Separation and synchronization of piecewise linear chaotic systems

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In this paper a topic regarding the synchronization of chaotic systems is dealt with: the case of separation and synchronization of many chaotic signals generated by different chaotic circuits and combined together is examined. In particular, an observer based strategy has been adopted, and an approach for the simultaneous stabilization of many Luenberger observers has been investigated to face the problem of separation and synchronization. The design strategy is based on linear matrix inequalities (LMIs). Indeed, the LMI problem is referred to have a solution if a dual optimization problem admits a solution. In our case the feasibility condition, if it does exist, allows us to establish that the separation and synchronization problem for the chosen circuit admits a solution. Some numerical simulations are reported. Further results refer to an experimental circuit showing the suitability of the approach. Furthermore, the use of the proposed scheme to transmit two or more information masked into two or more multiplexed chaotic signals and the design of suitable parameters through the introduced technique based on LMIs are discussed.

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I. INTRODUCTION

Chaotic signals are aperiodic signals generated by deterministic nonlinear systems which have a strong dependence on their initial conditions. Despite this characteristic, it is well known that two chaotic systems (and thus the signals they generate) can be synchronized [1]. The synchronization introduced by Pecora and Carroll [1] is complete and asymptotic, in the sense that asymptotically the two systems evolve identically. This concept of synchronization has been generalized in other works: generalized synchronization, phase synchronization and lag synchronization have been introduced [2]. In the following, complete synchronization will be considered.

Synchronization of chaotic systems has an important application in the field of secure communications [3,4]. In particular, the role of synchronization in chaos-based communication schemes has been studied in Ref. [3], where the advantages of using a chaotic carrier instead of a sinusoidal carrier in a digital communication system are discussed. In fact, in the case of digital communication, sinusoidal carriers have an optimal bandwidth efficiency and a relative ease of reconstruction of the original signal, but have an often high power spectral density which causes a high level of interference and enhances the probability of interception by other receivers, whereas chaotic carriers can solve these drawbacks.

Many schemes for complete synchronization have been proposed since the seminal work of Pecora and Carroll [1]: negative feedback [5], sporadic driving [6], active-passive decomposition [7,8], diffusive coupling and some other hybrid methods [9]. Moreover, both bidirectional and unidirectional couplings have been considered [2]. In the case of unidirectional coupling, the evolution of one of the coupled systems is unaltered by the other system. In this case, the two chaotic systems are called master and slave or drive and response systems.

In the negative feedback scheme [5], starting from the difference of two corresponding state variables (which are assumed measurable), an error signal is built and fed back into the slave system. In this paper the simultaneous synchronization of two groups of n chaotic systems instead of two chaotic systems is investigated by using a negative feedback scheme. In our case, thus, the master system is formed by n independent chaotic systems (i.e., n different systems which do not interact with each other). In general, the synchronization of two groups of such chaotic systems requires n independent feedback signals. In our case, instead, it is investigated if and under which conditions synchronization can be achieved by using only one feedback signal which depends on the chaotic systems of the master (i.e., it is for instance a linear combination of the state variables of the master chaotic systems). This problem is referred to as separation and synchronization of chaotic signals. A similar topic, referred as multiplexing of chaotic signals using synchronization, was investigated in Refs. [10,11]. In particular, Tsimring and Sushchik [10] investigate the simultaneous synchronization of chaotic maps, while Carroll and Pecora [11] discuss how to combine two chaotic systems with the multiplexing technique to make a communication system. However, in neither case is the synchronization of continuous-time flows shown. In our paper this is achieved with a technique and experimentally demonstrated with a circuit implementation of one of the examples shown. Moreover, a way to transmit different information on the different chaotic systems is introduced.

In this paper the problem of separation and synchronization for a class of chaotic systems, namely those with piecewise linear (PWL) nonlinearities, is approached with linear matrix inequalities (LMI) [12]. The proposed strategy allows us both to establish if separation and synchronization are possible with the considered chaotic systems and to design the master-slave circuit. A theoretical approach for master-slave systems made of n chaotic subsystems is discussed and, for the case of $n=2$, two numerical examples showing the separation of the two chaotic signals and the synchronization of the two pairs of chaotic systems are reported.

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Moreover, experimental results are discussed: they confirm the suitability of the approach even in the real case, when nonidentical systems are necessarily considered.

The solution to the problem of separation and synchronization, proposed in this paper, can be adopted in chaotic communication systems to mask two or more different information in two or more different chaotic signals. For instance, in order to increase the information carried on one channel it might be possible to use two different chaotic carriers transmitting two different information at the same time. In this case, the two chaotic systems of the slave must be synchronized to the two chaotic systems of the master, starting from only one signal containing the carriers and the data mapped on them. This can be achieved by applying the proposed separation and synchronization scheme.

The paper is organized as follows: in Sec. II the proposed negative feedback scheme for the synchronization of PWL chaotic circuits is introduced; in Sec. III the investigated cases are described and the results of numerical simulations are reported. In Sec. IV a physical implementation of the proposed scheme is reported by using field programmable analog arrays. Section V shows the use of the separation and synchronization scheme for communication. Section VI draws the conclusions of the paper.

II. SEPARATION AND SYNCHRONIZATION OF PWL CHAOTIC SYSTEMS

The synchronization scheme proposed to solve the separation and synchronization problem is based on negative feedback and, in particular, on the design of a nonlinear observer. The master system contains n different and uncoupled chaotic systems, and the slave is a copy of the master. In our approach the error signal used to synchronize the slave is obtained by comparing a linear combination of the master state variables with the same combination of the correspondent variables of the slave system. This error signal, weighted by suitable gains, is added to each state variable of the slave as in the negative feedback scheme for two chaotic systems [5]. Therefore, assuming that the equations of the master are

$$\dot{\mathbf{X}}_{\mathbf{m}} = f(\mathbf{X}_{\mathbf{m}}), \tag{1}$$

the slave equations will be

$$\dot{\mathbf{X}}_{\mathbf{s}} = f(\mathbf{X}_{\mathbf{s}}) + Ke, \tag{2}$$

where K is the gains vector and e is the (scalar) error signal. Assuming that the master is composed by n systems of order $m_1, m_2, \ldots, m_n$, then $\mathbf{X_m} \in \mathbb{R}^m$ with $m = m_1 + m_2 + \cdots + m_n$, $\mathbf{X_s} \in \mathbb{R}^m$ and $K \in \mathbb{R}^m$. This scheme is summarized in Fig. 1. In order to synchronize the master and slave systems, the error must asymptotically converge to zero. The slave system can be thus considered as an observer of the master system, so that the problem of separation and synchronization is equivalent to the design of an asymptotic observer in which gains must be calculated in order to ensure the stability of the error system.

As stated in the introduction, chaotic systems characterized by PWL nonlinearities are considered. In each region of the PWL, the systems of this class assume a different linear behavior switching through the PWL regions. Therefore, a PWL system is characterized by the set of its possible linearizations. Since, in each region, each linear system can be observed using the classical linear control techniques, our idea is to design an observer which simultaneously guarantees asymptotically stable error dynamics in each of these regions. Therefore, to solve the problem of separation and synchronization, the observer should be designed by solving a simultaneous stability problem. Before entering in the details of the proposed method, the LMI formulation of the asymptotic observer for linear systems is briefly recalled.

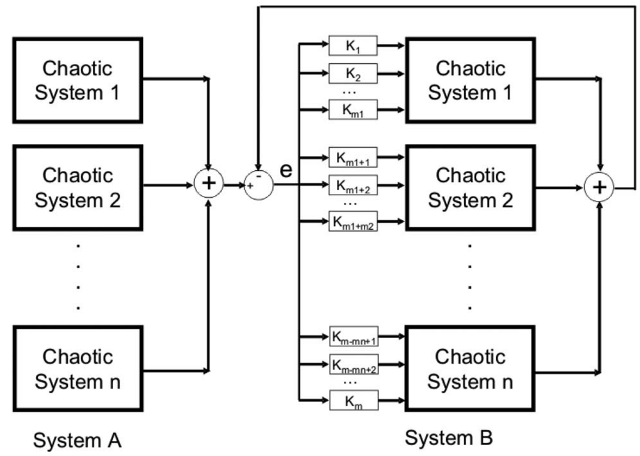


FIG. 1. Separation and synchronization scheme. 1-1′, 2-2′, and n-n' are identical systems starting from different initial conditions; the error is the difference between a linear combination of the state variables of the master system and those of the slave system.

A. LMI-based design of an asymptotic observer for linear systems

For a linear system an asymptotic observer can be designed by taking into account the following considerations. Let us assume that the system state equations are

$$\dot{\mathbf{X}} = A\mathbf{X}, \tag{3}$$

The observer is a dynamical system with the following equations:

$$\dot{\hat{\mathbf{X}}} = A\hat{\mathbf{X}} + Ke, \tag{4}$$

where K is the vector of observer gains and $e = C\mathbf{X} - C\hat{\mathbf{X}}$ with (A, C) the state matrices of the system. Figure 2 shows the block scheme of an asymptotic linear observer. The vector K of gain coefficients must be chosen in order to ensure the stability of the error system. This can be done by verifying the equation of the first Lyapunov criterion:

$$A_O^T P + P A_O = -\bar{Q}, \tag{5}$$

with P and $\bar{Q}$ positive definite matrices and $A_O = A - KC$ the state matrix of the error system.

This problem can be reformulated in terms of a system of two linear matrix inequalities (LMI) [12]:

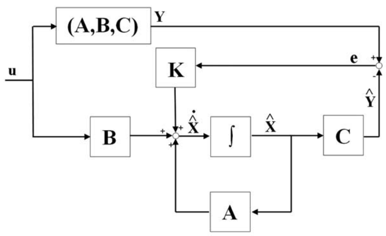


FIG. 2. Block scheme of a generic asymptotic linear observer. (A,B,C) are the system state matrices, K is the observer gains vector.

$$A_O^T P + PA_O < 0,$$

$$P > 0, \tag{6}$$

where $P>0$ $(P<0)$ indicates a positive (negative) definite matrix. It is possible to design the observer problem in terms of LMIs by substituting in (6) the state matrix of the error system

$$(A-KC)^T P + P(A-KC) < 0 \tag{7}$$

and thus defining $Q=PK$,

$$A^T P - C^T Q^T + PA - QC < 0,$$

$$P > 0. \tag{8}$$

With the values of P and Q obtained by solving the problem in (8), it is possible to calculate the gains vector as $K = P^{-1}Q$. This problem is feasible if system (A,C) is observable.

B. LMI-based design of the observer for PWL systems

In this section, the design of the observer for PWL systems is considered. To design the observer, one must repeat the considerations above for each of the possible regions in which the observed system and the observer may work. The advantages of the LMI approach is that other inequalities may be added to the problem so that a set of LMIs must be solved to find the gains able to simultaneously stabilize more than one system.

Let us define as $\mathbf{e_X}=\mathbf{X}-\hat{\mathbf{X}}$ the state estimation error. In general, the equation that describes the error system dynamics is

$$\dot{\mathbf{e}}_\mathbf{X} = A_i \mathbf{X} - A_j \hat{\mathbf{X}} - KC(\mathbf{X}-\hat{\mathbf{X}}), \tag{9}$$

where A_i and A_j, respectively, represent the linearization of the observed system and of the observer in ith or jth region of the PWL nonlinearity. The two matrices A_i and A_j are different when the two systems work in different regions of the PWL nonlinearity. Otherwise (i.e., when the observer works in the same region of the observed system), the matrices A_i and A_j are equal and the error system dynamic reduces to

$$\dot{\mathbf{e}}_\mathbf{X} = (A_i - KC)\mathbf{e}_\mathbf{X}. \tag{10}$$

In this situation the observer can be designed to be stable by solving the following LMI problem:

$$A_i^T P - C^T Q^T + PA_i - QC < 0, \quad i=1,\ldots,q,$$

$$P > 0, \tag{11}$$

where q is the number of regions of the considered PWL nonlinearity. This means that all the LMIs described in each region of the nonlinearity for $A_i=A_j$ must be solved. If the overall LMI problem is feasible, its solution leads to a gain vector K able to stabilize all the possible error dynamics.

This procedure permits the design of the observer able to reconstruct the dynamics of the observed system. Actually, a necessary condition to the stability of the error dynamics is imposed. In fact the error system is imposed to be stable only if the observed system and the observer are in the same PWL region. Otherwise, when the two systems are in different regions, the error dynamics are given by Eqs. (9).

Numerical simulations and experimental results, reported in the following sections, show that, provided that the eigenvalues of the error system (i.e., the eigenvalues of (A_i-KC) for $i=1,\ldots,q$) as designed by solving the stability problem are sufficiently fast, the necessary condition is sufficient for synchronization.

C. Formulation of the LMI for the separation and synchronization problem

The proposed approach is now applied to the problem of separation and synchronization. The slave system is designed as an observer for the master system. Let us consider the linearization of the master system described by the following equations:

$$\dot{\mathbf{X}}_1 = A_{i_1}^1 \mathbf{X}_1,$$

$$\dot{\mathbf{X}}_2 = A_{i_2}^2 \mathbf{X}_2,$$

$$\cdots,$$

$$\dot{\mathbf{X}}_\mathbf{n} = A_{i_n}^n \mathbf{X}_\mathbf{n}, \tag{12}$$

where $\mathbf{X_1},\ldots,\mathbf{X_n}$ are the state vectors of the chaotic systems forming the master (i.e., $\mathbf{X}=[\mathbf{X_1}\quad \mathbf{X_2}\quad \cdots \quad \mathbf{X_n}]^T$) and $A_{i_r}^r$ is the linearization of the rth chaotic subunit in the i_rth region of the PWL nonlinearity. Therefore, the state matrix of the overall system is a block matrix: the diagonal blocks are the state matrices of each linearized chaotic subsystem, the other blocks are zeros, as follows:

$$A_{(i_1,i_2,\ldots,i_n)} = \begin{pmatrix} A_{i_1}^1 & 0 & \ldots & 0 \\ 0 & A_{i_2}^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{i_n}^n \end{pmatrix}.$$

In the case of the separation and synchronization problem, one must write a linear matrix inequality, as in Eq. (11), for each of the possible matrices $A_{(i_1,i_2,\ldots,i_n)}$. Each possible combination of the matrices that could form the overall system, corresponding to each region of the PWL, has thus to be considered. The separation and synchronization problem may have a solution if the LMI problem (11) with $A_i = A_{(i_1,i_2,\ldots,i_n)}$ is feasible.

For example, in the case of $n=2$ and $q_1=q_2=2$, four LMIs must be considered in the LMI problem (11): in fact, each system has two possible linearizations and thus there are four possible combinations of them.

If the two subsystems of the master circuit are identical, the condition on the observability of the whole system is not respected, and this leads to an unfeasible LMI problem. The same conclusion, i.e., that synchronization is possible only in nonidentical multiplexed chaotic systems, was reached in Ref. [11] by considering the variational equation for the case of identical subsystems.

III. NUMERICAL RESULTS

In this section two numerical examples with $n=2$ are considered. Our procedure has been tested on several pairs of chaotic systems: here the numerical results obtained simulating two different pairs of chaotic systems for which the corresponding LMI problems are feasible are reported.

A. Separation and synchronization of a pair of Kennedy's oscillators

In the first example, the double-scroll-like chaotic oscillator described in Ref. [13], called in the following Kennedy's oscillator, is used. This chaotic system is characterized by a PWL nonlinearity that does not affect the linearization matrices as can be noticed from the state equations (i.e., $q_1 = q_2 = 1$),

$$\dot{x} = y,$$

$$\dot{y} = z,$$

$$\dot{z} = -a[x + y + z - \mathrm{sgn}(x)]. \tag{13}$$

In all the cases investigated, the error signal has been chosen as the sum of all the state variables: this means to choose $C = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$ in Eq. (11).

The first case consists of a pair of Kennedy's oscillators with two different values of the parameter a as follows:

$$\dot{x}_{1m} = y_{1m},$$

$$\dot{y}_{1m} = z_{1m},$$

$$\dot{z}_{1m} = -a_1[x_{1m} + y_{1m} + z_{1m} - \mathrm{sgn}(x_{1m})],$$

$$\dot{x}_{2m} = y_{2m},$$

$$\dot{y}_{2m} = z_{2m},$$

$$\dot{z}_{2m} = -a_2[x_{2m} + y_{2m} + z_{2m} - \mathrm{sgn}(x_{2m})]. \tag{14}$$

The slave dynamics are designed as described in the preceding section,

$$\dot{x}_{1s} = y_{1s} + k_1 e,$$

$$\dot{y}_{1s} = z_{1s} + k_2 e,$$

$$\dot{z}_{1s} = -a_1[x_{1s} + y_{1s} + z_{1s} - \mathrm{sgn}(x_{1s})] + k_3 e,$$

$$\dot{x}_{2s} = y_{2s} + k_4 e,$$

$$\dot{y}_{2s} = z_{2s} + k_5 e,$$

$$\dot{z}_{2s} = -a_2[x_{2s} + y_{2s} + z_{2s} - \mathrm{sgn}(x_{2s})] + k_6 e, \tag{15}$$

where $e = C(\mathbf{X_m} - \mathbf{X_s}) = x_{1m} + y_{1m} + z_{1m} + x_{2m} + y_{2m} + z_{2m} - (x_{1s} + y_{1s} + z_{1s} + x_{2s} + y_{2s} + z_{2s})$ and $a_1 = 0.8$, $a_2 = 0.6$.

In this case, there is only one linearized system, characterized by the state matrix

$$A_{1,1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -a_1 & -a_1 & -a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -a_2 & -a_2 & -a_2 \end{pmatrix}.$$

Solving the LMI problem for the simultaneous stability leads to the gains vector $K = [-1.5710 \quad -0.4173 \quad 1.0232 \quad 1.6569 \quad 1.6126 \quad -0.7322]^T$ which stabilizes the error system.

Figure 3 shows the temporal evolution of the six state variables of the master compared with the corresponding variables of the slave. In Fig. 3 the x_{1m} vs x_{1s} and x_{2m} vs x_{2s} plots are also shown. The logarithmic error for the state variables x_1 and x_2 is shown in Fig. 4.

The results obtained lead to the synchronization notwithstanding the condition imposed is only necessary. This is connected to the fast convergence characteristics of the observer system, which can be correlated to the unique set of eigenvalues (in this case, in fact, there is only one set of eigenvalues, i.e., those of the matrix $A_{1,1} - KC$). When the two systems are in the same region at the same time, the fast eigenvalues which drive the dynamics of the error system lead to a rapid convergence of the trajectories and imply that the two systems follow similar dynamics and enter the same PWL region at quite the same time.

This consideration is enforced by computer simulations: the eigenvalues of the error system (placed at $\lambda_{1,2} = -0.0527 \pm 0.9283i$, $\lambda_{3,4} = -0.6977 \pm 0.7317i$, $\lambda_{5,6} = -0.7357 \pm 0.1836i$) were moved towards the imaginary axis with the aim of reducing the speed of the error dynamics. In fact, it can be observed that, if eigenvalues with real part 10 times smaller are chosen, although the error system is still stable, the two systems do not synchronize anymore. In Fig. 5 the logarithmic absolute errors for the x_1 state variable in the two cases are shown.

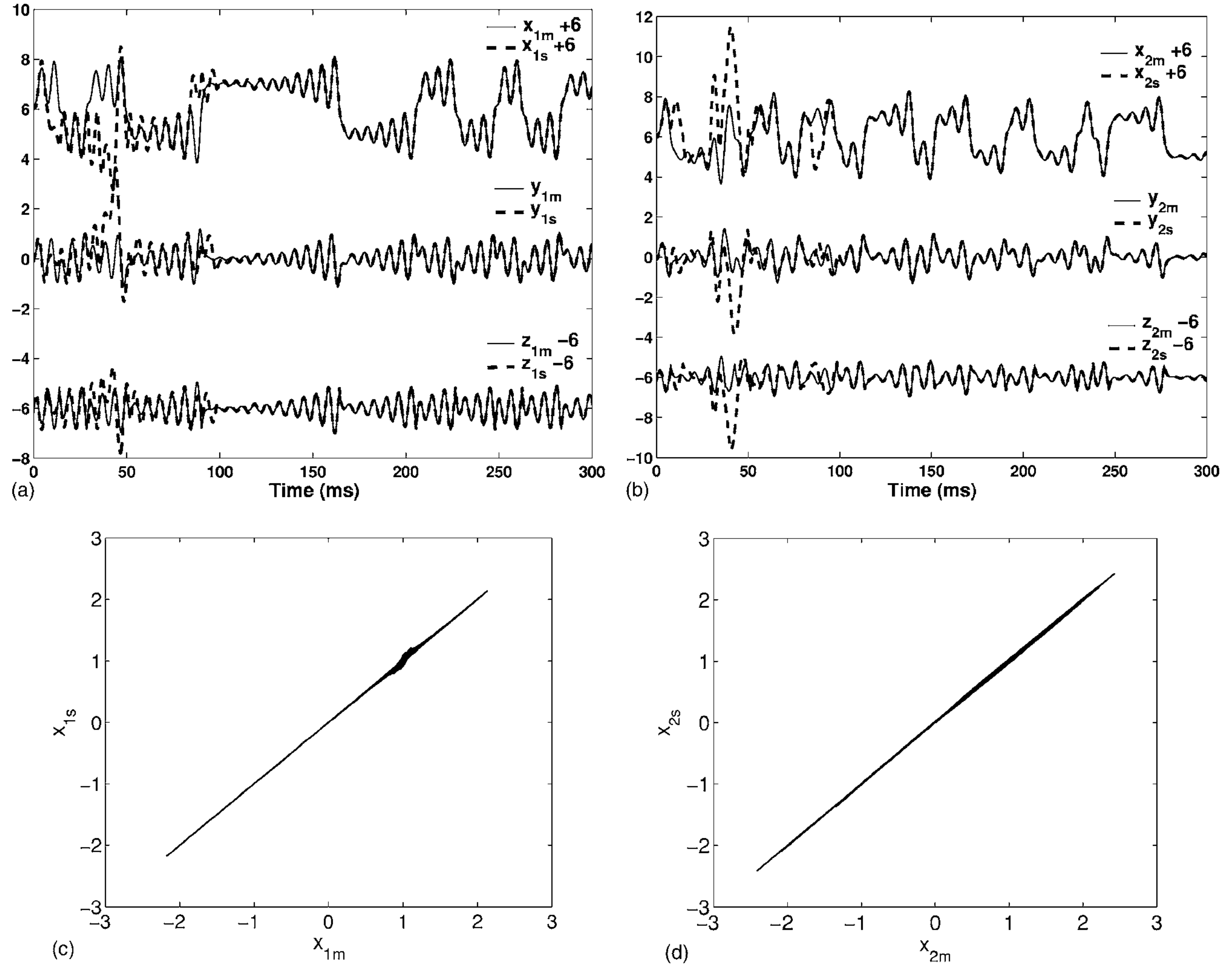


FIG. 3. Separation and synchronization of two Kennedy's oscillators. (a) Trend of $x_{1m}+6$, y_{1m} and $z_{1m}-6$ (continuous lines) compared to $x_{1s}+6$, y_{1s} and $z_{1s}-6$ (dotted lines), (b) trend of $x_{2m}+6$, y_{2m} and $z_{2m}-6$ (continuous lines) compared to $x_{2s}+6$, y_{2s} and $z_{2s}-6$ (dotted lines), (c) synchronization plot x_{1m} vs x_{1s}, (d) synchronization plot x_{2m} vs x_{2s}. The time scale of numerical results has been normalized to match the circuit time scale.

B. Separation and synchronization of a pair of a Chua's circuit and a Kennedy's oscillator

In order to test our scheme on other configurations the Chua's circuits family was investigated. Two sets of parameters applied to the generalized Chua's circuit equations [14,15], paired with the Kennedy's oscillator lead to a feasible LMI problem. In the following, one of those cases is reported.

The master consists of a Chua's circuit described by the equations

$$\dot{x} = k\alpha[y - x - h(x)],$$

$$\dot{y} = k(x - y + z),$$

$$\dot{z} = k(-\beta y - \gamma z) \tag{16}$$

with $h(x)=m_1x+0.5(m_0-m_1)(|x+1|-|x-1|)$ and of a Kennedy's oscillator. Therefore, the master equations are

$$\dot{x}_{1m} = k\alpha[y_{1m} - x_{1m} - h(x_{1m})],$$

$$\dot{y}_{1m} = k(x_{1m} - y_{1m} + z_{1m}),$$

$$\dot{z}_{1m} = k(-\beta y_{1m} - \gamma z_{1m}),$$

$$\dot{x}_{2m} = y_{2m},$$

$$\dot{y}_{2m} = z_{2m},$$

$$\dot{z}_{2m} = -a_2[x_{2m} + y_{2m} + z_{2m} - \mathrm{sgn}(x_{2m})]. \tag{17}$$

The slave is designed, like in the previous case, identical to the master except for the contribution of the error signal.

The chosen parameters are $C=[1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]$; $a=0.6$, $\alpha=-1.5591$, $\beta=0.0156$, $\gamma=0.1575$, $m_0=0.2439$, $m_1=0.0425$, $k=-1$. The two overall state matrices are

$$A_{1,1} = \begin{pmatrix} -k\alpha(1+m_1) & k\alpha & 0 & 0 & 0 & 0 \\ k & -k & k & 0 & 0 & 0 \\ 0 & -k\beta & -k\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -a & -a & -a \end{pmatrix}$$

$$A_{2,1} = \begin{pmatrix} -k\alpha(1+m_0) & k\alpha & 0 & 0 & 0 & 0 \\ k & -k & k & 0 & 0 & 0 \\ 0 & -k\beta & -k\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -a & -a & -a \end{pmatrix}$$

The LMI problem can be formulated as in Eq. (11) with $q_1=2$ and $q_2=1$. Numerical results prove that the problem is feasible and the calculated vector is $K=[-1.2581 \quad -0.0877 \quad 0.7821 \quad 1.2478 \quad 1.1465 \quad -0.5101]^T$, which stabilizes the error system.

Figure 6 shows the temporal evolution of the six state variables of the master compared with the corresponding variables of the slave. In Fig. 6 the x_{1m} vs x_{1s} and x_{2m} vs x_{2s} plots are also shown. The logarithmic error for the state variables x_1 and x_2 is shown in Fig. 7. As it can be noticed the two systems are perfectly synchronized.

IV. EXPERIMENTAL RESULTS: SEPARATION AND SYNCHRONIZATION OF TWO PAIRS OF CHAOTIC CIRCUITS

The results obtained by simulating the proposed synchronization scheme encouraged us to realize a physical implementation of the system. Field programmable analog array (FPAA) boards have been used for the realization of the four chaotic systems and the circuitry needed to calculate the error signal. FPAA, in fact, are an efficient approach to the realization of programmable analog nonlinear dynamics. In Ref. [16] the use of FPAA boards to implement chaotic circuits is described. In particular, the AN221E04 Anadigm board has been used.

This FPAA provides the user with a possibility of programming four configurable analog blocks (CABs), each of which has a limited number of resources. Since the technology is based on switched-capacitors, operational amplifiers, and several capacitors constitute the resources available in each CAB. These blocks can be programmed in a very effective way, by drawing circuital connections among the CABs and downloading the designed circuit through a serial connection between PC and FPAA boards. Moreover, at the software level, configurable analog modules (CAMs) can be used. These CAMs allow the FPAA to be programmed with a high-level design based on the use of standard circuital blocks, such as summing blocks based on operational amplifiers, multipliers, gains, comparators, integrators and so on.

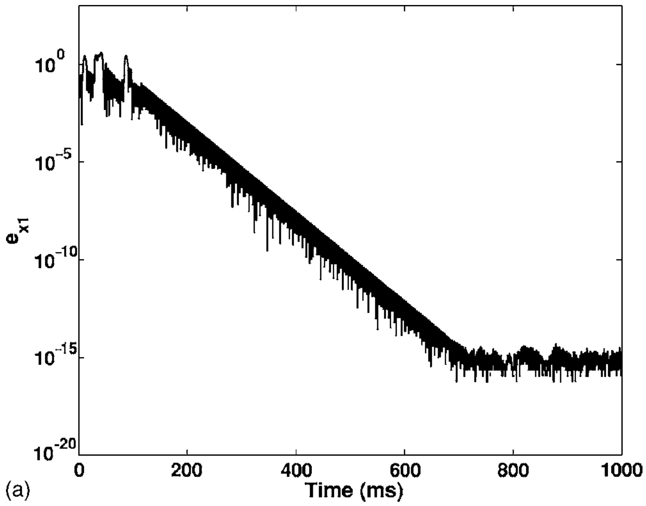
Following the guidelines described in Ref. [16], the boards can be suitably programmed to obtain the dynamics of the systems described in the preceding section, i.e., the Chua's circuit and the Kennedy's oscillator. The experimental chaotic attractors match the simulated ones.

Two FPAA based systems have been implemented and connected to experimentally investigate the problem of separation and synchronization. In particular, the first case described in the preceding section is discussed. The experimental results obtained agree with the numerical simulations carried out, showing the real possibility of separating and synchronizing two pairs of chaotic circuits through a unique feedback signal.

Figure 8 reports the oscilloscope traces showing the two synchronization plots x_{1m} vs x_{1s} and x_{2m} vs x_{2s}. In particular, in Fig. 8(a) the error signal is not fed back to the slave system and the two systems are not synchronized; switching on the error feedback, as shown in Fig. 8(b), the slave system follows the master dynamics. The synchronization is furthermore stressed in Fig. 9 where the trends of x_{2m} and x_{2s} are reported. The mismatches visible in Fig. 9 are due to the fact that a real case with circuits which necessarily have slightly different parameters is considered.

V. THE SEPARATION AND SYNCHRONIZATION SCHEME TO TRANSMIT TWO DIFFERENT INFORMATION ON TWO DIFFERENT CHAOTIC SIGNALS

In this section it is shown how to use the principle of separation and synchronization to transmit different informa-



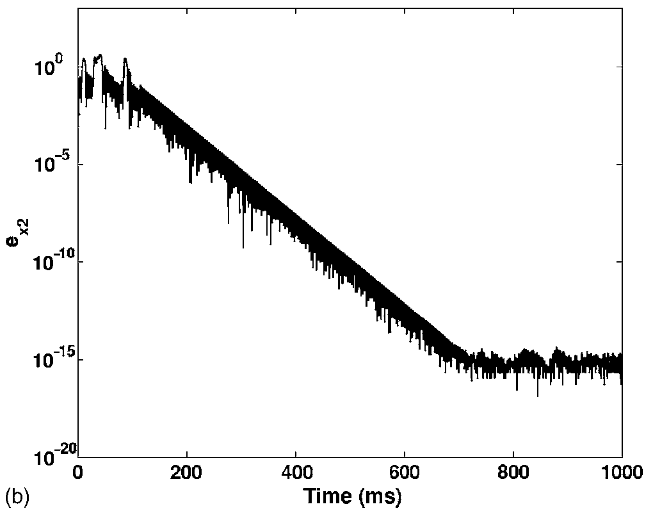


FIG. 4. Separation and synchronization of two Kennedy's oscillators. Semilogarithmic plot of the absolute error for (a) x_1 and (b) x_2.

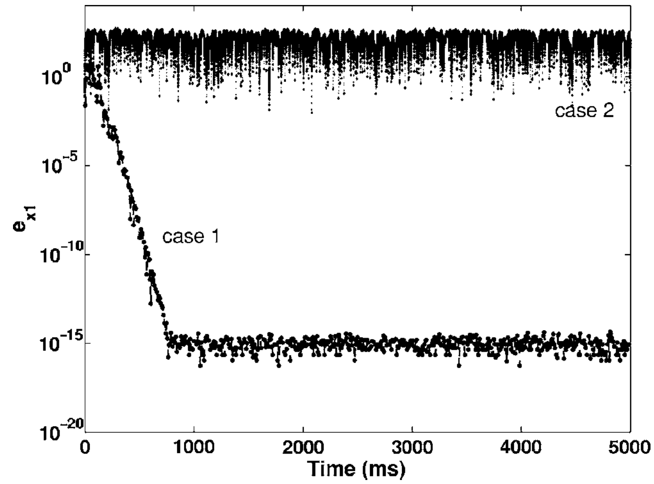


FIG. 5. Separation and synchronization of two Kennedy's oscillators: semilogarithmic plot of the absolute error for x_1 with respect to different values of the error system eigenvalues. In case 1 the eigenvalues are $\lambda_{1,2}=-0.0527\pm0.9283i$, $\lambda_{3,4}=-0.6977\pm0.7317i$, $\lambda_{5,6}=-0.7357\pm0.1836i$ as designed by solving the LMI problem. When the eigenvalues are fixed at $\lambda_{1,2}=-0.0053\pm0.9283i$, $\lambda_{3,4}=-0.0698\pm0.7317i$, $\lambda_{5,6}=-0.0736\pm0.1836i$ (case 2), the error does not converge to zero (the eigenvalues are moved towards the imaginary axis).

tion on the different multiplexed chaotic signals. Although the scheme is general, the case with a pair of chaotic systems will be referred to.

Since the possibility of separating and synchronizing two circuits constituted by a pair of chaotic oscillators was experimentally demonstrated as shown in the preceding section, the idea underlying chaotic switching [17] can be applied to implement a communication scheme based on separation and synchronization. This technique allows the transmission of a digital information. The master is switched into two chaotic attractors depending on the bit to be transmitted. The slave consists of two copies of the master circuit. Each of these circuits is able to synchronize with only one of the two master attractors. The bit transmitted is therefore identified on the basis of the synchronization error. The slave system with parameters such as those of the actual transmitted master attractor synchronizes, while the other slave system has a larger synchronization error.

The scheme proposed in this paper is shown in Fig. 10. The master consists of four chaotic systems: two Chua's circuits and two Kennedy's oscillators with different parameters. In particular, the first Chua's circuits, labelled in Fig. 10 as Chua1, is characterized by the following set of parameters: $\alpha_1=-1.5591$, $\beta_1=0.0156$, $\gamma_1=0.1575$, $m_{01}=0.2439$, $m_{11}=0.0425$, $k_1=-1$. The parameters of the second Chua's circuit (Chua2) are: $\alpha_2=-1.4246$, $\beta_2=0.0294$, $\gamma_2=0.3227$, $m_{02}=-0.0715$, $m_{12}=-0.1817$, $k_2=1$. The two Kennedy's oscillators differ from the value of the parameter a: $a=a_1=0.8$ for the first circuit and $a=a_2=0.6$ for the second circuit. These circuits are selected on the basis of two digital information to be transmitted. As shown in Fig. 10 the two digital information control two switches which select the circuits to be used to form the transmitted chaotic signal. Thus, the master has four possible configurations and the transmitted signal is given by the linear combination of the state variables of two circuits selected by the bits to be transmitted.

The slave system consists of four circuits, each one given by a pair of chaotic circuits (a Chua's circuit and a Kennedy's oscillator with the same parameters of the master systems). Each of these four systems refers to the separation and synchronization scheme shown in Fig. 1 and discussed above. In each of these systems an error signal is built and a set of gains K is designed by following the LMI approach discussed in Sec. II and in particular by applying Eqs. (11) for each of the four circuits C_1, C_2, C_3, and C_4. The error scalar signal is obtained as the linear combination of all the state variables of the two chaotic subsystems. The parameters K have been found, as already outlined, by applying the LMI approach which allows suitable parameters for each of the four circuits to be obtained. The following values have been obtained:

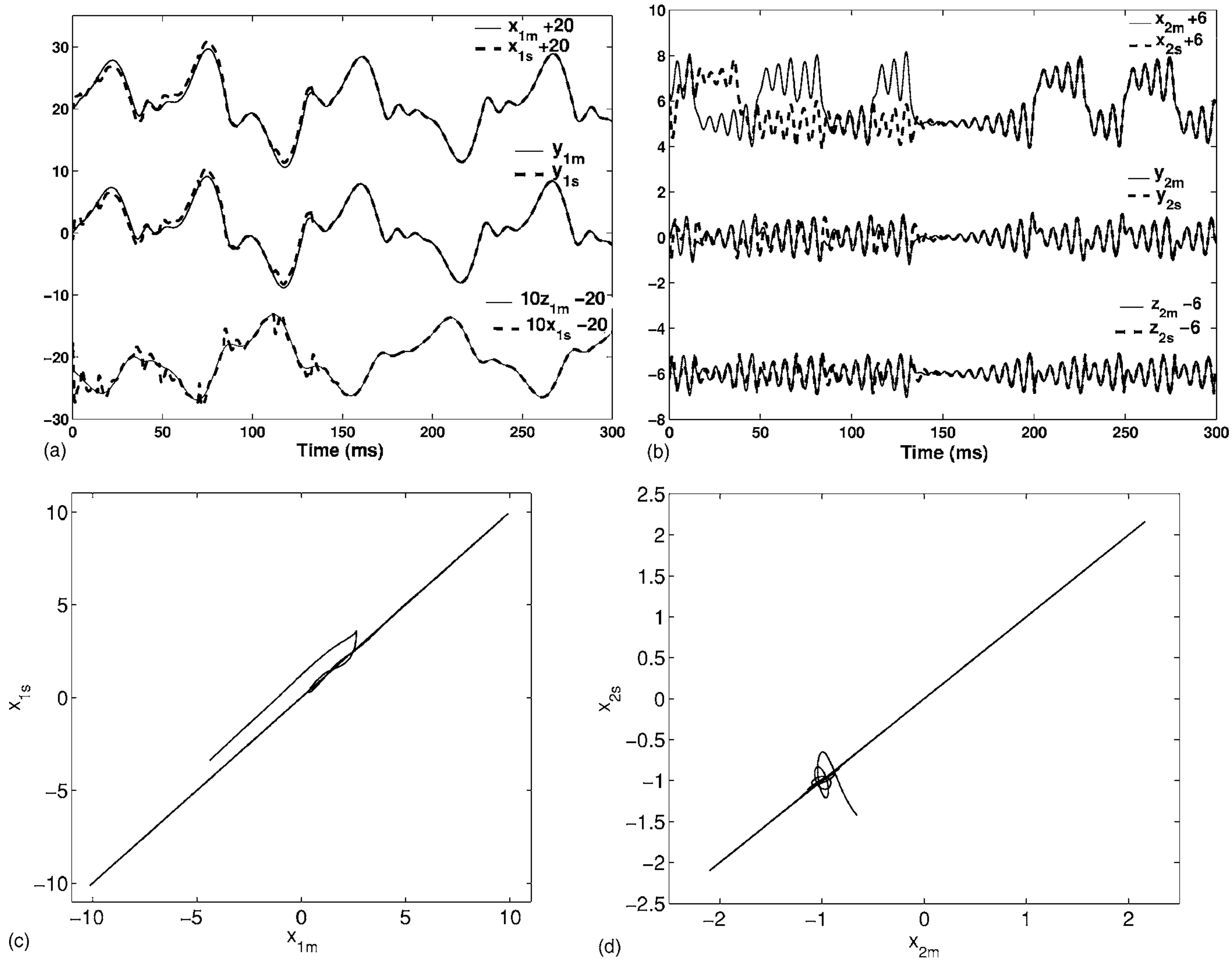


FIG. 6. Separation and synchronization of a pair composed by a Chua's circuit and a Kennedy's oscillator. (a) Trend of $x_{1m}+20$, y_{1m} and $z_{1m}-20$ (continuous lines) compared to $x_{1s}+20$, y_{1s} and $z_{1s}-20$ (dotted lines), (b) trend of $x_{2m}+6$, y_{2m} and $z_{2m}-6$ (continuous lines) compared to $x_{2s}+6$, y_{2s} and $z_{2s}-6$ (dotted lines), (c) synchronization plot x_{1m} vs x_{1s}, (d) synchronization plot x_{2m} vs x_{2s}.

$$K_1 = [5.1786 \quad 8.9838 \quad -2.9223 \quad -3.1920 \quad -4.0420 \quad 1.9599]^T$$

for the circuit indicated as C_1 (the circuit is made by a Kennedy's oscillator with $a_1=0.8$ and the Chua's circuit Chua1);

$$K_2 = [12.8282 \quad 6.1083 \quad -0.7062 \quad -8.6651 \quad -7.2247 \quad 5.3598]^T$$

for circuit C_2;

$$K_3 = [-1.2581 \quad -0.0877 \quad 0.7821 \quad 1.2478 \quad 1.1465 \quad -0.5101]^T$$

for circuit C_3;

$$K_4 = [4.2636 \quad 1.3301 \quad -0.2469 \quad -1.6775 \quad -1.3117 \quad 1.3809]^T$$

for circuit C_4.

Each of these four systems synchronizes with only one of the four possible master configurations. Let us suppose for instance that the sequences $S_{1t}=00\cdots$ and $S_{2t}=01\cdots$ should be transmitted. First the two bits $S_{1t}=0$ and $S_{2t}=0$ are transmitted. The circuit C_1 designed with parameters K_1 found by the LMI technique synchronizes. In fact, this is the circuit which corresponds to the bits 00 as shown in Fig. 10. The other circuits C_2, C_3, and C_4 do not synchronize. Then, the second pair of bits is transmitted ($S_{1t}=0$ and $S_{2t}=1$). In this case, the circuit C_1 does not synchronize, but C_2 (which cor-

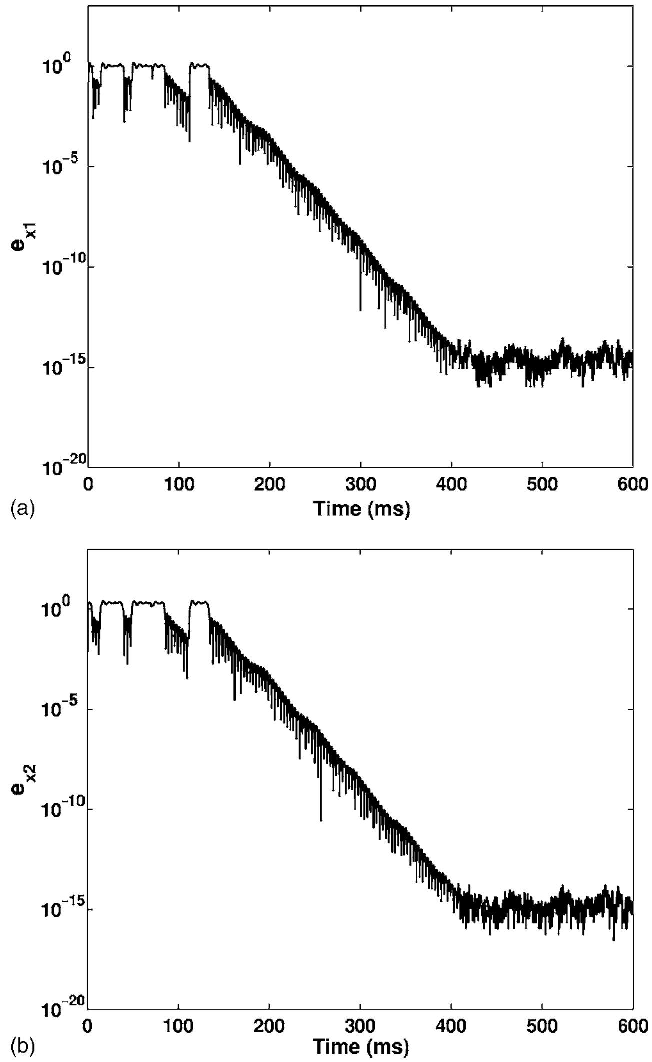


FIG. 7. Separation and synchronization of a pair composed by a Chua's circuit and a Kennedy's oscillator. Semilogarithmic plot of the absolute error for (a) x_1 and (b) x_2.

responds to bits 01) does. Circuits C_3 and C_4 do not synchronize.

The synchronization error easily allows the detection of the two transmitted bits. In particular, a very simple mechanism has been used: at each time step the circuit with minimum synchronization error (i.e., the scalar signal e in Fig. 1) is selected.

Figure 11 shows an example of the use of this communication scheme. The transmitted and detected bits are shown in Fig. 11(a). The detection system has good performance, and bit errors occur only at the switching of the attractors. The synchronization errors at each of the four slave systems are shown in Fig. 11(b), where the circuits have been numbered according to their order in Fig. 10. As it can be noticed, for instance in correspondence of the first two transmitted bits (S_{1_t}=0 and S_{2_t}=0), e_1 is the smallest value in almost the whole period $0 \leq t < 500$ ms. The same occurs for the other transmitted bits. Moreover, as it can be noticed in Fig. 11(a) bit errors, occur only when the attractors are switched.

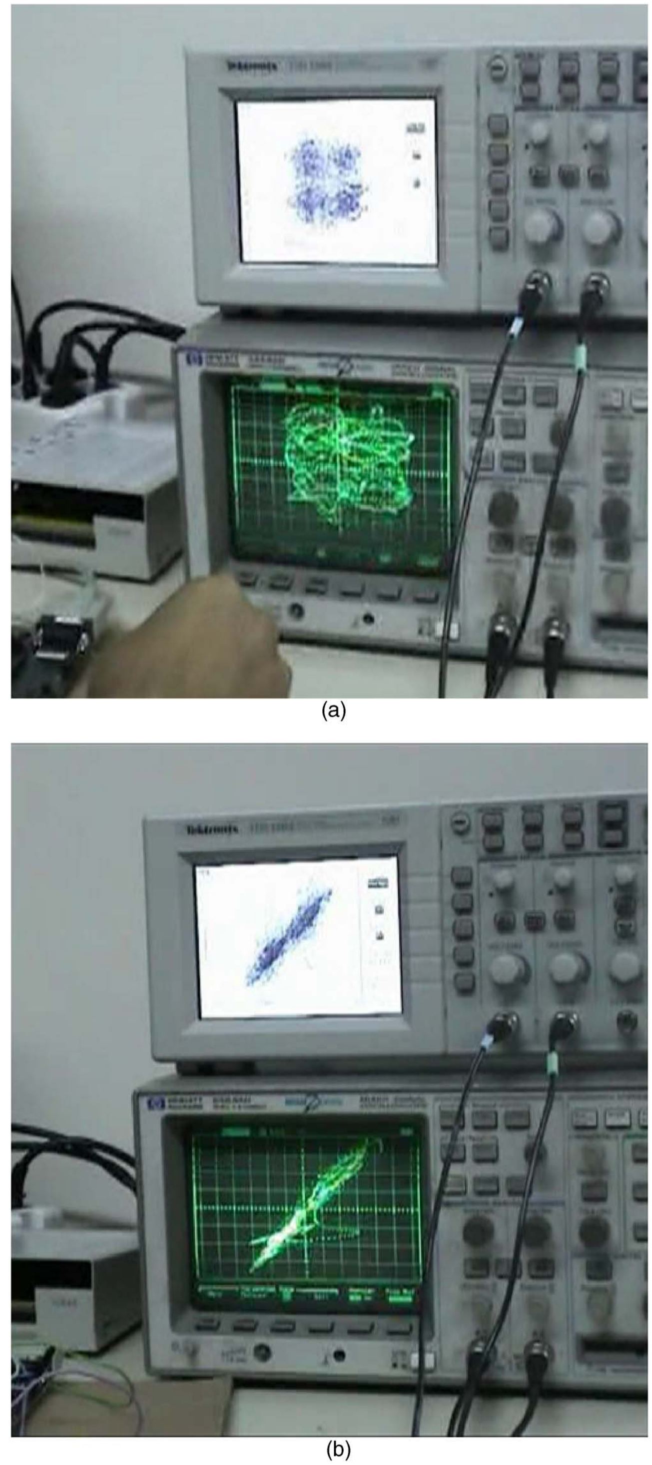


FIG. 8. (Color online) Separation and synchronization of a pair of Kennedy's oscillators. Synchronization plot, x_{1m} vs x_{1s} on the upper oscilloscope, x_{2m} vs x_{2s} on the lower oscilloscope. (a) Without feedback, there is no synchronization. (b) With feedback, separation and synchronization are achieved.

The performance of the proposed communication scheme when subject to noise have been characterized by calculating the bit error rate (BER) as a function of the signal-to-noise ratio (SNR). Gaussian noise has been added to the transmitted signal, and the variance of the noise has been changed to

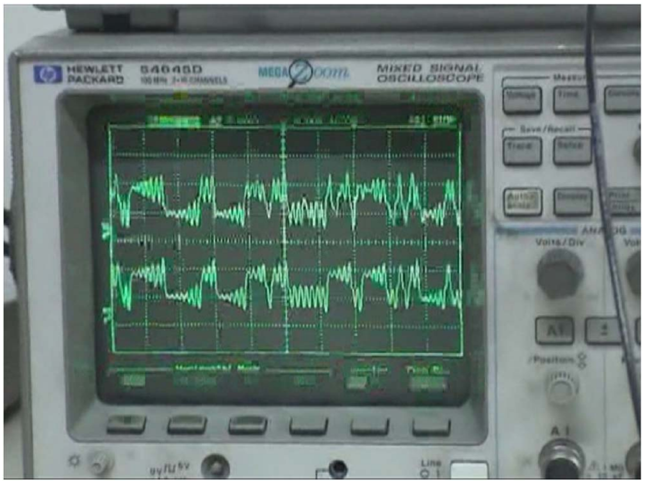


FIG. 9. (Color online) Trends of x_{2m} vs x_{2s} when the feedback is on. The two systems are synchronized except for some mismatches due to circuital parameter tolerances.

vary the SNR. BER has been evaluated by simulating the transmission of 1000 bits (i.e., 500 pairs of bits) for each value of the SNR. The received bit is calculated by taking into account the average of the signals e_1, e_2, e_3, and e_4 in the last 300 ms of the received signal (where the bit period has been fixed to $T=1$ s). An error occurs when the received bit does not match the transmitted one. It should be taken into account that, as shown in Ref. [18], under particular hypotheses, chaotic modulation schemes based on chaos shift keying can theoretically achieve the noise performance of binary phase-shift keying (BPSK). This requires the adoption of a coherent correlation receiver [19]. To take into account that a nonoptimal detection system has been adopted, SNR has been evaluated by considering the average power of the synchronization error. The probability of bit error is shown in Fig. 12, where the case of bipolar baseband signal with matched filter is also shown for comparison.

VI. CONCLUSIONS

In this paper, starting from the scheme of negative feedback [5], a synchronization scheme for chaotic systems is investigated, in which master and slave systems are constituted by n independent PWL chaotic systems and a unique scalar variable is transmitted. An approach based on the design of an asymptotic observer through the solution of an LMI problem has been introduced to find suitable values of

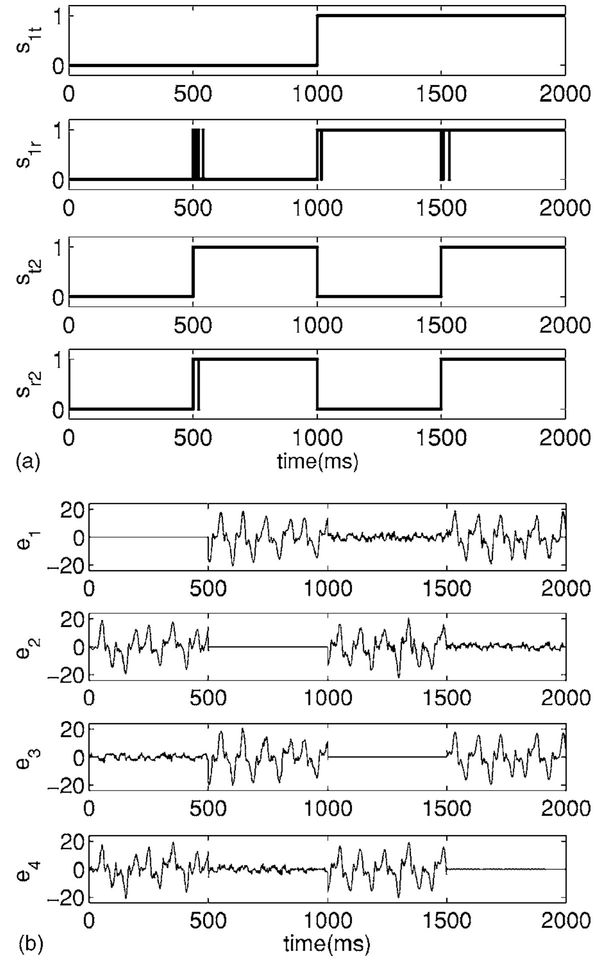


FIG. 11. (a) S_{1t} and S_{2t} are the transmitted information masked in the two different chaotic signals, while S_{1r} and S_{2r} are the detected signals using the separation and synchronization transmission scheme shown in Fig. 10. (b) e_1, e_2, e_3, and e_4 are the synchronization errors of the four slave systems C_1, C_2, C_3, and C_4.

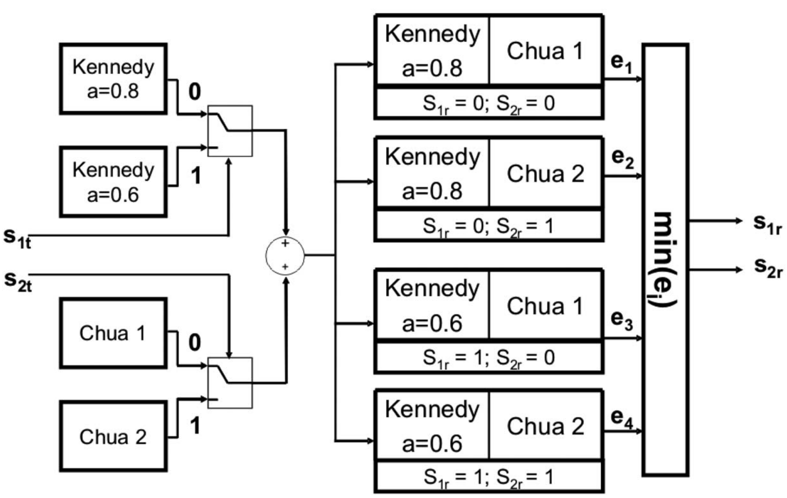


FIG. 10. Communication scheme based on the idea of separation and synchronization of chaotic signals.

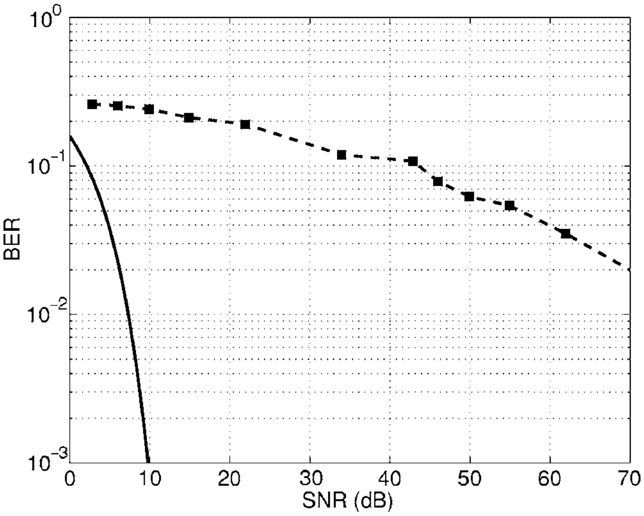


FIG. 12. Probability of bit error (BER) as a function of the signal-to-noise ratio (SNR) for the proposed communication scheme. For comparison the case of a bipolar baseband signal (with matched filter) is shown (solid line).

the feedback gains. In this way a necessary condition is imposed assuming that the two systems must converge if they are in the same PWL region. Computer simulations demonstrate that for the investigated cases our assumption ensures the synchronization.

Two different numerical examples are reported, one using a pair of Kennedy's oscillators with different parameters and the other using a pair formed by a Chua's circuit and a Kennedy's oscillator. In both cases, solving the correspondent LMI problem leads to correctly separate and synchronize the two different dynamics.

Experimental results are also shown. A physical implementation of the proposed scheme using FPAA boards confirming the possibility to correctly separate the information and to synchronize the slave dynamics to the master one has been provided.

This result opens the way to chaos-based communication systems, in which the transmission of two or more multiplexed chaotic signals on the same channel is possible, thus increasing the information transmitted in the communication channel, maintaining the security properties of chaotic communication.

The suitability of the LMI-based technique for separation and synchronization of chaotic signals is further remarked by the use of this approach for chaotic communication. The proposed technique allows the design of suitable parameters to implement a scheme in which two or more information can be masked into two or more multiplexed chaotic signals.

ACKNOWLEDGMENTS

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