24) are simultaneously true, then Eq. (13) gives $\theta = 1/2 \arctan(0/0)$ indeterminate form. The latter is an absolutely expected result as far as $P=1$ means that the linear amplitude anisotropy in Eq. (7) is absent. We proceed with the analysis of the expression Eq. (18) for the value of the circular amplitude anisotropy R. The following cases have to be analyzed:

1. R is real. It can be seen that

$$\begin{aligned}|t'_{11}|^2 + |t'_{12}|^2 + |t'_{21}|^2 + |t'_{22}|^2 \pm 2\,\mathrm{Im}(t'^*_{11} t'_{12} + t'^*_{22} t'_{21}) \\ = (\mathrm{Re}\, t'_{11} \pm \mathrm{Im}\, t'_{12})^2 + (\mathrm{Im}\, t'_{11} \mp \mathrm{Re}\, t'_{12})^2 \\ + (\mathrm{Re}\, t'_{21} \pm \mathrm{Im}\, t'_{22})^2 + (\mathrm{Im}\, t'_{21} \mp \mathrm{Re}\, t'_{22})^2 \geq 0,\end{aligned} \tag{25}$$

so that the radicand in Eq. (18) is always nonnegative yielding real values of R.

2. $0/0$ indeterminate form appears only if the matrix $\mathbf{T}'$ is null.

3. Solution uniqueness. It can be seen that if minus sign is chosen in Eq. (18) then $|R| > 1$, which is not a physically realizable value.

Note that we assumed thus far that the values of the elements of the matrices $\mathbf{T}$ and $\mathbf{T}'$ are arbitrary complex numbers. It allows us to state that Eqs. (16), (13), and (18) always give physically realizable values of parameters of the linear amplitude, P and θ, and circular amplitude, R [when the plus sign is taken in Eq. (18)], that satisfy Eq. (6).

Let us now come to the discussion of the values of the phase anisotropy, Eqs. (20)–(22). In the general case, as follows from Eq. (19), for three values of Δ, α, and ϕ we have eight equations (real and imaginary parts of the elements of $\mathbf{T}''$). It means that the expressions for the values of the phase anisotropy, Δ, α, and ϕ, can be obtained not in a unique form. Equations (20)–(22) are one of the possible solutions for the case $\phi \neq \pi/2$. However, only three of these equations are linearly independent and, as shown below, Eqs. (20)–(22) always give physically realizable values of Δ, α, and ϕ.

The case of interest here is an indeterminate form $0/0$ as can be seen from Eqs. (20)–(22). This possibility realizes in one of Eqs. (20)–(22) if

$$\begin{matrix} t''_{12} = t''_{21} \\ t''_{11} = -t''_{22} \end{matrix}, \tag{26a}$$

$$\begin{matrix} t''_{12} = -t''_{21} \\ t''_{11} = t''_{22} \end{matrix}, \tag{26b}$$

$$\begin{matrix} t''_{12} = -t''_{21} \\ t''_{11} = -t''_{22} \end{matrix}, \tag{26c}$$

giving the following realizations of the matrix $\mathbf{T}''$

$$\begin{pmatrix} \mathrm{Re}\, t''_{11} + \mathrm{i}\, \mathrm{Im}\, t''_{11} & \mathrm{Re}\, t''_{12} + \mathrm{i}\, \mathrm{Im}\, t''_{12} \\ \mathrm{Re}\, t''_{12} + \mathrm{i}\, \mathrm{Im}\, t''_{12} & -\mathrm{Re}\, t''_{11} - \mathrm{i}\, \mathrm{Im}\, t''_{11} \end{pmatrix}, \tag{27a}$$

$$\begin{pmatrix} \mathrm{Re}\, t''_{11} + \mathrm{i}\, \mathrm{Im}\, t''_{11} & \mathrm{Re}\, t''_{12} + \mathrm{i}\, \mathrm{Im}\, t''_{12} \\ -\mathrm{Re}\, t''_{12} - \mathrm{i}\, \mathrm{Im}\, t''_{12} & \mathrm{Re}\, t''_{11} + \mathrm{i}\, \mathrm{Im}\, t''_{11} \end{pmatrix}, \tag{27b}$$

$$\begin{pmatrix} \mathrm{Re}\, t''_{11} + \mathrm{i}\, \mathrm{Im}\, t''_{11} & \mathrm{Re}\, t''_{12} + \mathrm{i}\, \mathrm{Im}\, t''_{12} \\ -\mathrm{Re}\, t''_{12} - \mathrm{i}\, \mathrm{Im}\, t''_{12} & -\mathrm{Re}\, t''_{11} - \mathrm{i}\, \mathrm{Im}\, t''_{11} \end{pmatrix}. \tag{27c}$$

On the other hand, $\mathbf{T}''$ describes an optical system possessing phase anisotropy only and, therefore, is a unitary matrix of a general form [4]

$$\mathbf{T}'' = \begin{pmatrix} \mathrm{Re}\, t''_{11} + \mathrm{i}\, \mathrm{Im}\, t''_{11} & -\mathrm{Re}\, t''_{12} + \mathrm{i}\, \mathrm{Im}\, t''_{12} \\ \mathrm{Re}\, t''_{12} + \mathrm{i}\, \mathrm{Im}\, t''_{12} & \mathrm{Re}\, t''_{11} - \mathrm{i}\, \mathrm{Im}\, t''_{11} \end{pmatrix}. \tag{28}$$

It can be seen that Eqs. (27) do not satisfy Eq. (28). It should be noted that since the real and imaginary parts of the elements of the matrix Eq. (28) are not linearly independent, Eqs. (26a)–(26c) are valid only for the general case of simultaneous presence of both linear and circular phase anisotropy.

III. EXAMPLES

In order to demonstrate the consistency and self-descriptiveness of the proven theorem, we shall present in this section several modeling experiments. For this purpose, we synthesized the Jones matrices of three optical systems assuming given eigenpolarizations and eigenvalues and then analyzed the anisotropic properties of these systems.

If eigenpolarizations, $\chi_{1,2}$, and eigenvalues, $V_{1,2}$, of an optical system are known, then the elements of its Jones matrix, t_{ij}, are given by [8]

$$t_{11} = \frac{1}{\chi_1 - \chi_2}(V_2\chi_1 - V_1\chi_2),$$

$$t_{12} = \frac{1}{\chi_1 - \chi_2}(V_1 - V_2),$$

$$t_{21} = -\frac{\chi_1\chi_2}{\chi_1 - \chi_2}(V_1 - V_2),$$

$$t_{22} = \frac{1}{\chi_1 - \chi_2}(V_1\chi_1 - V_2\chi_2), \tag{29}$$

where

$$\chi_{1,2} = \frac{\cos\gamma\cos\varepsilon - \mathrm{i}\sin\gamma\sin\varepsilon}{\sin\gamma\cos\varepsilon + \mathrm{i}\cos\gamma\sin\varepsilon}.$$

Here, ε and γ are the ellipticity and azimuth of the large semiaxis of the polarization ellipse respectively.

Using Eq. (29), let us obtain the Jones matrices of the following three optical systems:

1. with eigenpolarizations $\chi_1=0.299-\mathrm{i}1.178$, $(\varepsilon_1=\pi/5,\ \gamma_1=\pi/7)$ $\chi_2=-0.203+\mathrm{i}0.797$ $(\varepsilon_2=-\pi/5,\ \gamma_2=\pi/7+\pi/2)$, and eigenvalues $V_1=0.6\exp(-\mathrm{i}40°)$, $V_2=0.2\exp(\mathrm{i}110°)$. The corresponding Jones matrix has a form

$$\mathbf{T}_1 = \begin{pmatrix} 0.145 - \mathrm{i}0.044 & 0.337 + \mathrm{i}0.181 \\ -0.210 - \mathrm{i}0.320 & 0.246 - \mathrm{i}0.154 \end{pmatrix}. \tag{30}$$

2. with eigepolarizations $\chi_1=0.299-\mathrm{i}1.178$, $(\varepsilon_1=\pi/5,\ \gamma_1=\pi/7)$ $\chi_2=0.035+\mathrm{i}0.728$ $(\varepsilon_2=-\pi/5,\ \gamma_2=\pi/7+\pi/3)$, and eigenvalues $V_1=0.6\exp(-\mathrm{i}40°)$, $V_2=0.2\exp(\mathrm{i}110°)$. The corresponding Jones matrix has a form

$$\mathbf{T}_2 = \begin{pmatrix} 0.088 - \mathrm{i}0.063 & 0.333 + \mathrm{i}0.231 \\ -0.248 - \mathrm{i}0.259 & 0.303 - \mathrm{i}0.135 \end{pmatrix}. \tag{31}$$

3. with eigepolarizations $\chi_1=0.299-\mathrm{i}1.178$, $(\varepsilon_1=\pi/5,\ \gamma_1=\pi/7)$ $\chi_2=0.299+\mathrm{i}1.178$ $(\varepsilon_2=-\pi/5,\ \gamma_2=\pi/7)$, and eigenvalues $V_1=0.6\exp(-\mathrm{i}40°)$, $V_2=0.2\exp(\mathrm{i}110°)$. The corresponding Jones matrix has a form

$$\mathbf{T}_3 = \begin{pmatrix} 0.123 - \mathrm{i}0.166 & 0.243 + \mathrm{i}0.224 \\ -0.360 - \mathrm{i}0.331 & 0.268 - \mathrm{i}0.032 \end{pmatrix}. \tag{32}$$

It can be seen that the eigepolarizations are elliptical and orthogonal for the case 1 and elliptical and nonorthogonal for the case 2. In case 3, the azimuths of both eigepolarizations coincide; they have, however, opposite rotation directions. For all three cases, the absolute value of ellipticity (the form of the polarization ellipse) and the eigenvalues are the same and chosen as arbitrary numbers. This would correspond to the general case when both amplitude (dichroism) and phase (birefringence) anisotropies are present.

Next, using the presentation Eq. (7) and Eqs. (13), (16), (18), and (20)–(22), we find the values of the anisotropy parameters characterizing the optical systems Eqs. (30)–(32),

1. $R=-0.485$, $P=0.777$, $\theta=64.3°$, $\Delta=28.1°$, $\alpha=9.7°$, $\phi=74.6°$,

2. $R=-0.487$, $P=0.698$, $\theta=-79.6°$, $\Delta=24.1°$, $\alpha=17.3°$, $\phi=75.7°$,

3. $R=-0.515$, $P=0.612$, $\theta=-31.0°$, $\Delta=14.6°$, $\alpha=4.7°$, $\phi=75.4°$.

It can be seen that the synthesized optical systems are characterized by all four anisotropy mechanisms. The fact that the optical systems have qualitatively different eigenpolarizations and the same eigenvalues has a different effect on the values of the corresponding anisotropy parameters.

IV. DISCUSSION AND CONCLUSIONS

Having derived the values of the anisotropy parameters in Sec. II A and shown that they always present physically realizable quantities in Sec. II B, we established the generalized matrix equivalence theorem formulated in Sec. I for the Jones matrices. Obviously, a similar theorem can be formulated for the Mueller matrices as well. Expressions for the elements of the Mueller matrix corresponding to the Jones matrix Eq. (7) are presented in Appendix B, and expressions for the anisotropy parameters, α, Δ, P, θ, ϕ, and R, in terms of the Mueller matrix elements are given in Appendix C.

Since we assumed arbitrary values of the Jones matrix elements in Secs. II A and II B the proven theorem has a general form. An advantage of this theorem over polar and singular decomposition is that the parameters used characterize physically realizable anisotropy mechanisms. This permits not only simple physical interpretation of the anisotropic properties of optical systems but also synthesis of the systems with arbitrary predetermined polarization properties.

The theorem [Eq. (7)] is a direct generalization of the first and second Jones theorems. It is gratifying to note that the need for such generalization was first claimed by Jones [17], although he was pessimistic about its feasibility. He wrote in Ref. [17]: "…It is desired to find some method of representing the properties of an arbitrary homogeneous crystal as a combination of number of simple properties. As a first attempt, one might try to find a simple way of factoring the matrix **M** of the crystal into the product of a finite number of simple **M**-matrices, each of which would represent a simple crystal property, such as circular dichroism, linear birefringence, or isotropic absorption. This effort fails, because the constants which specify the component matrices depend on the order in which the matrices are multiplied."

Matrix Eqs. (A1) and (B1) describe the generalized effect operators of anisotropic media on the polarized light in terms of the Jones and Mueller matrix formalisms, respectively. Therefore, Eqs. (A1) and (B1) may be regarded as generalized polarimetric matrix models of an arbitrary nondepolarizing medium or, in other words, as the basic relations of algebraic (or operator) optics of anisotropic media [18]. The anisotropy parameters Eqs. (13), (16), (18), and (20)–(22) are then the general solution of the inverse problem of crystal optics based on the models Eqs. (A1) and (B1).

Relying on Eq. (B1), it is possible to obtain the corresponding classes of the inverse problem for the models of homogeneous anisotropic media described by the incomplete Mueller matrices [19–21] that are measured in the methods of time-sequential and dynamical polarimeter [22]. This would permit an increase of both the speed and accuracy of polarimetric measurements.

In addition, Eqs. (A1) and (B1) determine the general matrix form of the polarization transfer equation [23] for homogeneous anisotropic media. Measurement of the Mueller matrices and determination of the anisotropy parameters for a corresponding number of directions in the studied medium can then be considered as a content of the method of Mueller tomography [8,24] for the given class of media.

Furthermore, the given proof of the generalized equivalence theorem clarifies the matter and physical interpretation of the nonuniqueness of the solutions of the inverse problem as well as the invariance of describing the anisotropic properties from electrodynamic point of view. Nonuniqueness of the inverse problem is a result of the fact that the matrices of the primitive anisotropic mechanisms [Eqs. (2)–(5)] are not interchangeable in Eq. (7), the latter gives several different ways in which the matrices [Eqs. (2)–(5)] can be multiplied [7,25].

The importance of the formulated problems was addressed in a number of earlier publications [6,7,18,25]. The models of the homogeneous anisotropic media available at the moment did not, however, allow their further development. The proven theorem provides a basis for a systematic investigation of these problems. This issue lies, however, outside the present discussion and is a subject of future work.

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APPENDIX A

Explicit form of the elements of the generalized Jones $\mathbf{T}^{\mathrm{Gen}}$ [Eq. (7)] as functions of the anisotropy parameters α, Δ, P, θ, ϕ, and R

$$\begin{aligned} t_{11} = {} & \{s_\phi c_\alpha s_\alpha[1-\exp(-\mathrm{i}\Delta)] + c_\phi[c_\alpha^2 + s_\alpha^2\exp(-\mathrm{i}\Delta)]\} \\ & \times[c_\theta^2 + s_\theta^2 P - \mathrm{i}R(1-P)c_\theta s_\theta] \\ & + \{s_\phi[s_\alpha^2 + c_\alpha^2\exp(-\mathrm{i}\Delta)] + c_\phi c_\alpha s_\alpha[1-\exp(-\mathrm{i}\Delta)]\} \\ & \times[(1-P)c_\theta s_\theta + \mathrm{i}R(c_\theta^2 + s_\theta^2 P)], \end{aligned}$$

$$\begin{aligned} t_{21} = {} & \{c_\phi c_\alpha s_\alpha[1-\exp(-\mathrm{i}\Delta)] - s_\phi[c_\alpha^2 + s_\alpha^2\exp(-\mathrm{i}\Delta)]\} \\ & \times[c_\theta^2 + s_\theta^2 P - \mathrm{i}R(1-P)c_\theta s_\theta] \\ & + \{c_\phi[s_\alpha^2 + c_\alpha^2\exp(-\mathrm{i}\Delta)] - s_\phi c_\alpha s_\alpha[1-\exp(-\mathrm{i}\Delta)]\} \\ & \times[(1-P)c_\theta s_\theta + \mathrm{i}R(c_\theta^2 + s_\theta^2 P)], \end{aligned}$$

$$\begin{aligned} t_{12} = {} & \{s_\phi c_\alpha s_\alpha[1-\exp(-\mathrm{i}\Delta)] + c_\phi[c_\alpha^2 + s_\alpha^2\exp(-\mathrm{i}\Delta)]\} \\ & \times[(1-P)c_\theta s_\theta - \mathrm{i}R(s_\theta^2 + c_\theta^2 P)] \\ & + \{s_\phi[s_\alpha^2 + c_\alpha^2\exp(-\mathrm{i}\Delta)] + c_\phi c_\alpha s_\alpha[1-\exp(-\mathrm{i}\Delta)]\} \\ & \times[s_\theta^2 + c_\theta^2 P + \mathrm{i}R(1-P)c_\theta s_\theta], \end{aligned}$$

$$\begin{aligned} t_{22} = {} & \{c_\phi c_\alpha s_\alpha[1-\exp(-\mathrm{i}\Delta)] - s_\phi[c_\alpha^2 + s_\alpha^2\exp(-\mathrm{i}\Delta)]\} \\ & \times[(1-P)c_\theta s_\theta - \mathrm{i}R(s_\theta^2 \\ & + c_\theta^2 P)] + \{c_\phi[s_\alpha^2 + c_\alpha^2\exp(-\mathrm{i}\Delta)] - s_\phi c_\alpha s_\alpha[1-\exp(-\mathrm{i}\Delta)]\} \\ & \times[s_\theta^2 + c_\theta^2 P + \mathrm{i}R(1-P)c_\theta s_\theta], \end{aligned} \tag{A1}$$

where $s_x \equiv \sin x$, $c_x \equiv \cos x$ $(x = \alpha, \phi, \theta)$.

APPENDIX B

Explicit form of the elements of the generalized Mueller matrix, $\mathbf{M}^{\mathrm{Gen}}$, corresponding to the generalized Jones matrix $\mathbf{T}^{\mathrm{Gen}}$ [Eq. (7)] as functions of the anisotropy parameters α, Δ, P, θ, ϕ, and R

$$m_{11} = (1+P^2)(1+R^2),$$

$$\begin{aligned} m_{21} = {} & (1-P^2)(1-R^2)\{c_{2\theta}[c_{2\phi}(c_{2\alpha}^2 + s_{2\alpha}^2 c_\Delta) \\ & + (1-c_\Delta)s_{2\alpha}c_{2\alpha}s_{2\phi}] + s_{2\theta}[s_{2\phi}(s_{2\alpha}^2 + c_{2\alpha}^2 c_\Delta) \\ & + (1-c_\Delta)s_{2\alpha}c_{2\alpha}c_{2\phi}]\} + 2(1+P^2)s_\Delta s_{2\phi-2\alpha}, \end{aligned}$$

$$\begin{aligned} m_{31} = {} & (1-P^2)(1-R^2)\{c_{2\theta}[-s_{2\phi}(c_{2\alpha}^2 + s_{2\alpha}^2 c_\Delta) \\ & + (1-c_\Delta)s_{2\alpha}c_{2\alpha}c_{2\phi}] + s_{2\theta}[c_{2\phi}(s_{2\alpha}^2 + c_{2\alpha}^2 c_\Delta) \\ & - (1-c_\Delta)s_{2\alpha}c_{2\alpha}s_{2\phi}]\} + 2R(1+P^2)s_\Delta c_{2\phi-2\alpha}, \end{aligned}$$

$$m_{41} = (1 - P^2)(1 - R^2)s_\Delta s_{2\alpha-2\theta} + 2R(1 + P^2)c_\Delta,$$

$$m_{12} = (1 - P^2)(1 + R^2)c_{2\theta},$$

$$\begin{aligned} m_{22} = \{&[c_{2\theta}^2(1 + P^2) + 2Ps_{2\theta}^2][c_{2\phi}(c_{2\alpha}^2 + s_{2\alpha}^2 c_\Delta) \\ &+ (1 - c_\Delta)c_{2\alpha}s_{2\alpha}s_{2\phi}] + (1 - P)^2 c_{2\theta}s_{2\theta}[s_{2\phi}(s_{2\alpha}^2 + c_{2\alpha}^2 c_\Delta) \\ &+ (1 - c_\Delta)c_{2\alpha}s_{2\alpha}c_{2\phi}]\}(1 - R^2) \\ &+ 2R(1 - P^2)s_{2\phi-2\alpha}c_{2\theta}s_\Delta, \end{aligned}$$

$$\begin{aligned} m_{32} = \{&[c_{2\theta}^2(1 + P^2) + 2Ps_{2\theta}^2][- s_{2\phi}(c_{2\alpha}^2 + s_{2\alpha}^2 c_\Delta) \\ &+ (1 - c_\Delta)c_{2\alpha}s_{2\alpha}c_{2\phi}] + (1 - P)^2 c_{2\theta}s_{2\theta}[c_{2\phi}(s_{2\alpha}^2 + c_{2\alpha}^2 c_\Delta) \\ &- (1 - c_\Delta)c_{2\alpha}s_{2\alpha}s_{2\phi}]\}(1 - R^2) \\ &+ 2R(1 - P^2)c_{2\phi-2\alpha}c_{2\theta}s_\Delta, \end{aligned}$$

$$\begin{aligned} m_{42} = (1 - R^2)\{&s_{2\alpha}[c_{2\theta}^2(1 + P^2) + 2Ps_{2\theta}^2] \\ &- (1 - P)^2 s_{2\theta}c_{2\theta}c_{2\alpha}\}s_\Delta + 2R(1 - P^2)c_{2\theta}c_\Delta, \end{aligned}$$

$$m_{13} = (1 - P^2)(1 + R^2)s_{2\theta},$$

$$\begin{aligned} m_{23} = \{&[s_{2\theta}^2(1 + P^2) + 2Pc_{2\theta}^2][s_{2\phi}(s_{2\alpha}^2 + c_{2\alpha}^2 c_\Delta) \\ &+ (1 - c_\Delta)c_{2\alpha}s_{2\alpha}c_{2\phi}] + (1 - P)^2 c_{2\theta}s_{2\theta}[c_{2\phi}(c_{2\alpha}^2 + s_{2\alpha}^2 c_\Delta) \\ &+ (1 - c_\Delta)c_{2\alpha}s_{2\alpha}s_{2\phi}]\}(1 - R^2) \\ &+ 2R(1 - P^2)s_{2\phi-2\alpha}s_{2\theta}s_\Delta, \end{aligned}$$

$$\begin{aligned} m_{33} = \{&[s_{2\theta}^2(1 + P^2) + 2Pc_{2\theta}^2][c_{2\phi}(s_{2\alpha}^2 + c_{2\alpha}^2 c_\Delta) \\ &- (1 - c_\Delta)c_{2\alpha}s_{2\alpha}s_{2\phi}] \\ &+ (1 - P)^2 c_{2\theta}s_{2\theta}[- s_{2\phi}(c_{2\alpha}^2 + s_{2\alpha}^2 c_\Delta) + (1 \\ &- c_\Delta)c_{2\alpha}s_{2\alpha}c_{2\phi}]\}(1 - R^2) + 2R(1 - P^2)s_{2\phi-2\alpha}s_{2\theta}s_\Delta, \end{aligned}$$

$$\begin{aligned} m_{43} = (1 - R^2)\{&(1 - P)^2 s_{2\theta}c_{2\theta}s_{2\alpha} \\ &- c_{2\alpha}[s_{2\theta}^2(1 + P^2) + 2Pc_{2\theta}^2]\}s_\Delta + 2R(1 - P^2)s_{2\theta}c_\Delta, \end{aligned}$$

$$m_{14} = 4RP,$$

$$m_{24} = 2P(1 + R^2)s_{2\phi-2\alpha}s_\Delta,$$

$$m_{34} = 2P(1 + R^2)c_{2\phi-2\alpha}s_\Delta,$$

$$m_{44} = 2P(1 + R^2)c_\Delta, \tag{B1}$$

where $c_x \equiv \cos x$, $s_x \equiv \sin x$.

APPENDIX C

Solution of the inverse problem for the generalized Mueller matrix, $\mathbf{M}^{\mathrm{Gen}}$ [Eq. (B1)]

$$\theta = \frac{1}{2}\arctan\frac{m_{13}}{m_{12}}, \tag{C1}$$

$$P = \frac{(m_{11} - m_{12}\cos 2\theta - m_{13}\sin 2\theta)^2}{m_{11}^2 - (m_{12}\cos 2\theta - m_{13}\sin 2\theta)^2}, \tag{C2}$$

$$R = \frac{m_{11}' \pm \sqrt{(m_{11}')^2 - (m_{14}')^2}}{m_{14}'}, \tag{C3}$$

where m_{ij}' are the elements of the Mueller matrix $\mathbf{M}'$ corresponding to the Jones matrix $\mathbf{T}'$ [Eq. (17)],

$$\alpha = -\frac{1}{2}\arctan\frac{m_{42}''}{m_{43}''} \quad \text{if} \quad m_{43}'' \neq 0, \tag{C4}$$

$$\alpha = \frac{1}{2}\arctan\frac{m_{32}''}{m_{22}''} \quad \text{if} \quad m_{43}'' = 0, \tag{C5}$$

$$\Delta = \arctan\frac{m_{42}''}{m_{44}''\sin 2\alpha} = \arctan\frac{-m_{43}''}{m_{44}''\cos 2\alpha}, \tag{C6}$$

$$\phi = \frac{1}{2}\arctan\frac{m_{23}'' - m_{32}''}{m_{22}'' + m_{33}''} = -\frac{1}{2}\arctan\frac{m_{34}''m_{42}'' - m_{24}''m_{43}''}{m_{34}''m_{43}'' + m_{24}''m_{42}''}, \tag{C7}$$

where m_{ij}'' are the elements of the Mueller matrix $\mathbf{M}''$ corresponding to the Jones matrix $\mathbf{T}''$ [Eq. (19)].

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