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Collapse of periodic orbits in a driven inelastic particle system

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The dynamical behavior of a one-dimensional inelastic particle system with particles of unequal mass traveling between two walls is investigated. The system is driven by adding energy at one of the walls while the other wall is stationary and does not add energy. By deriving analytic solutions for the periodic orbits of this system, we show that there are a countable infinity of critical mass ratios at which the particle dynamics become highly degenerate in the following sense. As the mass ratio passes through these critical points, large numbers of stable periodic orbits can collapse onto a single trivial orbit. We show that the widely studied equal-mass systems represent one of these critical points and are therefore such a degenerate case. We also show that in the elastic limit the number of orbits that collapse onto the single trivial orbit can become arbitrarily large.

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I. INTRODUCTION

Mechanical systems in which discrete masses interact through inelastic collisions are extremely widespread in industrial applications. Examples include vibration hammers, pile drivers, compacting machinery, and many others. Such systems can often exhibit chattering behavior in which large numbers of interactions between the masses occur in a relatively short time. When such systems are driven, they may exhibit periodic behavior and the control of such behavior will frequently be crucial in operating machines. Obtaining stable and consistent operating conditions that give rise to simple repeatable periodic behavior that avoids chattering can be very important in controlling machine wear and noise production.

Following the groundbreaking works of Tonks [1] and computer simulations [2,3], the statistical dynamics of one-dimensional particle chains has been widely studied [4]. In recent years there has been significant interest in one-dimensional models of granular materials, which has led to a number of valuable insights. Mehta and Luck [5] and Luck and Mehta [6] showed that a single particle moving on a vibrating plate can give rise to dynamical behavior, such as abrupt termination of period-doubling sequences. In the absence of boundaries, Cipra *et al.* [7] have considered the stability of one-dimensional inelastic collapse. There have also been a number of studies regarding the development of equations to describe such systems at the continuum scale [8–13], and it is now recognized that one-dimensional systems behave fundamentally differently from particle systems in two and three dimensions [14]. Although the principle focus of most of these works has been macroscopic properties, it has become clear that a fundamental understanding of the microscale dynamics is crucial [15,16]. In this vein, Du *et al.* [8] showed that equal-mass particles placed between a vibrating wall and a stationary reflecting wall lead to the following phenomena. The particle nearest the oscillating wall moves rapidly, whereas the remaining particles are typically trapped close to the stationary wall. The particle-scale dynamics underlying this phenomenon in equal-mass particle systems has been studied by Yang [17] who showed that the periodic orbits may not be unique, and this makes such systems difficult to control. One-dimensional models have also proved useful when considering signal transmission in granular materials [18–20].

In reality, the interaction of particles of unequal mass is generally unavoidable. We note that in some situations mass variations may not lead to qualitatively different behavior [21], but in harmonic oscillator chains rich behavior may occur [22,23]. Therefore, a natural extension is to consider systems with particles of unequal mass. It seems natural that the type of motion observed in equal-mass systems, in which most of the particles are trapped against the stationary wall, may not occur when the masses differ. This raises the important question as to whether the dynamics in the unequal-mass case will exhibit any qualitative differences with the equal-mass case. If any qualitative changes occur, the limit as the masses become equal will be of special interest. Of particular concern in such dynamical systems is whether unequal masses can give rise to nonuniqueness and instability of periodic orbits.

In this paper, we show that the well-studied simple orbit in equal-mass systems is highly degenerate and hides many complicated dynamical features of unequal-mass systems. Even infinitesimal differences in the masses of the particles can give rise to multiple stable periodic orbits that do not exist in the equal-mass case. We show that the number of stable periodic orbits becomes infinite as the coefficient of restitution tends to unity. We carefully examine the limit as the masses become equal and show that, in this limit, a large number of periodic orbits collapse onto a simple orbit. Here, the phenomenon of *orbit collapse* represents the coalescence of several orbits onto a single orbit at a critical parameter value. This phenomenon is unrelated to *inelastic collapse* [7,24] in which particles experience an infinite number of collisions in finite time.

II. FORMULATION AND PHENOMENA

The purpose of this paper is to study orbit collapse in many-particle systems. However, before considering the general system, we will study a two-particle system that has

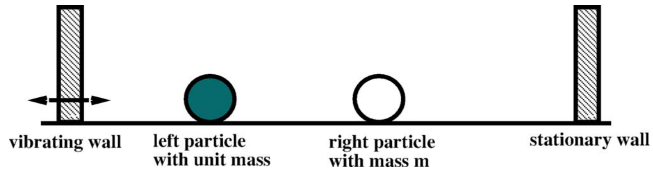

FIG. 1. (Color online) A schematic representation of a two particle unequal-mass system.

simple analytic solutions and exhibits many of the dynamical features of many-particle systems. In particular, in Sec. VI, we will show that there exists a simple relation between the orbit collapse phenomenon in many-particle systems and that in two-particle systems. We therefore focus on the motion of two particles constrained on a line between two walls.

In granular materials, it is often the case that energy is added through vibrations at one of the boundaries [25,26]. Thus, we assume that the left wall executes a periodic motion and the right wall is fixed. We will refer to the left and right walls as the "oscillating wall" and the "stationary wall," respectively (see Fig. 1). For simplicity we adopt a "saw-tooth" motion [27] for the oscillating wall, in which the wall moves with a constant speed v over a distance a before executing an instantaneous jump back to its starting position. This implies that any collision between a particle and the oscillating wall must occur when the wall is moving with speed v. We will further assume that the distance a is much smaller than the size of the domain, and thus, at leading order, all collisions with the oscillating wall must occur at the same location.

Since the physical size of the particles does not play a role in one-dimensional motion, we consider point particles. We choose scales such that the distance between the walls, the speed of the oscillating wall, and the mass of the left particle are all unity. The mass of the right particle is m.

Collisions between particles are dissipative with a constant coefficient of restitution e. We will denote the coefficient of restitution between the right particle and the stationary wall as e_w. It is straightforward to perform our analysis for an arbitrary coefficient of restitution for collisions between the left particle and the oscillating wall, but for simplicity we will take this coefficient to be unity.

When the right particle hits the stationary wall, it simply bounces inelastically and the velocity is updated using the operator R defined by

$$R\begin{pmatrix} v_L \\ v_R \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -e_w \end{pmatrix}\begin{pmatrix} v_L \\ v_R \end{pmatrix} = \begin{pmatrix} v_L \\ -e_w v_R \end{pmatrix}, \tag{1}$$

where v_L and v_R are the velocities of the left and right particles, respectively. When the left particle hits the oscillating wall it gains kinetic energy. In the frame of reference of the stationary wall, the left particle bounces back elastically from the oscillating wall and the velocity is updated using the operator L defined by

$$L\begin{pmatrix} v_L \\ v_R \end{pmatrix} = \begin{pmatrix} 2 - v_L \\ v_R \end{pmatrix}. \tag{2}$$

When two particles collide, momentum is conserved and their new relative velocity is just $-e$ times their old relative velocity. The velocities are hence updated using the operator C defined by

$$C\begin{pmatrix} v_L \\ v_R \end{pmatrix} = \frac{1}{1+m}\begin{pmatrix} 1-em & m(1+e) \\ 1+e & m-e \end{pmatrix}\begin{pmatrix} v_L \\ v_R \end{pmatrix}. \tag{3}$$

Numerical simulations for this system were performed by calculating the times for each of the three possible collisions (particle-particle, particle-stationary wall and particle-oscillating wall) and finding the minimum among these three collision times. The particle positions and velocities are then updated using the appropriate choice among the three collision rules given by the equations (1)–(3).

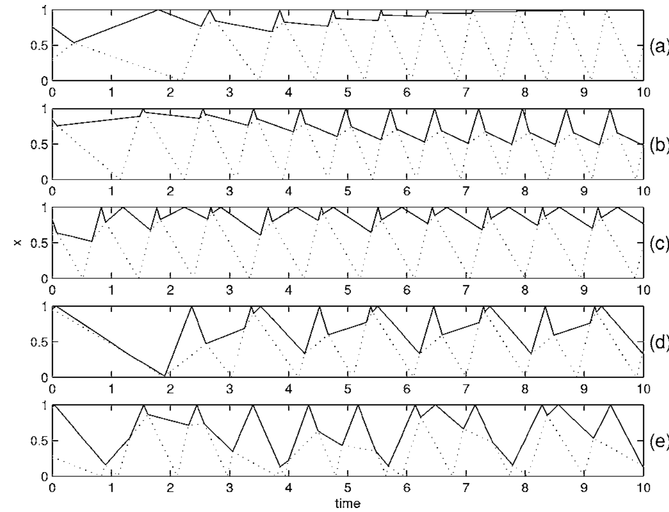

FIG. 2. The trajectories of two inelastic particles with coefficient of inelasticity $e=0.5$. The mass ratios m in (a–e) are 1, 1.1, 0.7, 0.7, and 1.6, respectively. The horizontal axis represents dimensionless time, and the vertical axis represents the dimensionless location in the domain. The oscillating wall is located at zero, and the stationary wall at 1. The trajectory of the right particle is plotted with a solid line, and the trajectory of the left particle is plotted with a dotted line. (a) has sequence *CRCL*, (b) also has sequence *CRCL*, (c) has sequence *RCRCL*, (d) has sequence (*RCRCL*)(*CRCL*), and (e) does not evolve to any low-order periodic orbit during the time of the simulation.

A. Mechanical description of periodic orbits

We begin by presenting some examples of simple orbits that give an indication of the types of phenomena that can occur. We also give a mechanical explanation of some of the orbit transitions that occur if the mass ratio of the two particles is varied. These orbits will serve as the building blocks for analyzing more complicated orbits.

In Fig. 2, we show examples of trajectories that occur for $e=0.5$ and various mass ratios m. For two particles of equal mass, $m=1$, after a few initial collisions the dynamics rapidly adopt the following simple sequence, which is made up of four collisions [Fig. 2(a)]. The right particle hits the stationary wall, bounces back inelastically from it, and collides with the left particle. Then, the left particle moves toward the oscillating wall, hits it, bounces back with increased kinetic energy, and then hits the right particle. This sequence is one

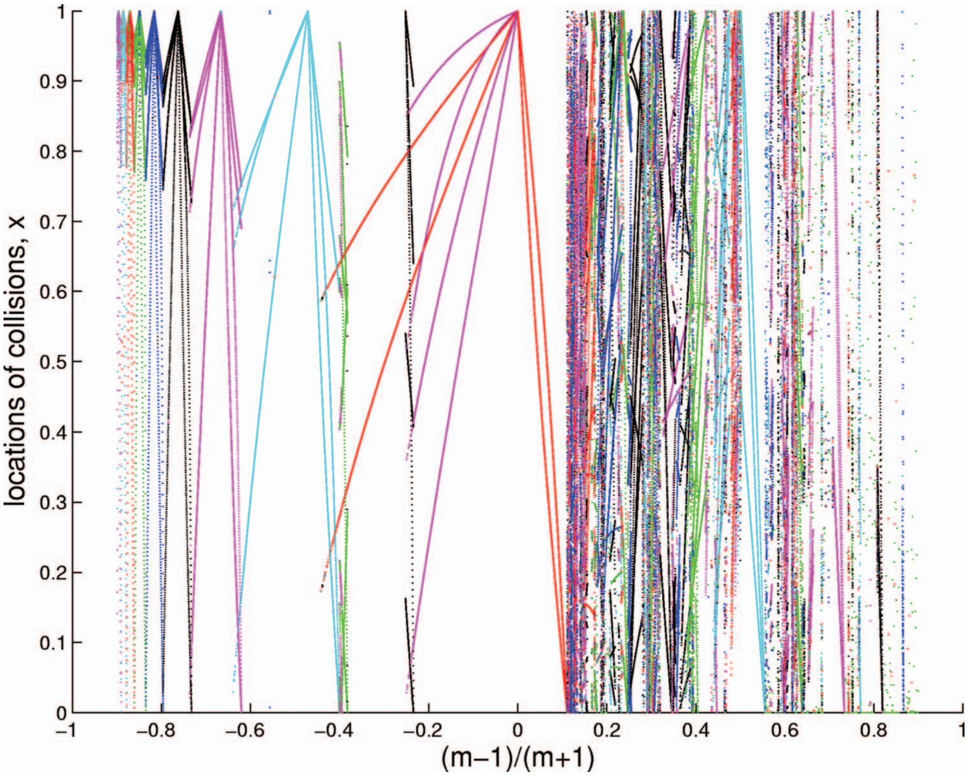

FIG. 3. (Color) The dimensionless locations of interparticle collisions are plotted against the modified mass ratio $(m-1)/(m+1)$ for $e=0.5$. Since orbits typically have a number of interparticle collisions, orbits with different numbers of interparticle collisions are distinguished by using different colors. The phenomenon of orbit collapse refers to the situation in which different orbits merge into a single orbit at certain critical mass ratios. An example of this phenomenon occurs at $(m-1)/(m+1)=0$, where the orbits containing two interparticle collisions (red curves) and four interparticle collisions (magenta curves) that exist for $(m-1)/(m+1)$ slightly below zero collapse onto an orbit contain two interparticle collisions (red curves) that exists for $(m-1)/(m+1)$ slightly above zero.

period of the motion, and the period repeats. Each sequence can be represented symbolically as *CRCL*, where the sequence of collisions is read from right to left. After each period of the motion, the right particle moves closer to the stationary wall than before. For all initial conditions, the right particle will eventually be trapped against the stationary wall. The right particle will then have zero velocity except at the instant when it is involved in collisions.

This periodic collision sequence *CRCL* also occurs in the system in which the left particle is slightly heavier than the right particle, that is $m>1$ [see Fig. 2(b)]. The only difference is that the right particle is no longer trapped against the stationary wall. As we will show later, in unequal mass systems, the right particle has sufficient momentum to move away from the stationary wall after colliding with it. Therefore, the inter-particle collisions occur at a finite distance away from the walls. This is in contrast to the equal-mass case. As m decreases towards unity, the locations of both of the inter-particle collisions will become increasingly close to the stationary wall. When $m=1$, all inter-particle collisions will occur at the stationary wall.

However, when m is slightly smaller than unity, the sequence *CRCL* can not be a periodic orbit. The reason that the *CRCL* sequence exists when $m \geqslant 1$, but does not exist for $m<1$ can be understood by considering the velocity of the right particle after the last interparticle collision, C, in the *CRCL* sequence. In the $m=1$ case, the resulting velocity of the right particle is zero and so the next collision cannot be between the right particle and the stationary wall. However, the situation is different for $m<1$. After the final interparticle collision in the sequence, the right particle will have a nonzero velocity traveling toward the stationary wall. Each collision sequence moves the locations of both of the interparticle collisions closer to the stationary wall. Eventually, following a *CRCL* collision block, the right particle will be sufficiently close to the stationary wall and have sufficient speed to collide with the stationary wall before any further interparticle collision can occur. Hence, an extra collision operator R needs to be added to the sequence and the *CRCL* orbit can no longer exist.

When m is slightly smaller than unity, different initial conditions evolve to one of many possible orbits. We illustrate this phenomenon in Figs. 2(c) and 2(d) for the case of $e=0.5$. For the simpler orbit [Fig. 2(c)], the dynamics is similar to the case of $m \geqslant 1$ [Fig. 2(a) and 2(b)], but an extra collision with the stationary wall is needed. This yields an *RCRCL* orbit.

The second periodic orbit [Fig. 2(d)] contains four interparticle collisions and is given by $(RCRCL)(CRCL)$. Here we have grouped the collisions using brackets to emphasize that this sequence is composed of the two subsequences corresponding to Figs. 2(b) and 2(c). The determination of orbits that are composed of more collisions will be presented in Sec. III.

For larger values of m, the observed motion is not as simple and there is no straightforward method to determine if the motion is aperiodic or periodic with an extremely long period. An example of this is shown in Fig. 2(e). In the rest of this paper, we will only consider motion that is periodic.

B. Description of the bifurcation structure of orbits

To further investigate the nature of the periodic orbits, we fix $e=0.5$ and vary m. For a given value of m, we choose a variety of random initial conditions and perform simulations until a periodic orbit is achieved. In Fig. 3, we plot the locations x at which interparticle collisions occur in periodic orbits against the mass ratio $(m-1)/(m+1)$. The oscillating wall is located at $x=0$, and the stationary wall is located at $x=1$. Since a single orbit will typically involve a number of interparticle collisions, each orbit is represented by a number of points. For example, in Fig. 3, when $(m-1)/(m+1)=-0.6$, there exists an orbit with three interparticle collisions that occur at locations $x=0.31$, 0.74, and 0.77. We plot orbits that contain different numbers of interparticle collisions using different colors. This allows us to easily distinguish orbits with different numbers of interparticle collisions when multiple orbits occur at a given mass ratio. For example, in the range $-0.2<(m-1)/(m+1)<0$ two distinct periodic orbits coexist, an orbit with two interparticle collisions (shown in red) and an orbit with four interparticle collisions (shown in magenta). We note that this representation does not contain all the information required to reconstruct the orbit because the collision sequence is not always obvious; but as we will see below, this method of presenting the data allows us to understand a number of important features of the dynamics.

From Fig. 3, it is clear that this apparently simple mechanical system can have extremely complicated dynamics. For $(m-1)/(m+1)<1/9=0.1111$, the dynamics is relatively simple and most of the observed orbits contain relatively few collisions. However, for $(m-1)/(m+1)>1/9$, there are typically a large number of long orbits that coexist. In Sec. III, we will show that this transition occurs when $(m-1)/(m+1)=(1-e)^2/(1+e)^2=1/9$.

From Fig. 3, we also see that as we vary the mass ratio m the collision locations can change discontinuously in four distinct ways. The first type of transition occurs when all interparticle collisions are located at the stationary wall. An example of this type of transition can be seen at $(m-1)/(m+1)=0$. For $0<(m-1)/(m+1)<0.1$, we observe a single orbit with two interparticle collisions (the two red lines). This is the orbit with a collision sequence *CRCL* as shown in Fig. 2(b). As m decreases toward unity, both interparticle collisions become increasingly close to the stationary wall, as one would expect from the discussion in Sec. II A

For values of $(m-1)/(m+1)$ slightly below zero, which corresponds to m slightly below unity, we observe two possible orbits. One of them has two interparticle collisions represented by the two red lines that exist approximately in the range $-0.45<(m-1)/(m+1)<0$. This corresponds to the *RCRCL* orbit [Fig. 2(c)]. The other has four interparticle collisions represented by the four magenta lines that exist approximately in the range $-0.25<(m-1)/(m+1)<0$. This corresponds to the (*RCRCL*)(*CRCL*) orbit [Fig. 2(d)]. For both of these orbits, the positions of all the interparticle collisions become increasingly close to the stationary wall as m increases toward unity and both orbits no longer exist for $m>1$. This represents a collapse of two periodic orbits (*RCRCLCRCL* and *RCRCL*) onto a single orbit (*CRCL*). Later we will show that, at certain parameter values, an arbitrarily large number of periodic orbits can collapse onto a single trivial orbit.

This type of transition occurs at an infinite set of other critical mass ratios in the range $(m-1)/(m+1)<0$, such as $(m-1)/(m+1)=-0.45,-0.65,-0.77,-0.81,\ldots$. It is obvious that, in any periodic orbit, the left particle must experience at least one collision with the oscillating wall, which will propel it to the right. Eventually, it must change direction and return to the oscillating wall. In order to change the direction of the left particle, the right particle must repeatedly collide between the stationary wall and left particle. The lighter the right particle is, the smaller the change of the momentum in each interparticle collision will be. Consequently, more collisions between the right particle and the right wall will be needed to turn around the left particle. For example, in the vicinity of $(m-1)/(m+1)=-0.5$, there are three cyan curves in Fig. 3 that represent orbits with three interparticle collisions. As m decreases through the critical value of $(m-1)/(m+1)=-0.5$, a *CRCRCL* orbit changes into a *RCRCRCL* orbit. Similar behavior can also be observed for the magenta orbit near $(m-1)/(m+1)=-0.7$, where a *CRCRCRCL* orbit changes to an *RCRCRCRCL* orbit. In fact, there are an infinite number of such critical values where a $(CR)^N CL$ orbit changes to an $R(CR)^N CL$ orbit for any integer N. For the value of e chosen for the figure, there is only a single orbit on either side of the critical values $(m-1)/(m+1)=-0.5,-0.7,\ldots$. Thus, unlike the transition at $(m-1)/(m+1)=0$, there is no collapse of multiple orbits onto a single orbit. However, for values of e closer to unity, we can observe multiple orbits collapsing onto a single orbits at critical values of $(m-1)/(m+1)$ in a similar way to the transition that occurs at $(m-1)/(m+1)=0$.

The second type of transition is that in which interparticle collisions occur at the oscillating wall. An example of this type of transition can be seen when $(m-1)/(m+1)$ is ~ 0.1, where the two red lines represent a *CRCL* orbit. As $(m-1)/(m+1)$ increases, the velocity of the left particle following the first interparticle collision in the *CRCL* sequence becomes smaller. This means that interparticle collisions occur increasingly close to the oscillating wall. At a critical value of m, the velocity of the left particle following the first interparticle collision becomes zero and the left particle becomes trapped against the oscillating wall. For values of m above the critical value, this orbit can no longer exist.

A second example of this type of transition occurs near $(m-1)/(m+1)=0.55$, where the three cyan lines represent a *CLCRCL* orbit. A third example is the orbit represented by the four magenta lines in the vicinity of $(m-1)/(m+1)=0.7$. This corresponds to the *CLCLCRCL* orbit. In fact, careful examination shows that these orbits are part of an infinity family of orbits of the type $CR(CL)^N$. These orbits have multiple collisions with the oscillating wall and only one collision with the stationary wall. All of these orbits share the property that as m increases toward a critical value, all interparticle collisions occur at the oscillating wall. However, unlike the first type of transition, this transition does not simply add an extra collision with the oscillating wall to obtain a low-order orbit, but instead a transition to very high-order periodic orbits occurs.

The third type of transition occurs when some, but not all, of the interparticle collisions occur at the stationary wall. An example of such an orbit can be seen from the black curves in the vicinity of $(m-1)/(m+1)=-0.25$. This orbit contains five interparticle collisions and has the form *CRCLCRCRCL*. This orbit contains two collisions between the left particle and the oscillating wall and three collisions between the right particle and the stationary wall. This transition differs from the first type of transition in that at the critical value of m the velocities of the particles near the stationary wall are always nonzero. As m decreases through the critical value, rather than adding a collision between the right particle and the stationary wall, these orbits cease to exist for values of m below the critical value.

The fourth type of transition occurs when orbits disappear even when all interparticle collisions occur at finite distances away from both walls. This can be clearly seen in Fig. 3, where below a critical value of $(m-1)/(m+1) \simeq -0.45$, the *RCRCL* orbit no longer exists. At this transition point all interparticle collisions occur at finite distances away from the walls. Hence, this transition is fundamentally different from the other three types of transitions. In Secs. III we will show that at this transition point the orbit becomes unstable to small perturbations.

III. ANALYTIC CONSTRUCTION OF PERIODIC ORBITS

If the particles execute a given periodic sequence of collisions, one simply needs to solve a set of linear equations to determine the velocities and locations of the collisions. The nonlinearity in the system arises exactly because the sequence of collisions is unknown. This is because finding the appropriate collision requires choosing the minimum time among the three possible collisions (R, L, and C), and this is inherently nonlinear. However, not all collision sequences will be feasible, since collision sequences must only contain collisions between objects that are moving relatively toward each other. Even if the sequence gives consistent velocity directions, one still needs to check whether the order of collisions is consistent. In the case of two particles, this essentially reduces to checking whether collisions between the two particles occur between the two walls.

For analytical simplicity, we consider the case $e_w=1$ in which the stationary wall is elastic. The extension to general values of e_w is straightforward, but the formulas become considerably more complicated while the phenomenon remains the same. When $e_w=1$, the operator L can be written as

$$L\begin{pmatrix} v_L \\ v_R \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} - R\begin{pmatrix} v_L \\ v_R \end{pmatrix}. \quad (4)$$

In a system with two particles, the structure of allowable periodic orbits is relatively simple. Any sequence must include at least one collision between particles and at least one collision with each wall. Between any two interparticle collisions, there are three possibilities: a collisions with the oscillating wall (L), a collision with the stationary wall (R), or collisions with both walls (LR or RL). In determining the velocities, the blocks RL and LR are equivalent and, henceforth, we denote both by RL. This gives rise to only three basic building blocks from which any sequence must be constructed, which we denote as $\{CL, CR, CRL\}$. In addition, the dynamics for any collision sequence constructed from a number of repetitions of a subsequence must exhibit the repeated dynamics of the subsequence. For example, the dynamics of an *RCLRCL* orbit must be made up of two identical sets of the solution of an *RCL* orbit. This property is just a simple consequence of the linearity of the operators R, L, and C.

Given a periodic sequence S, the periodicity of the velocities requires

$$S\begin{pmatrix} v_L \\ v_R \end{pmatrix} = \begin{pmatrix} v_L \\ v_R \end{pmatrix}. \quad (5)$$

Using (1), (3), and (4), this linear system can be rewritten as

$$(I - S')\begin{pmatrix} v_L \\ v_R \end{pmatrix} = S''\begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

where I is the identity matrix and S' and S'' are matrices constructed from products of R and C. The matrix C has two eigenvalues, $-e$ and 1, and one can show that all eigenvalues of S' must have magnitude less than unity if $e<1$. Therefore, $I-S'$ must have eigenvalues with a magnitude larger than zero, and thus, $\det(I-S') \neq 0$. Hence, the linear system for the velocities must always have a unique solution. However, not all sequences are consistent because the order of collisions assumed in a sequence may be incompatible with the collision rules. Therefore, we need to check that the particle velocities are consistent with the collision sequence and that the location of all collisions are within the domain.

Given a collision sequence that contains n interparticle collisions, we solve the above linear system and subsequently determine the particle velocities after the ith collision, which we denote by $v_L^{(i)}$ and $v_R^{(i)}$. For a given collision block $\{CL, CR, CRL\}$, we can then compute the transit times from one collision location to the next for each of the two particles. We denote the location of the ith interparticle collision by $p^{(i)}$, the time between the ith and the $(i+1)$th interparticle collisions by $t^{(i)}$, and the velocities of the left and right particles after the ith collision by $v_L^{(i)}$ and $v_R^{(i)}$. For the RC collision block, the transit time for the left particle to move directly from $p^{(i)}$ to $p^{(i+1)}$ is simply $t^{(i)}=[p^{(i+1)} - p^{(i)}]/v_L^{(i)}$. Since the right particle will collide with the stationary wall, its transit time is $t^{(i)}=[(1-p^{(i+1)})+(1-p^{(i)})]/v_R^{(i)}$. By eliminating $t^{(i)}$ from the above two expressions, we obtain an expression for the location of the $(i+1)$th interparticle collision in terms of the location of the ith interparticle collision

$$p^{(i+1)} = \left(\frac{v_R^{(i)} - v_L^{(i)}}{v_L^{(i)} + v_R^{(i)}}\right)p^{(i)} + \frac{2v_L^{(i)}}{v_L^{(i)} + v_R^{(i)}}.$$

Similar expressions can be obtained for the collision blocks CL and RCL. In general, for a given collision sequence, the location of each interparticle collision can be written as a simple linear function of the location of the previous interparticle collision

$$p^{(i+1)} = A_{i+1}p^{(i)} + b_i \quad \text{for} \quad i = 1, \ldots, n, \quad (6)$$

where $A_{n+1}=A_1$ and

$$A_i = \frac{v_R^{(i)} - v_L^{(i)}}{v_L^{(i)} + v_R^{(i)}}, \quad b_i = \frac{2v_L^{(i)}}{v_L^{(i)} + v_R^{(i)}}, \quad \text{for CR collisions}$$

$$A_i = \frac{(v_L^{(i)} - v_R^{(i)})(v_L^{(i)} - 2)}{v_R^{(i)}(v_L^{(i)} + v_R^{(i)} + 2)}, \quad b_i = 0, \quad \text{for CL collisions}$$

$$A_i = \frac{(v_R^{(i)} - v_L^{(i)})(v_L^{(i)} - 2)}{v_L^{(i)}(v_R^{(i)} - v_L^{(i)} + 2)},$$

$$b_i = \frac{2(2 - v_L^{(i)})}{(v_R^{(i)} - v_L^{(i)} + 2)}, \quad \text{for RCL collisions.} \tag{7}$$

After substituting the velocities obtained by solving (5) into (7), A_i and b_i become functions of m and e only.

If the sequence represents a periodic orbit, then $p^{(n+1)} = p^{(1)}$. Thus, after composing all contributions from all collisions, we obtain $p^{(1)} = Ap^{(1)} + b$, where $A = \Pi_{i=1}^{n} A_i$ and $b = \Sigma_{i=1}^{n} (\Pi_{j=0}^{i-1} A_j) b_i$ are constants that depend only on m and e. Here we define $A_0 = 1$. Finally, even if the locations are consistent, we still need to check that the periodic orbit is stable which requires $|A| < 1$.

The nature of the instability can be understood by considering the simple case of an *CRL* orbit. For this orbit, (6) reduces to

$$p^{(2)} = Ap^{(1)} + b \text{ and } A = \frac{\dfrac{1}{(Cv)_R} + \dfrac{1}{-(Cv)_L}}{\dfrac{1}{v_L} + \dfrac{1}{-v_R}}, \tag{8}$$

where b is a constant that is independent of $p^{(1)}$ and $p^{(2)}$. Stability requires that $|A| < 1$. After colliding with the left particle and the stationary wall, the right particle will have velocity $(RCv)_R = -(Cv)_R$. Since the orbit is periodic, we have $v_R = -(Cv)_R$. On the other hand, the oscillating wall adds energy to the left particle. Therefore, after the collision with the oscillating wall, $|v_L|$ must be greater than its magnitude before the wall collision, $|(Cv)_L|$, i.e., $|v_L| > |(Cv)_L|$. Since $v_L > 0$ and $(Cv)_L < 0$, we conclude that $-(Cv)_L < v_L$. Hence, (8) implies that $A < -1$ and the return map (8) will give rise to growth of any initial perturbation away from the fixed location.

We now briefly explain the physical mechanism that underlies this instability. Essentially, the instability occurs because the time interval between two interparticle collisions is dominated by the time needed for the left particle to move to the oscillating wall. Thus, a small movement in the initial location of the collision towards (away from) the oscillating wall implies that the left particle will collide with the oscillating wall slightly earlier (later). Consequently, the left particle will achieve the higher velocity earlier (later). Since the right particle travels with a fixed speed, the next interparticle collision will occur on the other side of the equilibrium point at a distance farther from the equilibrium point. The process repeats itself, and the distance from the fixed point will grow. Therefore, instability can develop in the locations of the collisions.

For longer orbits, the mechanism is similar. The stability depends on the magnitude of the product $A = A_1 A_2 A_3 \ldots A_n$. Components of the orbits where the instability grows ($|A_i| > 1$) can be offset by components of the orbit where the instability decays ($|A_i| < 1$) to yield an overall stable orbit.

In the following, we give some simple examples of this method of constructing periodic orbits. The simplest conceivable orbit contains three collisions, namely, *CRL*. The above argument demonstrated that this orbit is unstable for all values of m and e. It is also trivial to show analytically that *CRL* is unstable following the procedure outlined above.

We now focus on the next few simple orbits. There is only one orbit that contains four collisions, namely, *CRCL*. Using the above procedure, it is straightforward to show that the velocities and locations of the collisions are consistent and the solution is stable if $1 \leqslant m < e(1+e^2)/(2e)$. We now give a simple physical explanation of this restriction to illustrate the different types of transitions that can occur. If the lower bound is violated, i.e., if $m < 1$, then the right particle will collide with the stationary wall before experiencing an interparticle collision. If the upper bound is violated, i.e., if $m > (1+e^2)/(2e)$, the left particle will collide with the oscillating wall before experiencing the interparticle collision. It is also straightforward to check that the solution will be stable only when $m < (1+e^2)/(2e)$.

Following the same procedure, we can study periodic orbits that contain five collisions. There are two possible five-collision orbits: one is *RCLCL*, which can never be both consistent and stable; and the other is *RCRCL*, which can be realized if the following three conditions are satisfied:

$$m < 1, \quad m \geqslant \frac{1-e}{1+e} \equiv M_1 \quad \text{and} \quad m > \frac{1+e^2}{1+4e+e^2} \equiv M_2.$$

The first condition ensures the velocity of the right particle is moving toward the stationary wall after the second interparticle collision. The second condition ensures that the left particle collides with the oscillating wall before experiencing an interparticle collision. The third condition ensures that the orbit is stable. For $e \geqslant \sqrt{2} - 1$ (as is the case in Fig. 3, where $e = 0.5$), the stability condition is the relevant lower bound for m. The *RCRCL* orbit becomes unstable and disappears at a critical value of $m = M_2$. We note that the orbit disappears with all interparticle collisions occurring at finite distances away from the walls and is an example of the fourth type of transition. However, for $e < \sqrt{2} - 1$, the consistency condition is the relevant lower bound and, thus, orbits of this type will end with an interparticle collision occurring at the oscillating wall at $m = M_1$.

Higher-order periodic orbits can be analyzed in the same way. We will pursue such orbits in detail in Sec. IV for the collapse point near $m = 1$ and briefly study some of the other collapse points in Sec. V.

IV. MULTIPLE ORBITS NEAR THE COLLAPSE POINT AT $m = 1$

The numerical simulations shown in Figs. 2(c) and 2(d) have illustrated that multiple periodic orbits exist for values of $m<1$. The two orbits [shown in Figs. 2(c) and 2(d)] are given by $RCRCL$ and $(RCRCL)(CRCL)$, respectively. As $m\to 1$, the locations of all interparticle collisions in both orbits tend to the stationary wall. For values of m slightly greater than one, the two orbits change and undergo the following transformations $RCRCL\to CRCL$ and $(RCRCL)\times(CRCL)\to(CRCL)(CRCL)$. Since the problem is linear, repeated orbits are identical to their building blocks, and thus, the two orbits $CRCL$ and $(CRCL)(CRCL)$ are the same.

We are naturally led to consider an orbit (for $m>1$) made up of N blocks of $(CRCL)$, which we denote as $(CRCL)^N$. As m decreases below unity, we know that this orbit is no longer consistent. However, by replacing one or more of the N blocks with $(RCRCL)$, we may be able to obtain a consistent and stable orbit. We begin by considering orbits in which we replace only one of the $(CRCL)$ blocks with $RCRCL$ to obtain a collision sequence of the form $(RCRCL)(CRCL)^{N-1}$. Here we have used the fact that since the orbit is periodic, it is irrelevant which $CRCL$ block is replaced by $CRCRL$. Using the method developed in Sec. III, we can construct such orbits and check their consistency and stability.

We denote the vector $\mathbf{v}=[v_L,v_R]^T$ and $\mathbf{2}=[2,0]^T$, where T denotes transpose. If we consider the operators R and C to be matrices, then the operator L can be written as $\mathbf{2}-R\mathbf{v}=R(\mathbf{2}-\mathbf{v})$. The condition that the velocities must be periodic is given by $(RCRCL)(CRCL)^{N-1}\mathbf{v}=\mathbf{v}$. Multiplying both sides by R, we obtain $(CRCL)^N\mathbf{v}=R\mathbf{v}$. Replacing the first L operator by $R(\mathbf{2}-\mathbf{v})$ yields $(CRCL)^{N-1}(CRCR)(\mathbf{2}-\mathbf{v})=R\mathbf{v}$. After repeatedly replacing the L operators, we obtain

$$[R-(-RCRC)^N]\mathbf{v}=\sum_{i=0}^{N-1}(-RCRC)^i\mathbf{2}.$$

After solving this equation for $\mathbf{v}$, we can calculate the locations of the collisions. For arbitrary values of m, the solution can be expressed symbolically, but the expression is quite complicated and omitted here. We are particularly interested in understanding why the orbits collapse near the point $m=1$ and the qualitative difference between $m>1$ and $m<1$. We therefore consider the case when $m-1$ is small and expand all quantities in powers of $m-1$. After calculating the locations at which interparticle collisions occur, we see that as $e\to 1$, the location of the final interparticle collision in the only $(RCRCL)$ block will have the largest value. This is because each of the $(CRCL)$ blocks pushes the locations of the interparticle collisions toward the right.

After some algebra, we find that in the limit $m\to 1$, the stability condition is given by $A=-e^N$. Thus, this type of orbit is always stable, and it is easy to show that the velocities are always consistent with the collisions. Thus, all that remains to be checked is that the interparticle collisions occur between the walls. Some algebra reveals that the location of the first interparticle collision in the sequence occurs at a location given by

$$p^{(1)}=1+(m-1)S^{(1)}+O(m-1)^2,$$

where

$$S^{(1)}=\frac{(1+e)^3[1-e^{2N}-2Ne^N(1-e)]}{4e(1-e)^2(1+e^N)^2}.$$

One can show that $S^{(1)}>0$ for $0<e<1$, and thus, in the limit as m tends to unity from below, $p^{(1)}<1$. Therefore, in this limit in which $m<1$, the location of the first interparticle collision will occur between the two walls. However, for $m>1$, the location will occur behind the stationary wall and, therefore, orbits of this type can never occur for $m>1$.

Similarly, we find that the location at which the final collision in the sequence occurs (that is the $2N$th collision) is given by

$$p^{(2N)}=1+(m-1)S^{(2N)}+O(m-1)^2,$$

where

$$S^{(2N)}=\frac{(1+e)^3[e^{2N}-1+2Ne^{N-1}(1-e)]}{4(1-e)^2(1+e^N)^2}.$$

For the orbit to be consistent when $m<1$ (that is, all the collisions occur between the two walls), we require that $S^{(2N)}>0$. Some algebra also reveals that if $S^{(2N)}>0$ and $S^{(1)}>0$, then all other interparticle collisions will be located between the locations of the first and last interparticle collisions. Hence, a necessary and sufficient requirement for consistency is $m<1$ and $S^{(2N)}>0$.

The slope $S^{(2N)}\to N/2>0$ in the limit $e\to 1$, and thus, for e sufficiently close to unity, $S^{(2N)}>0$ and the orbit will be consistent. The slope can only change sign at the zeros of the polynomial,

$$1-e^{2N}-2Ne^{N-1}(1-e).$$

For $N=1$, we immediately see that the $RCRCL$ orbit will exist for all values of e. In the case of $N=2$, we can show that the $(RCRCL)(CRCL)$ orbit can only exist for values of $e>\sqrt{2}-1=0.414\ldots$. For larger values of N, we must solve a polynomial with degree larger than 4, and thus, in general, no analytic expression for the threshold can be obtained. However, as $N\to\infty$ this polynomial has a root near unity. A straightforward asymptotic expansion gives the critical value of the restitution coefficient

$$e_N=1-\frac{3}{N^2}+O\left(\frac{1}{N^4}\right).$$

Asymptotically, e_N is an increasing function of N, and thus, if the Nth orbit exists, then all lower orbits must also exist. Hence, asymptotically, the number of periodic orbits will be at least of order $\sqrt{3/(1-e)}$ as $e\to 1$. Therefore, as $e\to 1$, the number of periodic orbits that exist for $m<1$ will tend to infinity. However, when $m=1$, all periodic orbits collapse onto one orbit, namely, $CRCL$.

V. MULTIPLE ORBITS NEAR OTHER COLLAPSE POINTS

Figure 3 shows that the phenomenon of orbits collapsing that occurs at $m=1$ also occurs for an infinite set of mass ratios. These collapse points can be seen in Fig. 3 near $(m-1)/(m+1)=-0.45$, or equivalently $m=0.38$ (three cyan curves), $(m-1)/(m+1)=0.65$, or equivalently $m=0.21$ (four magenta curves) and at an infinite number of similar points as $m\to 0$. A detailed investigation of the orbits shows that at these transitions the $(CR)^M CL$ orbit changes to $(CR)^M CRL$. As in the $m=1$ case, for e sufficiently close to one, we can also observe the phenomenon of orbits of the type $[(CR)^M CL]^N$ changing to orbits of type $(CR)^M CRL[(CR)^M CL]^{N-1}$. For $M=2$, which represents the $CRCRCL$ orbit, we find that the collapse point occurs at

$$m=\frac{1+3e+e^2-2\sqrt{e(1+e)^2}}{e^2+e+1}.$$

As $e\to 1$, this collapse point tends to $m=1/3=0.3333\ldots$. For $M=3$, which represents the $CRCRCRCL$ orbit, we find that the collapse point occurs at

$$m=\frac{e^2+4e+1-2\sqrt{2e}(1+e)}{e^2+1}.$$

As $e\to 1$, this collapse point tends to $m=3-2\sqrt{2}=0.1716\ldots$. For $M=4$ and 5, the expressions for the critical points as functions of e require the solution of high degree polynomials, but in the limit as $e\to 1$ the critical mass ratios are $1-2/\sqrt{5}=0.1055\ldots$ and $7-4\sqrt{3}=0.0718\ldots$, respectively. For $M\geqslant 6$, even the limiting values require the solution of high degree polynomials, and thus, no analytic expressions can be obtained. Nevertheless, numerically it poses no challenges, and we can show that the critical mass ratio tends to zero as M increases.

Furthermore, for $M=2$, we can show that the $N=2$ orbit can only exist if $e>(3-\sqrt{8})^{1/3}=0.55567\ldots$. For all higher values of M and N, the determination of these critical points requires the solution of high degree polynomials. For any given value of M the behavior for larger N can be determined numerically and is similar to the $m=1$ case. We also find that the critical value of the restitution coefficient increases toward unity as N increases. Thus, as $e\to 1$, we still obtain an infinite number of periodic orbits in the vicinity of the collapse points. We note that at these collapse points the masses of the particles are not equal.

We now turn our attention to the second type of transition. For $m>1$, there are also an infinite number of critical values, but in this case the locations of all interparticle collisions occur against the oscillating wall. These orbits have the form $CR(CL)^M$. For $M=1$, the critical value is at $m=(1+e^2)/(2e)$ and can be seen in Fig. 3 near $(m-1)/(m+1)=1/9=0.1111$ (two red curves). For $M=2$, the critical value is at $m=(1+e+e^2)/e$ and can be seen in Fig. 3 near $(m-1)/(m+1)=5/9=0.5555$ (three cyan curves). Longer orbits with larger values of M require the solution of high degree polynomials, and no analytic expressions can be obtained.

As the mass ratio tends to these critical values from below, the velocity of the left particle following the final interparticle collision in the periodic sequence tends to zero. This means that the left particle becomes trapped increasingly close to the oscillating wall. If the mass ratio exceeds this critical value, then the collision location moves behind the wall. To resolve this, by analogy with the collapse points at the stationary wall, one might naively expect that the $CR(CL)^M$ orbit would add an extra collision with the oscillating wall to form a $CRL(CL)^M$ orbit. However, a calculation similar to the one in Sec. IV shows that such orbits can never be realized. Therefore, the type of transitions described in Sec. IV, in which an infinite set of periodic orbits collapse at the stationary wall, cannot exist at the oscillating wall.

It is easy to understand the fundamental difference between the case in which the collapse point occurs at the stationary wall and that at the oscillatory wall. As the mass ratio decreases toward any of the collapse points at the stationary wall, the velocity of the right particle following the final interparticle collision in the sequence tends to zero. If we slightly decrease the mass ratio below the collapse point, then the collision sequence will no longer be consistent and another collision with the stationary wall will be added. The velocity of the right particle after reflection is close to zero since its incoming velocity was close to zero. Hence, adding this collision does not result in any discontinuous change in momentum. We therefore expect that the addition of the extra collision would give rise to a collision sequence whose collisions were close to that of the degenerate orbit.

However, for a collapse point at the oscillating wall, the situation is quite different. As the mass ratio increases toward any of the collapse points at the oscillating wall, the velocity of the left particle following the final interparticle collision in the sequence tends to zero. But when the system adds another collision with the oscillating wall, the left particle gains a large momentum impulse, and thus, the orbit will experience a large discontinuous change in energy. We therefore expect the resulting orbit to be quite different from the degenerate orbit, and such orbits may not be consistent or stable.

Finally, we consider the third type of transition. There also exist some orbits in which some of the interparticle collisions occur at the stationary wall and the remaining interparticle collisions occur a finite distance from the wall. An example of this type of orbit is the five black curves near $(m-1)/(m+1)=-0.25$ in Fig. 3. This type of transition is qualitatively different from the first type of transition in which all interparticle collisions occur at the stationary wall. This is because the right particle always has nonzero velocity for the third type of transition but attains zero velocity for the first type of transition. For the third type of transition, the addition of an R operator to the collision sequence will cause a discontinuous change in momentum, which leads to a discontinuity in the collision locations. We therefore expect any resulting orbit to be quite different from the degenerate orbit. That is why these isolated orbits end abruptly with no apparent continuation. For the first type of transition, the addition of an R operator will not cause any discontinuous momentum change, and thus, the collision locations will be the same.

VI. MULTIPLE PARTICLES

We now consider a system containing P particles with different masses. We index the particles from left to right by $i=1,\ldots,P$, let the mass of the ith particle be m_i, and denote the collision operator between the ith and $(i+1)$th particles as C_i. We will assume that the inelasticity in collisions between the ith and the $(i+1)$th particle is characterized by a constant coefficient of restitution, e_i. We also assume that collisions between the Pth particle and the stationary wall have coefficient of restitution e_w.

Numerical simulations show that systems with more than two particles exhibit similar behavior to the system with two particles. To illustrate this, we choose an example with four particles and take $e_1=e_2=e_3=0.8$ and $e_w=1$. If we choose the mass ratios m_{i+1}/m_i to be certain critical ratios, we obtain the situation in which all interparticle collisions collapse against the stationary wall. In this case, the critical mass ratios are $m_1/m_2=0.9523\ldots$, $m_2/m_3=0.9756\ldots$, and $m_3/m_4=1$. If we choose the mass ratios to be slightly larger than these critical ratios, we always obtain periodic orbits that contain exactly eight collisions. The order of the collisions is always the same and is given by $C_1C_2C_3RC_3C_2C_1L$. If we choose any of the mass ratios m_{i+1}/m_i to be slightly smaller than the ratios required for collapse, we obtain a situation with many more orbits. For example, if we choose all three ratios to be 1% larger than the critical ratios, we obtain periodic orbits that contain 17, 18, 25, 26, 28, 29, 32, 33, 70, and 132 collisions. As e_1, e_2, and e_3 get closer to unity, the number of possible orbits increases dramatically. Similar behavior is found for systems containing other numbers of particles.

It is straightforward to generalize the results for two-particle systems to the case where the coefficient of restitution between the right particle and the stationary wall can take a general value e_w. In this case, the simplest orbit collapse occurs at

$$m=\frac{e_we+1}{e+e_w}.$$

We use this result to derive the mass ratios required to cause collapse for P particles and use this to choose the mass ratios so that the resulting orbits have the simplest possible structure with the fewest possible number of collisions.

Motivated by the results for two particles, we can consider a system containing P particles and choose the two particles closest to the stationary wall to have the mass ratio

$$\frac{m_P}{m_{P-1}}=\frac{e_we_{P-1}+1}{e_{P-1}+e_w}.$$

Numerical simulations show that the particle closest to the stationary wall will rapidly become closely trapped against the stationary wall. One can show that, once the particle is trapped against the wall, the system essentially has a wall with coefficient of restitution $e_{P-1}e_w$. Therefore, the system of P particles and stationary wall with coefficient of restitution e_w is effectively reduced to a system of $P-1$ particles and a stationary wall with coefficient of restitution $e_{P-1}e_w$. Next, the $(P-1)$th particle will also collapse onto the effective wall if the mass ratio between the $(P-1)$th and $(P-2)$th particle is

$$\frac{m_{P-1}}{m_{P-2}}=\frac{e_we_{P-1}e_{P-2}+1}{e_{P-2}+e_we_{P-1}}.$$

This then leads to a system that effectively contains $P-2$ particles with a stationary wall with coefficient of restitution $e_{P-1}e_{P-2}e_w$. We can follow the same procedure to find masses for which all particles collapse. Without loss of generality, we choose $m_1=1$, and then total collapse (in which $P-1$ particles are trapped against the wall) will occur for

$$m_i=\prod_{j=1}^{i}\frac{1+e_w\prod_{k=j-1}^{P-1}e_k}{e_{j-1}+e_w\prod_{k=j}^{P-1}e_k}.$$

For ratios larger than these collapse ratios, we obtain the simple unique orbit analogous to the two particle case, but if any of the ratios are smaller than the collapse ratios, we obtain nonunique orbits, which may contain large numbers of collisions. Therefore, we have shown the phenomenon of orbit collapse is not a special feature of two-particle systems, but a generic feature in many-particle systems.

The case of P particles of equal mass has been carefully considered by Yang [17]. He found that nonunique orbits existed in this case. This corresponds exactly to the case where the mass ratios m_{i+1}/m_i are slightly smaller than the ratios required for collapse. As e tends to unity, the mass ratios required for collapse tend to unity and, thus, the system with equal masses is exactly that described above. This means that when e tends to unity, the number of possible periodic orbits can also be arbitrarily large in multiparticle systems.

VII. CONCLUSION

In this paper, we have considered the behavior of one-dimensional inelastic particle systems with arbitrary masses that travel between a vibrating wall and a stationary wall. We have demonstrated the following generic behavior that is not evident in widely studied equal-mass systems.

1. It is well known that if all of the particles in the system have the same mass then the particle nearest the oscillating wall traps, the remaining particles in the vicinity of the stationary wall [8,17]. As the coefficient of restitution tends to unity the locations of all inter-particle collisions tend to the stationary wall. In general, this is not the case in unequal-mass systems.

2. We have shown that large numbers of periodic orbits can collapse onto simple orbits at certain critical mass ratios. These collapse points correspond to mass ratios for which all the interparticle collisions in the orbit occur at the stationary wall. By deriving analytic solutions for these orbits we have shown that as the coefficient of restitution tends to unity the number of periodic orbits that collapse onto the simple orbit tends to infinity.

3. For a fixed value of e there are a countably infinite number of collapse points. These critical mass ratios at which the collapse phenomenon occur depend on the value

of e, except for equal-mass systems, which represent a collapse point for all values of e. Therefore, systems with particles of equal mass are highly degenerate.

4. As one varies the mass ratio and/or the restitution coefficient, there are critical parameter values at which orbits can end abruptly. This can occur in four different ways. First, all interparticle collisions occur at the stationary wall. In this case, the velocities of the particle nearest to the stationary wall tends to zero. Adding an additional collision with the stationary wall therefore leads to a continuous change in momentum and orbits change continuously. At these critical values, multiple orbits can collapse onto a single simple orbit. Second, interparticle collisions can occur at the oscillating wall. This leads to a discontinuous change in momentum as an extra collision with the oscillating wall is added since the oscillating wall adds energy and momentum to the system. Third, some (but not all) of the interparticle collisions occur at the stationary wall. In this case, the velocities of the particles near the stationary wall do not tend to zero. This leads to a discontinuous change in momentum, and thus, orbits do not change continuously. Fourth, the orbit can become unstable to small changes in the locations of the collisions. This differs from the first three reasons because in this case orbits end abruptly without any interparticle collision occurring at any of the walls.

The analysis of two-particle systems plays a crucial role in allowing us to understand orbit collapse in many-particle systems. In fact, the critical mass ratios required for orbit collapse in many-particle systems can be derived directly from the two-particle system by appropriately adjusting the inelasticity in particle-wall collisions. With knowledge of the mass ratios required for orbit collapse, we can select mass ratios that give operating conditions with simple unique periodic orbits. This gives simple and direct criteria for avoiding chattering and obtaining simple repeatable periodic behavior in particle systems.

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[1] L. Tonks, Phys. Rev. **50**, 955 (1936).

[2] E. Fermi, J. Pasta, and S. Ulam, Los Alamos Report No. LA-1940, 1955 (unpublished).

[3] M. Tabor, *Chaos and Integrability in Nonlinear Dynamics: An Introduction* (Wiley, New York, 1989).

[4] E. A. Jackson, *Perspectives of Nonlinear Dynamics* (Cambridge University Press, Cambridge, England, 1990).

[5] A. Mehta and J. M. Luck, Phys. Rev. Lett. **65**, 393 (1990).

[6] J. M. Luck and A. Mehta, Phys. Rev. E **48**, 3988 (1993).

[7] B. Cipra, P. Dini, S. Kennedy, and A. Kolan, Physica D **125**, 183 (1999).

[8] Y. Du, H. Li, and L. P. Kadanoff, Phys. Rev. Lett. **74**, 1268 (1995).

[9] K. Geisshirt, P. Padilla, E. Praestgaard, and S. Toxvaerd, Phys. Rev. E **57**, 1929 (1998).

[10] E. L. Grossman and B. Rooman, Phys. Fluids **8**, 3218 (1996).

[11] C. Vamos, N. Suciu, and A. Georgescu, Phys. Rev. E **55**, 6277 (1997).

[12] T. Zhou, Phys. Rev. Lett. **80**, 3755 (1998).

[13] T. Zhou, Phys. Rev. E **58**, 7587 (1998).

[14] L. P. Kadanoff, Rev. Mod. Phys. **71**, 435 (1999).

[15] S. Sen, T. R. Krishna Mohan, and J. M. M. Pfannes, Physica A **342**, 336 (2004).

[16] S. Sen, J. M. M. Pfannes, and T. R. Krishna Mohan, J. Korean Phys. Soc. **46**, 571 (2005).

[17] J. Yang, Phys. Rev. E **61**, 2920 (2000).

[18] T. R. Krishna Mohan and S. Sen, Phys. Rev. E **67**, 060301(R) (2003).

[19] V. Nesterenko, J. Appl. Mech. Tech. Phys. **5**, 733 (1983).

[20] S. Sen, M. Manciu, and J. D. Wright, Phys. Rev. E **57**, 2386 (1998).

[21] M. Manciu, S. Sen, and A. J. Hurd, Physica D **157**, 226 (2001).

[22] M. H. Lee, J. Florencio, and J. Hong, J. Phys. A **22**, L331 (1989).

[23] D. Vitali and P. Grigolini, Phys. Rev. A **39**, 1486 (1989).

[24] S. McNamara and W. R. Young, Phys. Fluids A **4**, 496 (1992).

[25] H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys. **68**, 1259 (1996).

[26] H. K. Pak and R. P. Behringer, Phys. Rev. Lett. **71**, 1832 (1993).

[27] S. McNamara and J. L. Barrat, Phys. Rev. E **55**, 7767 (1997).