Front propagation sustained by additive noise

M. G. Clerc, C. Falcón,* and E. Tirapegui
Departamento de Física, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 487-3, Santiago, Chile
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The effect of noise in a motionless front between a periodic spatial state and an homogeneous one is studied. Numerical simulations show that noise induces front propagation. From the subcritical Swift-Hohenberg equation with noise, we deduce an adequate equation for the envelope and the core of the front. The equation of the core of the front is characterized by an asymmetrical periodic potential plus additive noise. The conversion of random fluctuations into direct motion of the core of the front is responsible of the propagation. We obtain an analytical expression for the velocity of the front, which is in good agreement with numerical simulations.

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I. INTRODUCTION

The description of macroscopic matter—i.e., matter composed of a large number of microscopic constituents—is usually done using a small number of coarse-grained or macroscopic variables. When spatial inhomogeneities are considered these variables are spatiotemporal fields whose evolution is determined by deterministic partial differential equations (PDEs). This reduction is possible due to a separation of time and space scales, which allows a description in terms of the slowly varying macroscopic variables, which are in fact fluctuating variables due to the elimination of a large number of fast variables whose effect can be modelized including suitable stochastic terms, *noise*, in the PDE. The influence of noise in nonlinear systems has been the subject of intense experimental and theoretical investigations in the last decades [1–17]. Far from being merely a perturbation to the idealized deterministic evolution or an undesirable source of randomness and disorganization, noise can induce specific and even counterintuitive dynamical behavior. The most well-known examples in zero-dimensional systems are noise-induced transitions [1,7,9] and stochastic resonance (see the review in [2] and references therein). More recently, examples in spatially extended system were found, such as, noise-induced phase transitions [3–5,12], noise-induced patterns [13–15], stochastic spatiotemporal intermittency [16], and noise-induced traveling waves [17]. Here, we present another robust effect of noise in extended systems: *the motion of a static front connecting a stable homogeneous state with a stable inhomogeneous (spatially periodic) state due to additive noise*. A first preliminary discussion of this effect was done by the present authors in a recent Letter [18], and the aim of this article is to study and characterize the universal mechanism which is at the origin of the front motion in the presence of noise.

The concept of front propagation emerged in the field of population dynamics [19], and the interest in this type of problems has been growing steadily in chemistry, physics, and mathematics. In physics, front propagation plays a central role in a large variety of situations, ranging from reaction diffusion models, to general pattern-forming systems (see the review in [20] and references therein). A front solution is a solution which links spatially two extended states. One of the most studied front solutions is the front connecting a stable uniform state with an unstable one: the Fisher-Kolmogorov-Petrovsky-Piskunov (FKPP) front [21]. The speed of propagation of this type of front is not unique, and it is fixed by the initial conditions [22]. Another well-known type of front, the *normal front*, connects two stable uniform states. The speed of this kind of front is unique, and for a variational system it is proportional to the difference of free energy between the two uniform states. In Fig. 1 the dashed curve represents the typical behavior of the speed of a normal front as a function of an arbitrary parameter. Note that the speed of the front is zero only at the Maxwell point where both states have the same energy. This picture is modified when one considers a front connecting an spatially periodic state with a uniform one, which is the case of interest for us here. In this case the speed is zero not only in one point but in a whole interval of variation of the relevant parameter, the *pinning range* [23], and additive noise will induce front motion [18]. In Fig. 1 the solid line represents the typical speed of these fronts and the

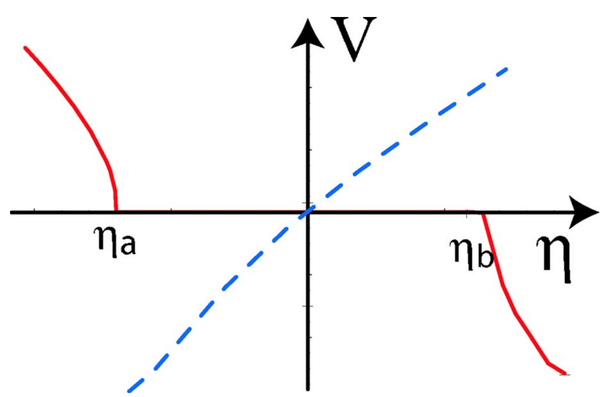

FIG. 1. (Color online) Speed of the front as function of one parameter. The dashed curve depicts the typical behavior of the speed of a normal front as function of arbitrary parameter, and the solid curve represents the speed of a front that links a spatial periodic state and uniform one. For the sake of simplicity the origin represents the Maxwell point. The pinning range is depicted by the interval between η_a and η_b.

*Present address: Laboratoire de Physique Statistique, École Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex 05, France.

interval $[\eta_a, \eta_b]$ represents the pinning range. The effect of additive noise on the speed of a "normal front" is just a random fluctuation of its speed. On the other hand, the influence of multiplicative noise in a front, which will not occupy us in this paper, has been extensively studied in the literature, particularly concerning the issue of velocity selection [24].

The paper is organized as follows. In Sec. II, we present several examples of numerical observations of motion of fronts induced by additive noise. In Sec. III, we use a prototype model which exhibits this type of front to derive an adequate equation for the envelope of the front solution: an amended amplitude equation, which includes a resonant term coming from the additive noise whose origin is discussed in Appendix A and also nonresonant terms. It turns out that the contributions of the noise and of the nonresonant terms are of the same kind, giving rise to an asymmetric potential with denumerable stable equilibria in the equation for the core of the front which is derived in Sec. IV. In this equation the dominant contribution to the potential near threshold comes from the noise term. In Sec. V, we obtain an analytical expression for the mean velocity of the front using Dinkyn's equation, which is proportional to Kramer's rate in the weak-noise limit. This expression is in good agreement with numerical simulations. In Sec. VI, we summarize our results. In Appendix A, we show that, as stated and used in the text in Sec. III, noise is always resonant in the sense of the *stochastic* normal form. In Appendix B, we give the technical details involved in the derivation of the equation for the core of the front. And finally in Appendix C, we show that the mean value of the derivative of the phase remains bounded, which is a necessary consistency condition of our approach.

II. NUMERICAL OBSERVATIONS OF ADDITIVE NOISE-INDUCED FRONT PROPAGATION

In order to illustrate the generic nature of additive noise-induced front propagation, we consider the effect of additive noise over several dynamical systems which have fronts linking a spatially periodic solution and a uniform one.

(a) *Lifshitz normal form.* A prototype model that exhibits coexistence of a spatially periodic solution and a uniform state is the Lifshitz normal form [25]

$$\partial_t u = \eta + \mu u - u^3 + \nu \partial_{xx} u - \partial_{xxxx} u + d u \partial_{xx} u + c(\partial_x u)^2 + \sqrt{\Delta}\zeta(x,t) \tag{1}$$

This model describes the dynamics close to the confluence of a bistability of homogeneous states and a spatial bifurcation—that is, near a critical point of codimension 3, called a Lifshitz point. Here, μ is the bifurcation parameter, η accounts for the asymmetry between the two homogeneous states, and the term $\partial_{xxxx} u$ describes a superdiffusion, accounting for the short-distance repulsive interaction, whereas the terms proportional to d and c are, respectively, the nonlinear diffusion and convection, $\zeta(x,t)$ is a Gaussian white noise with zero mean value and correlation $\langle \zeta(x,t)\zeta(x',t')\rangle = \delta(x-x')\delta(t-t')$, and Δ represents the intensity of the noise. Recently, this model has been used to describe the complex dynamics observed in a liquid-crystal light valve with optical feedback [25]. In the region corresponding to the pinning range of the above model, the system exhibits a motionless front that connects the spatially periodic state with the uniform one [cf. Fig. 2(a)]. When we consider the effect of additive noise, we notice that it induces the invasion of one of the states over the other one, or vice versa, depending on the region of parameters we initially choose. This situation is depicted in Fig. 2.

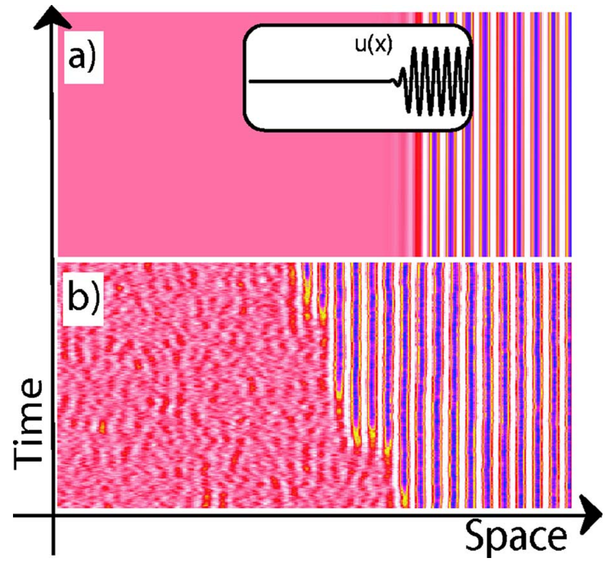

FIG. 2. (Color online) Spatiotemporal evolution of Eq. (1), with time running up. The gray scale is proportional to field u. The inset is the initial condition. The parameters have been chosen as $\eta=-0.044$, $\mu=-0.0126$, $\nu=-1.0$, $c=0.177$, and $d=0.2$ (a) $\delta=0.0$, and (b) $\delta=0.9$.

(b) *Population dynamics.* In order to take into account the long-range effect of the environment one can consider nonlocal models to describe the population dynamics. In this type of models the emergence of self-organized structures and patterns is well known [26,27]. A minimal model that exhibits the coexistence of a spatial periodic state and a uniform one is the variational nonlocal Nagumo model [28].

$$\partial_t u(x,t) = \partial_{xx} u + u(\alpha - u)(1-u) + u^3 - u(x,t)\int_\Omega u(x',t)^2 f_\sigma(x,x')d^2x', \tag{2}$$

where $u(x,t)$ is the local density and α is the adversity parameter which accounts for the complications of development of the species under study. The adversity characterizes the equilibrium point and can always be chosen to satisfy $0 \leq \alpha \leq 1$ without loss of generality. The function $f_\sigma(x,x')$ is the influence function, characterized by a range σ and normalized in the domain Ω under study. For simplicity, we consider the environment to be homogeneous and isotropic. Then $f_\sigma(x,x') = f_\sigma(x-x')$, with $f_\sigma(z)$ even, and $\int_\Omega f_\sigma(x,x')dx' = 1$. In the extreme local limit $\sigma \to 0$, one has $f_\sigma(x,x') = \delta(x-x')$, and Eq. (2) reduces to the Nagumo model [26]. Let us now consider the simple influence functions $f_\sigma(z) = \theta(\sigma+z)\theta(\sigma-z)/2\sigma$, where $\theta(z)$ is the Heaviside func-

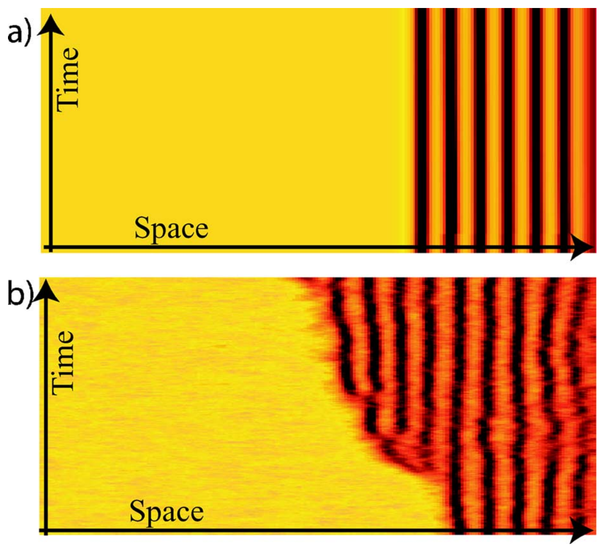

FIG. 3. (Color online) Spatiotemporal evolution the population density $u(x,t)$ for the model (2), with time running up, $\alpha=0.35$, $\sigma=4\%$ of total system size, system size=400 points, (a) without noise, and (b) with additive noise. The gray scale is proportional to the population density.

tion. The dynamics is described by the parameters $\{\alpha, \sigma$ and Eq. (2) can be written as

$$\partial_t u = -\frac{\delta \mathcal{F}[u]}{\delta u},$$

where the Lyapunov functional $\mathcal{F}[u]$ has the form

$$\mathcal{F}[u] = \int_\Omega \left\{ \frac{1}{2}(\partial_x u)^2 + \frac{\alpha}{2}u^2 - \frac{(\alpha+1)}{3}u^3 \right\} dx + \frac{1}{4}\int_\Omega \int_\Omega u^2 u'^2 f_\sigma(x,x') dx dx' .$$

Hence, the dynamics of model (2) is of the relaxation type and the stationary states are local minima of $\mathcal{F}[u]$.

The model (2) exhibits a motionless front that connects the spatially periodic state with the uniform one [cf. Fig. 3(a)]. Note that these motionless front solutions are not the global minimum of the Lyapunov functional $\mathcal{F}[u]$; however, they are local minima of this functional and the population only can spread if one adds energy to the system. For instance, if we consider the effect of additive noise, the periodic population state can invade the unpopulated state ($u=0$). In Fig. 3, we depict the spread of population due to the presence of additive noise in the variational nonlocal Nagumo model.

(c) *Subcritical Swift-Hohenberg equation.* A prototype model used in pattern-forming system is the Swift-Hohenberg equation [20]. Initially, this model was used to describe the onset of Rayleigh-Bénard convection [20]. From the point of view of dynamical systems theory, this model describes the confluence of a subcritical stationary and a spatial bifurcation. Generalizations of this model have been used intensively to account for pattern formation in several systems (see the review in [20] and reference therein). We shall consider here the subcritical Swift-Hohenberg equation, which exhibits the coexistence between a uniform state and an spatially periodic one. In the presence of additive noise this equation reads

$$\partial_t u = \varepsilon u + \nu u^3 - u^5 - (\partial_{xx} + q^2)^2 u + \sqrt{\eta}\zeta(x,t), \qquad (3)$$

where $u(x,t)$ is an order parameter, $\varepsilon - q^4$ is the bifurcation parameter, q is the wave number of the periodic spatial solution, ν is the control parameter of the type of bifurcation (supercritical or subcritical), $\zeta(x,t)$ is a Gaussian white noise with zero mean value and correlation $\langle \zeta(x,t)\zeta(x',t')\rangle = \delta(x-x')\delta(t-t')$, and η represents the intensity of the noise. In the pinning range of the model above the system exhibits a motionless front that connects the spatially periodic state with the uniform one [cf. Fig. 4(a)]. When one considers the effect of additive noise, depending where is the control parameter inside the pining range, it induces on average that one of the states invades the other one. This situation is shown in Fig. 4, and in Fig. 5 we show the speed of the front of this model (3) with and without additive noise.

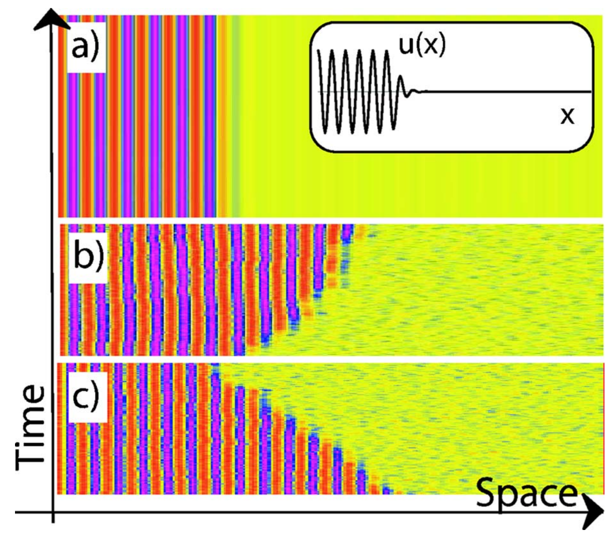

FIG. 4. (Color online) Spatiotemporal evolution of Eq. (3), with time running up. The gray scale is proportional to field u. The inset is the initial condition. The parameters have been chosen as $\nu=1.0$, $q=0.7$, (a) $\epsilon=-0.16$ and $\eta=0.0$, (b) $\epsilon=-0.16$ and $\eta=0.4$, and (c) $\epsilon=-0.177$ and $\epsilon=-0.16$.

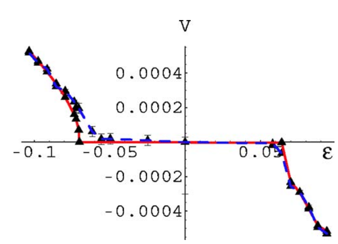

FIG. 5. (Color online) Mean velocity of the front with and without noise. The thick and dashed curves are the average velocity of the front of Eq. (3) for $\varepsilon=-0.16$, $\nu=1.0$, $q=0.7$, $\eta=0.0$, and $\eta=0.01$, respectively.

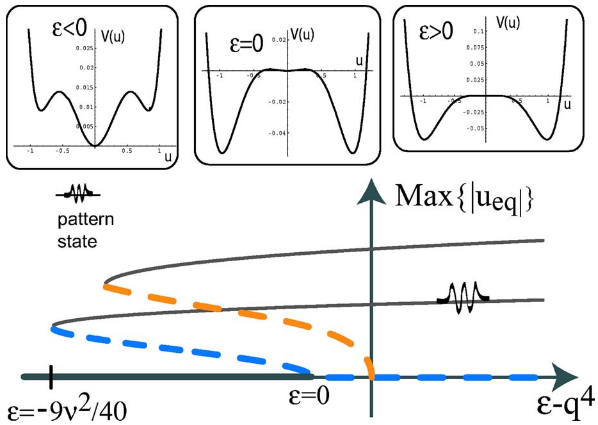

FIG. 6. (Color online) Bifurcation diagram of the subcritical Swift-Hohenberg model (3); the horizontal axis represents the control parameter and the vertical one represents the maximum value of the absolute value of the equilibrium state. As solid lines are depicted the stable states and as dashed lines the unstable ones. The model (3) exhibits the coexistence between two uniform states and spatial periodic one. In the insets are illustrated the local potentials $V(u)$ for negative, zero, and positive ϵ, respectively.

In brief, we have considered three different dynamical systems, which exhibit a motionless front solution linking a spatially periodic state and a uniform one. When additive noise is taken into account generically one state starts to invade the other one by means of a noise-induced front propagation. In order to figure out the mechanism of this propagation in the next section we shall study in detail the dynamics of the subcritical Swift-Hohenberg equation with additive noise at the onset of spatial instability.

III. AMPLITUDE EQUATION AND EVOLUTION OF THE CORE OF THE FRONT

In order to understand the mechanism through which additive noise induces front propagation, we consider a prototype model that exhibits this type of front, the *subcritical Swift-Hohenberg equation with noise*, Eq. (3). This model reads

$$\partial_t u = -\frac{\delta \mathcal{F}}{\delta u} + \sqrt{\eta}\zeta'(x,t), \tag{4}$$

where the free energy has the form

$$\mathcal{F} = \int dx \left\{ -\frac{(\varepsilon - q^4)u^2}{2} - \frac{\nu u^4}{4} + \frac{u^6}{6} - q^2(\partial_x u)^2 - \frac{(\partial_{xx}u)^2}{2} \right\}.$$

Hence, the dynamics of the above model is characterized by the minimization of this free energy. In Fig. 6, we show the typical bifurcation diagram observed in the subcritical Swift-Hohenberg model. Note that the system exhibits coexistence between different homogeneous states and the spatial periodic one. For small ε and μ, the system shows spatially periodic solutions with small amplitude (proportional to $\sqrt{\nu}$), and consequently the amplitude equation will be an adequate framework for our study. The free energy has a local potential

$$V(u) = -\frac{(\varepsilon - q^4)u^2}{2} - \frac{\nu u^4}{4} + \frac{u^6}{6},$$

which characterizes the stability properties of the uniform states.

A. Amplitude equation

In the limit of small ν, we look for a solution $u(x,t)$ in Fourier modes putting $u = A_0 e^{iqx} + \bar{A}_0 e^{-iqx} + \cdots$, with A_0 of order $\nu^{1/2}$. Replacing in the subcritical Swift-Hohenberg equation (3), we find that A_0 satisfies

$$\varepsilon A_0 + 3\nu |A_0|^2 A_0 - 10|A_0|^4 A_0 = 0. \tag{5}$$

For small and negative ν and $-9\nu^2/40 < \varepsilon < 0$ (cf. Fig. 6), the system exhibits coexistence between a stable homogenous state $u=0$ and a periodic spatial one: $u = \sqrt{\nu}\{\sqrt{2(1+\sqrt{1+40\varepsilon/9\nu})}\cos[q(x-x_0)]\} + o(\nu^{5/2})$, where x_0 is an arbitrary number, related to the symmetry of translation. In this region of the space of parameters, we find then a front solution between these two states. This type of solution is a heteroclinic curve of the spatial dynamical system ($\partial_t u = 0$) associated with the above model [29]. A front between an homogeneous and a spatially oscillating state can be described by an envelope $A(X,T)$, which is introduced through the ansatz

$$u = A(X = |\varepsilon|^{1/2}x, T = |\varepsilon|t)e^{iqx} + \text{c.c.} + W(X,T), \tag{6}$$

where $W(X,T)$ is a small function of the order of $\nu^{5/2}$, $A \sim \nu^{1/2}$, and $\nu \sim |\varepsilon|^{1/2}$. Introducing the above ansatz in Eq. (3), linearizing in W, and considering the dominating order ($\nu^{5/2}$), we obtain

$$\begin{aligned}(\partial_x^2 + q^2)^2 W = (\varepsilon A + 3\nu|A|^2 A - 10|A|^4 A)e^{iqx}(4|\varepsilon|q^2\partial_{XX}A \\ - |\varepsilon|\partial_T A)e^{iqx} + (\nu A^3 - 5|A|^2 A^3)e^{i3qx} - A^5 e^{4iqx} \\ + \frac{\sqrt{\eta}}{2}\zeta'(x,t)e^{-iqx} + \text{c.c.},\end{aligned}$$

where the self-adjoint operator $L = (\partial_x^2 + q^2)^2$ has a nontrivial kernel characterized by the eigenfunctions $\{e^{iqx}, e^{-iqx}\}$. Taking $W=0$ at this order we obtain an equation for $A(X,t)$ which contains nonresonant terms

$$\begin{aligned}\partial_t A(X,t) = \Bigg[\varepsilon A + 3\nu A|A|^2 - 10A|A|^4 + 4|\varepsilon|q^2\partial_{XX}A \\ + \frac{\sqrt{\eta}}{2}\zeta'(x,t)e^{-iqx} \Bigg] + (\nu A^3 - 5A^4\bar{A})e^{2iqx} - A^5 e^{4iqx}.\end{aligned}$$

In this equation the terms in square brackets $[\cdots]$, except the noise term, are the ones in the usual normal form, which is obtained from the solvability condition for W in the previous equations—i.e., that the right-hand side is orthogonal to the kernel $L = (\partial_x^2 + q^2)^2$. The noise term is included there since it is always resonant in the sense that it cannot be removed by a stochastic nonlinear and nonsingular close to the identity change of variables as we discuss in Appendix A following Refs. [33–37], and the rest of the terms are the nonresonant terms up to this order which can be eliminated by a deterministic nonlinear and nonsingular close to the identity change of variables. Defining

$$A(x,t)=\sqrt{\frac{3\nu}{10}}B(y,\tau),\quad x=\frac{2\sqrt{10}q}{3\nu}y,\quad t=\frac{10}{9\nu^2}\tau,$$

the envelope equation reads

$$\partial_\tau B(y,\tau)=\sigma B+B|B|^2-B|B|^4+\partial_{yy}B+\left(\frac{1}{9\tilde{\nu}}B^3-\frac{1}{2}B^4\bar{B}\right)e^{2iqy/\alpha\sqrt{|\varepsilon|}}-\frac{1}{10}B^5e^{2iqy/\alpha\sqrt{|\varepsilon|}}+\frac{\beta\sqrt{\eta}}{2\alpha|\varepsilon|^2}e^{-iqy/\alpha\sqrt{|\varepsilon|}}\zeta(y,\tau),\qquad(7)$$

where the new noise ζ is proportional to ζ' and has mean value zero and correlation $\langle\zeta(y,\tau)\zeta(y',t')\rangle=\delta(y-y')\delta(t-t')$. One has

$$\sigma=\frac{10\varepsilon}{9\nu^2},\quad \nu=\sqrt{|\varepsilon|}\tilde{\nu},\quad \alpha=\frac{3\tilde{\nu}}{2\sqrt{10}q},\quad \beta=\frac{10^{1/4}10^2}{81\tilde{\nu}^4}.$$

If in the above amplitude equation we eliminate all the terms with an explicit dependence on the spatial variable y, we obtain the normal form without noise:

$$\partial_\tau B(y,\tau)=\sigma B+B|B|^2-B|B|^4+\partial_{yy}B=-\frac{\delta\mathcal{F}[B,\bar{B}]}{\delta\bar{B}},\qquad(8)$$

which is variational, as indicated, with the functional

$$\mathcal{F}[B,\bar{B}]=-\int dy\left[\sigma|B|^2+\frac{1}{2}|B|^4-\frac{1}{3}|B|^6-|\partial_y B|^2\right].$$

By minimizing this free energy functional, we find that the system has five uniform states, three of which are stable: $B=0$ and $B_\pm=\pm\sqrt{(1+\sqrt{1+4\sigma})/2}$ (cf. Fig. 6). It is then straightforward to show that the previous equation (the normal form without noise) has front solutions connecting two homogeneous stable states, $B=0$ and $B=B_\pm=\pm\sqrt{(1+\sqrt{1+4\sigma})/2}$, when $-\frac{1}{4}\le\sigma<0$, and these fronts will be stationary only when the free energy for both states is the same—i.e., when the system is in the Maxwell point. As already stated all the terms which were eliminated to arrive at Eq. (8) are nonresonant (except the noise term), in the sense that they can be eliminated by a nonlinear and nonsingular change of variables near the bifurcation point [33], and hence are usually neglected. As we shall see these terms can give an explanation to the locking phenomena and the pinning range [31]. Nevertheless, it should be pointed out that if we include only the noise term which is always resonant [34–37] (Appendix A), this would be enough to explain the locking phenomena, the pinning range, and the motion of the front, since this term

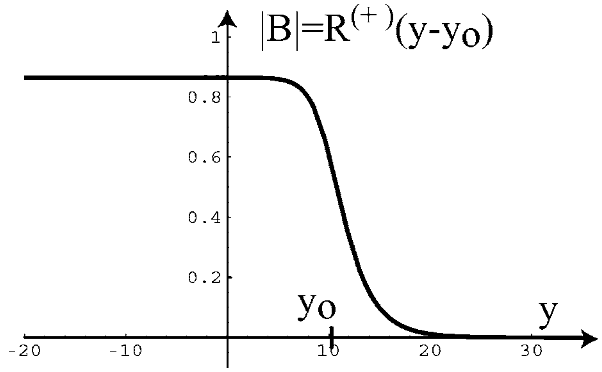

FIG. 7. Motionless front solution of Eq. (8), computed at the Maxwell point. As the vertical axis is represented the modulus of amplitude B as a function of the position. y_0 represents the position of the core of the front.

dominates the similar terms coming from the nonresonant terms near the bifurcation point [see Eq. (12)]. The nonresonant terms and the noise term give exponentially small contributions due to the fast oscillating exponentials and can be treated perturbatively in the amplitude equation.

We can calculate the Maxwell point from $\mathcal{F}[B,\bar{B}]$, and we obtain $\sigma_M=-3/16$. We put now $\sigma=\sigma_M+\delta\sigma$ in the complete equation for $B(y,\tau)$, which we treat as the variational part in the Maxwell point plus small terms which can be treated as perturbations. One has

$$\partial_\tau B(y,\tau)=\left(-\frac{3}{16}B+B|B|^2-B|B|^4+\partial_{yy}B\right)+\left\{\delta\sigma B+\left(\frac{1}{9\tilde{\nu}}B^3-\frac{1}{2}B^4\bar{B}\right)e^{2iqy/\alpha\sqrt{|\varepsilon|}}-\frac{1}{10}B^5e^{4iqy/\alpha\sqrt{|\varepsilon|}}+\frac{\beta\sqrt{\eta}}{2|\varepsilon|^2}e^{-iqy/\alpha\sqrt{|\varepsilon|}}\zeta(y,\tau)\right\},$$

where the small terms are inside the curly brackets $\{\cdots\}$ and they include the noise term. The unperturbed equation for $B(y,\tau)$ at the Maxwell point has the exact stationary front solutions

$$B^{(\pm)}(y-y_0)=R_0^{(\pm)}(y-y_0)e^{i\varphi},$$

where φ is an arbitrary constant phase, y_0 stands for the position of the core of the front, and $R_0^{(\pm)}(y-y_0)$ is given by

$$R_0^{(\pm)}(y-y_0)=\sqrt{\frac{3/4}{1+e^{\pm\sqrt{3/4}(y-y_0)}}}.$$

From now on, we shall work with the front $R_0^{(+)}(y-y_0)$ which goes from zero at $y=-\infty$ to the value $\sqrt{3/4}$ at $y=+\infty$ and we shall simply write R_0 for this solution which is depicted in Fig. 7. We put $B(y,\tau)=R(y,\tau)e^{i\Theta(y,\tau)}$ in the complete equation for $B(y,\tau)$, and we obtain

$$\partial_\tau R(y,\tau) + i\partial_\tau \Theta(y,\tau)R(y,\tau) = \left(-\frac{3}{16}R + R^3 - R^5 + \partial_{yy}R\right) + [2i\partial_y R\partial_y\Theta + iR\partial_{yy}\Theta - R(\partial_y\Theta)^2] + \left\{\delta\sigma R + \left(\frac{1}{9\tilde{\nu}}R^3 e^{2i\Theta} - \frac{1}{2}R^5 e^{4i\Theta}\right)e^{2iqy/\alpha\sqrt{|\varepsilon|}} - \frac{1}{10}R^5 e^{4i\Theta}e^{4iqy/\alpha\sqrt{|\varepsilon|}} + \frac{\beta\sqrt{\eta}}{2|\varepsilon|^2}e^{-iqy/\alpha\sqrt{|\varepsilon|}}\zeta(y,\tau)\right\}.$$

B. Nonresonant terms

In order to solve the above equation we make the ansatz

$$R(y,\tau) = R_0[y - y_0(\tau)] + \tilde{\varepsilon}\rho(y,y_0(\tau)),$$

$$\Theta(y,\tau) = \tilde{\varepsilon}\Theta_1(y,y_0(\tau)),$$

where $\tilde{\varepsilon}$ is a small parameter, and we have promoted the coordinate y_0 of the core of the front to a function of time $y_0(\tau)$. We replace our ansatz in the previous equation, where we assume that the small terms in the square brackets are $O(\tilde{\varepsilon})$ as well as the time derivative $dy_0(\tau)/d\tau \equiv \dot{y}_0$, and we equate the real and imaginary parts at $O(\tilde{\varepsilon})$, obtaining the equations

$$-R_{0y}(y-y_0(\tau))\dot{y}_0 = \tilde{\varepsilon}L(y-y_0)\rho + \delta\sigma R_0(y-y_0(\tau)) + \left(\frac{1}{9\tilde{\nu}}R_0^3 - \frac{1}{2}R_0^5\right)\cos\left(2q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right) - \frac{1}{10}R_0^5\cos\left(4q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right) + \frac{\beta\sqrt{\eta}}{2|\varepsilon|^2}\cos\left(q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right)\zeta(y,\tau) \quad (9)$$

and

$$\frac{d}{dy}[R_0(y-y_0(\tau))^2\Theta_{1y}] = -\left(\frac{1}{9\tilde{\nu}}R_0^4 - \frac{1}{2}R_0^6\right)\sin\left(2q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right) - \frac{1}{10}R_0^6\sin\left(4q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right) - \frac{\beta\sqrt{\eta}}{2|\varepsilon|^2}R_0(y-y_0(\tau))\sin\left(q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right)\zeta(y,\tau), \quad (10)$$

where

$$R_{0y}(y) \equiv \frac{dR_0}{dy}, \quad R_{0yy}(y) \equiv \frac{d^2R_0}{dy^2}, \quad \Theta_{1y} \equiv \partial_y\Theta_1(y,y_0(\tau))$$

and

$$L(y-y_0) \equiv -\frac{3}{16} + 3R_0(y-y_0)^2 - 5R_0^4 + \partial_{yy}$$

is the operator obtained through linearization of the variational equation for $B(y,\tau)$ around the front. The function $R_0(y-y_0(\tau))$ satisfies the equation

$$-\frac{3}{16}R_0 + R_0^3 - R_0^5 + \partial_{yy}R_0 = 0.$$

Taking the derivative with respect to y one shows

$$L(y-y_0)R_{0y}(y-y_0) = 0.$$

The operator $L(y-y_0)$ is self-adjoint in the scalar product

$$\{f(y),g(y)\} = \int dy f(y)^* g(y),$$

where $f(y)^*$ stands for the complex conjugate of $f(y)$. We multiply the equation for $y_0(\tau)$ by $R_0(y-y_0(\tau))$, and we integrate over y. We obtain the solvability condition (putting $\tilde{\varepsilon}=1$)

$$-\{R_{0y},R_{0y}\}d_\tau y_0(\tau) = \{R_{0y},L\rho\} + \delta\sigma\{R_{0y},R_0\} + \int dy R_{0y}(y-y_0(\tau))\left(\frac{1}{9\tilde{\nu}}R_0^3 - R_0^5\right)\cos\left(2q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right) - \frac{1}{10}\int dy R_{0y}R_0(y-y_0(\tau))^5\cos\left(4q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right) + \frac{\beta\sqrt{\eta}}{2|\varepsilon|^2}\int dy R_{0y}(y-y_0(\tau))\cos\left(q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right)\zeta(y,\tau). \quad (11)$$

One has $\{R_{0y},R_{0y}\}=3/4=1/a$ and $\{R_{0y},R_0\}=3/8$. On the other hand, since L is self-adjoint, one has $\{R_{0y},L\rho\} = \{LR_{0y},\rho\}=0$ and we obtain an equation for $y_0(\tau)$ of the form ($\sqrt{\tilde{\eta}} \equiv a\beta\sqrt{\eta}/2|\varepsilon|^2$)

$$\dot{y}_0(\tau) = A(y_0(\tau)) - \sqrt{\tilde{\eta}}\int dy R_{0y}\cos\left(q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right)\zeta(y,\tau), \quad (12)$$

with

$$A(y_0(\tau)) \equiv -\frac{3}{8}a\,\delta\sigma + a\int dy\left[R_{0y}\left(-\frac{1}{9\tilde{\nu}}R_0^3 + R_0^5\right)\cos\left(2q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right)\right] + \frac{a}{10}\int dy R_{0y}R_0(y-y_0(\tau))^5\cos\left(4q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right).$$

In the equation for $y_0(\tau)$ the product of a function of the stochastic process $y_0(\tau)$ with the white noise $\zeta(y,\tau)$ is not defined due to the singular properties of the noise. We define it considering $\zeta(y,\tau)$ as the limit of a physical noise with time correlation proportional to a symmetric function $\Delta_\mu(\tau-\tau')$ of width μ, where μ is much smaller than the characteristic times of variation of the macroscopic physical variables, which tends to $\delta(\tau-\tau')$ when μ tends to zero. This leads to the Stratonovich interpretation for the undefined product [38]. In Appendix B we show that this gives a supplementary drift which is added to $A(y_0(\tau))$ and we transform the noise term to obtain (neglecting an exponentially small contribution to the last term)

$$\dot{y}_0(\tau) = A(y_0(\tau)) + \sqrt{\frac{\tilde{\eta}}{2a}}\xi(\tau) - \frac{\tilde{\eta}}{2}\int dy R_{0y}R_{0yy}\cos\left(2q\frac{y}{\alpha\sqrt{|\varepsilon|}}\right),$$

where $\xi(\tau)$ is a Gaussian white noise of zero mean value and correlation $\langle\xi(\tau)\xi(\tau')\rangle=\delta(\tau-\tau')$. If we make the change of variables $y'=y-y_0(\tau)$ in the last integral and in the integrals in the definition of $A(y_0(\tau))$, we obtain the equation (see Appendix B for the calculation)

$$d_\tau y_0(\tau) = -\frac{3}{2}a\delta\sigma\sqrt{K_1^2+K_2^2}\cos\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\varepsilon|}}+\phi\right) + \sqrt{\frac{\tilde{\eta}}{2a}}\zeta(\tau) + e^{-c2q/\sqrt{|\varepsilon|}}, \quad (13)$$

where $c\equiv\sqrt{4\pi/3\alpha}$. In the last equation K_1 and K_2 are not exponentially small (for small $|\varepsilon|$) and we have neglected a term $O(e^{-c4q/\sqrt{|\varepsilon|}})$. We have

$$K_1 = e^{c2q/\sqrt{|\varepsilon|}}\left[\mathrm{Re}(K) - \frac{9\tilde{\eta}}{128}\frac{q}{\alpha\sqrt{|\varepsilon|}}\,\mathrm{Im}I\right],$$

$$K_2 = e^{c2q/\sqrt{|\varepsilon|}}\left[\mathrm{Im}(K) + \frac{9\tilde{\eta}}{128}\frac{q}{\alpha\sqrt{|\varepsilon|}}\,\mathrm{Re}I\right],$$

$$\cos\phi = \frac{K_1}{\sqrt{(K_1)^2+(K_2)^2}},$$

$$\sin\phi = \frac{K_2}{\sqrt{(K_1)^2+(K_2)^2}}, \quad (14)$$

with

$$K = a\int dy\left[-\frac{1}{9\tilde{\nu}}R_{0y}(y)R_0(y)^3 + \frac{1}{2}R_{0y}(y)R_0(y)^5\right]e^{i2qy/\alpha\sqrt{|\varepsilon|}} = O(e^{-c2q/\sqrt{|\varepsilon|}}),$$

and

$$I = \int dy\frac{e^{-2\sqrt{(3/4)}y}}{(1+e^{-\sqrt{(3/4)}y})^3}e^{i2qy/\alpha\sqrt{|\varepsilon|}} = O(e^{-c2q/\sqrt{|\varepsilon|}}).$$

We have obtained then in Eq. (13) our final equation for the core of the front which tells us that the coordinate $y_0(\tau)$ of the core is a stochastic diffusion process defined by this equation. In the next section, we shall study Eq. (13) and show that it is at the origin of the motion of the front. We take care now of Eq. (10) which involves the phase Θ_1. There is no solvability condition here, and we have then to show that Θ_1 can be calculated and is bounded. Since $R_{0y}(y-y_0(\tau))$ vanishes for $y\to-\infty$, we integrate Eq. (10) from $-\infty$ to y to obtain

$$\begin{aligned}R_0(y-y_0(\tau))^2\Theta_{1y}(y,y_0(\tau)) &= \int_{-\infty}^{y}dy'\left[-\frac{1}{9\tilde{\nu}}R_0(y'-y_0(\tau))^4+\frac{1}{2}R_0^6\right]\sin\left(2q\frac{y'}{\alpha\sqrt{|\varepsilon|}}\right)\\ &+\int_{-\infty}^{y}dy'\frac{1}{10}R_0(y'-y_0(\tau))\sin\left(4q\frac{y'}{\alpha\sqrt{|\varepsilon|}}\right)\\ &+\frac{\sqrt{\tilde{\eta}}}{a}\int_{-\infty}^{y}dy' R_0(y'-y_0(\tau))\sin\left(q\frac{y'}{\alpha\sqrt{|\varepsilon|}}\right)\zeta(y',\tau).\end{aligned}$$

In the last term of this equation, we have the same problem as in Eq. (11)—i.e., an undefined product of a function of the process $y_0(\tau)$ with the white noise $\zeta(y,\tau)$ which we interpret in the Stratonovich sense. After a long calculation done in Appendix C, we obtain

$$\begin{aligned}R_0(y-y_0(\tau))^2\Theta_{1y} &= -\frac{1}{16\tilde{\nu}}|S^{(1)}(y-y_0(\tau))|\cos\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\varepsilon|}}-\varphi^{(1)}\right)\\ &+\frac{27}{128}|S^{(2)}(y-y_0(\tau))|\cos\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\varepsilon|}}-\varphi^{(2)}\right)\\ &+\frac{27}{640}|S^{(3)}(y-y_0(\tau))|\cos\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\varepsilon|}}-\varphi^{(3)}\right)\\ &-\frac{9\tilde{\eta}}{256a}|I(y-y_0(\tau))|\cos\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\varepsilon|}}-\varphi\right)\\ &+\frac{\sqrt{\tilde{\eta}}}{a}\left[\int_{-\infty}^{y}dy' R_0(y'-y_0(\tau))\times\sin\left(q\frac{y'}{\alpha\sqrt{|\varepsilon|}}\right)\zeta(y',\tau)\right]_{\gamma_1(0)},\end{aligned}$$

where $\gamma_1(0)$ in the noise term means that in a time discretization $y_0(\tau)$ has to be evaluated at the beginning of the time interval (prepoint discretization) [38], which corresponds to the Ito prescription, and consequently the mean value of this term vanishes. It is shown in Appendix C that for all values of y one has that $|S^{(j)}(y-y_0(\tau))|$ is bounded by $O(\sqrt{|\varepsilon|})$, $j=1,2,3$, while $|I(y-y_0(\tau))]$ is bounded by an exponentially small quantity. If we take then mean value of the above equation, we conclude that $\langle R_0(y-y_0(\tau))^2\Theta_{1y}\rangle$ remains bounded everywhere.

IV. EVOLUTION OF THE CORE OF THE FRONT

The evolution equation (13) for the core of the front can be written in the form

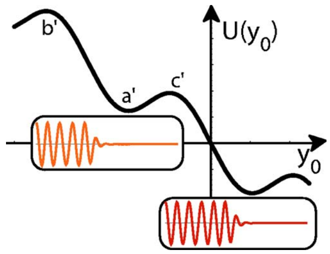

FIG. 8. (Color online) Schematic representation of the potential $U(y_0)$, when $0>\Delta>\Delta_-$ and $|\Delta|<|\Gamma|$. $\{a',b',c'\}$ are fixed points. The insets represent two equilibrium states of Eq. (3).

$$\dot{y}_0=-\frac{\partial U(y_0)}{\partial y_0}+\sqrt{\frac{\tilde{\eta}}{2a}}\xi(\tau) =\Delta+\Gamma\cos\left(\frac{2q}{\alpha\sqrt{|\varepsilon|}}y_0(\tau)+\phi\right)+\sqrt{\frac{\tilde{\eta}}{2a}}\xi(\tau), \quad (15)$$

where the potential has the expression

$$U(y_0)=\frac{3}{2}a\,\delta\sigma y_0-b\sin\left(2q\frac{y_0}{\alpha\sqrt{\varepsilon}}\right),$$

with $\Delta\equiv-\frac{3}{2}a\,\delta\sigma$, $\Gamma\equiv e^{-c2q/\alpha\sqrt{|\varepsilon|}}\sqrt{(K_1)^2+(K_2)^2}$, and $b\equiv\frac{\alpha\sqrt{|\varepsilon|}}{2q}\Gamma$.

Due to the interaction of the large scale with the small scale underlying the spatial periodic solution, the dynamics of the core of the front is modified with terms which are exponentially small and periodic in space. The above deterministic system is characterized by the spatial periodic state invading the homogeneous one with a well-defined velocity when $\Delta<0$ and $|\Delta|>|\Gamma|$. Increasing Δ, the system exhibits a simultaneous transition to infinite saddle nodes for $|\Delta_-|\equiv|\Delta|=|\Gamma|$. For $\Delta>\Delta_-$ and $|\Delta|<|\Gamma|$, the system has an infinite number of stable equilibria. Each equilibrium point represents a static front with different bumps (see Fig. 8). Increasing further Δ, all critical points disappear by the saddle node when $\Delta>0$ and $|\Delta_+|\equiv|\Delta|=|\Gamma|$. For $\Delta>\Delta_+$ the homogeneous state invades the spatial periodic one with a well-defined velocity. Therefore, for $\Delta_-<\Delta<\Delta_+$ (pinning range) the system exhibits the locking phenomena.

We now consider the effect of noise in Eq. (15). Due to the asymmetry of the potential, the system does not have a global stationary state and continuously converts the random fluctuations in directed motion of the front; i.e., the noise induces front propagation. This type of phenomena is well known as a Brownian motor [32]. One can easily understand the origin of this phenomena: if initially y_0 is inside the basin of attraction Ω of a fixed point; the front just fluctuates around the fixed point during a time of the order of the mean first passage time to $\partial\Omega$; the border of Ω; after this time the system makes a transition to the basin of attraction of the nearest stable fixed point separated from the first one by the lowest-energy barrier. This behavior is repeated in this new basin of attraction, and the final result is a directed motion of the front. Since the energy thresholds for jumping to the right or to the left are different, the probability of jumping to the side with the highest-energy threshold will be exponentially small with respect to the probability of jumping to the other side and this determines the direction of motion of the front.

V. MEAN VELOCITY OF THE FRONT: ANALYTICAL RESULTS

From the above analysis, we can estimate the mean velocity of the core of the front:

$$\langle v\rangle=\frac{\pi\alpha\sqrt{|\epsilon|}}{q}\left(\frac{1}{\tau_+}-\frac{1}{\tau_-}\right),$$

where $\pi\alpha\sqrt{|\epsilon|}/q$ is the distance between the two successive fixed points and $\{\tau_-,\tau_+\}$ are the escape times to move to the basins of attraction of the left or right fixed point, respectively. To calculate these escape times, we use Dynkin's equation or mean first passage time (MFPT) equation [39,40]. This equation describes the evolution of the stochastic variable first passage time (FPT) $\tau(\Omega,\partial\Omega;x_0)$, for a given domain Ω with boundary $\partial\Omega$ and initial condition x_0. The mean first passage times to each side of the basin of attraction where x_0 lies are called $\{\tau_-,\tau_+\}$, or escape times. Dynkin's equation for Eq. (15)) is [39,40]

$$\frac{\tilde{\eta}}{4a}\frac{d^2\tau}{dy_0^2}-\frac{\partial U[y_0]}{\partial y_0}\frac{d\tau}{dy_0}=-1, \quad (16)$$

with boundary condition $\tau(a')=\tau(b')\equiv0$, where a' and b' are two successive maxima of the potential (cf. Fig. 8). Integrating this equation, we compute the escape times. They have the expression

$$\left(\frac{1}{\tau_+}-\frac{1}{\tau_-}\right)^{-1}=\frac{2}{\theta}\int_{c'}^{b'}\int_{c'}^{y}e^{2[U[y]-U[z]]/\theta}dydz -\frac{2}{\theta}\int_{c'}^{a'}\int_{c'}^{y}e^{2[U[y]-U[z]]/\theta}dydz\left[\frac{\int_{c'}^{a'}e^{2U[y]/\theta}dy}{\int_{c'}^{b'}e^{2U[y]/\theta}dy}\right], \quad (17)$$

where c' is a maximum of the potential $U(y_0)$ (see Fig. 8) and, in this case, $\theta\equiv\tilde{\eta}/4a$. In the limit of weak noise, the expression for the mean velocity is

$$\langle v\rangle=\frac{2\sqrt{|\epsilon|}}{qa\sqrt{\partial_{yy}U(a')|\partial_{yy}U(c')}}e^{-[U(c')-U(a')]/\theta} \times\left(1-\sqrt{\frac{|\partial_{yy}U(c')|}{|\partial_{yy}U(b')|}}e^{-[U(b')-U(c')]/\theta}\right).$$

From the above expression one can find that in this limit the velocity is proportional to Kramer's rate. Numerically, we have measured the front velocity for different values of the noise intensity and we obtain good agreement with the theoretical prediction, as is shown in Fig. 9. It is important to remark that $U(y_0)$ is a function of the noise intensity. For finite noise intensity this dependence is dominant in the

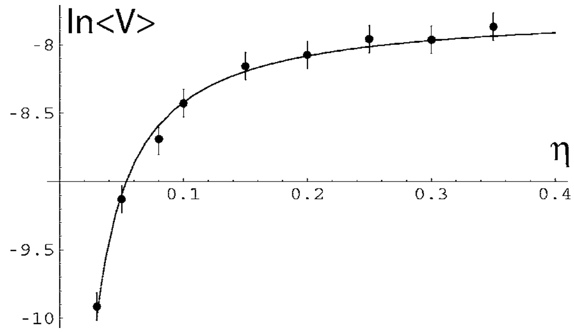

FIG. 9. Logarithm of the mean velocity of the front as a function of noise intensity. The solid lines are the analytical formula of the mean velocity and the dotted lines are the numerical measuring of the mean velocity of the front of Eq. (3).

terms K_1 and K_2, in the limit of $\epsilon \rightarrow 0$. Hence for finite noise intensity one only needs to consider the terms coming from the noise to explain the locking phenomena and the induced front propagation.

VI. DISCUSSION AND CONCLUSION

In order to understand the mechanism of noise-induced front propagation we have considered the subcritical Swift-Hohenberg equation. This model allows us to obtain analytical expressions for the mean velocity of the front. For an arbitrary model it is thorny to obtain explicit formulas for the front velocity, since in general we do not have access to explicit expressions of spatial periodic solutions. Given a system that exhibits locking phenomena between a spatial periodic state and a homogeneous state, we expect to find, close to a spatial bifurcation, an amended envelope equation since the spatial periodic solutions in the onset of the bifurcation are harmonic and coexists with the homogeneous state. Hence, one can use an ansatz similar to Eq. (6), and noticing that the envelope satisfies the symmetries $\{x \rightarrow -x, A \rightarrow \bar{A}\}$ and $\{x \rightarrow x + x_0, A \rightarrow A e^{iqx_0}\}$ [31], we can conclude that the amplitude equation has a form similar to Eq. (7) with real coefficients which can be written in the form

$$\partial_\tau A = f(|A|^2)A + \partial_{yy}A + \sum_{m,n} g_{mn} A^m \bar{A}^n e^{iq(1+n-m)x/\varepsilon}, \quad (18)$$

where the terms which have explicit exponential are nonresonant and rapidly varying in space. However, it is precisely due to these terms that we can explain the locking phenomena [30,31]. In the presence of additive noise in the original problem, we shall have an additive noise in Eq. (18), since noise is always resonant (cf. Appendix A) and the final equation will be of the form of Eq. (7), and once again the noise term will give the dominant contributions to the locking effect and to the motion of the front. We can conclude then that the noise induces front propagation due to the asymmetry of the core of the front potential and the lack of a global stationary state. Another way to understand this phenomenon is that noise prefers to create or remove a bump, because the necessary perturbations to nucleate or destroy a bump are different.

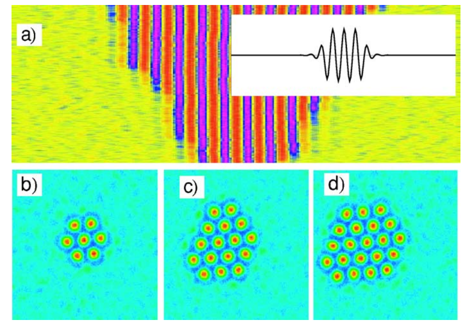

FIG. 10. (Color online) Noise-induced propagation of the interface of localized patterns. (a) Numerical simulations of the generalized Swift-Hohenberg model in one-extended system with additive noise. The inset is the initial condition. (b), (c), and (d) are different snapshots of the sequence time for numerical simulations of the generalized Swift-Hohenberg model in a 2D extended system with noise.

The existence, stability properties, and bifurcation diagrams of localized patterns in the pinning range in one-dimensional extended systems have recently been studied [29,41], from a dynamical point of view and front interaction, respectively. When we consider the effects of noise on these solutions, we expect, due to our previous discussion, propagation of the interface of these localized patterns. In Fig. 10 we show, in one and two extended dimensions, the noise-induced propagation of one state into the other. In one spatial dimension, one can then understand the localized pattern solutions as the interaction of two fronts [41]. In two-dimensional spatial systems, the understanding of the phenomena is in progress.

From the above results, one realizes that the localized patterns are unstable in nature—that is, in the presence of noise. The velocity of propagation of the interfaces and fronts is proportional to Kramer's rate. Therefore, experimentally, one can observe these localized patterns, when noise is weak enough, for long intervals of time, as metastable states.

In summary, we have studied the effect of internal noise in a motionless front that links a periodic spatial state with a homogeneous one. Noise induces front propagation; that is, one extended state invades the other one. In order to explain the mechanism of this phenomenon we have consider a prototype model of pattern forming, the subcritical Swift-Hohenberg equation with noise. From this model, we deduce an amended amplitude equation for the envelope and the core of the front. The equation of the core of the front is characterized by an asymmetrical periodic potential plus additive noise. The conversion of random fluctuations into direct motion of the core of the front is responsible for the propagation. We have obtained an analytical expression for the speed of the front, which is in good agreement with numerical simulations.

ACKNOWLEDGMENTS

The simulation software DIMX developed by P. Coullet and collaborators at the laboratory INLN in France has been used for all the numerical simulations. M.G.C. and E.T. acknowledge the support of FONDECYT Projects Nos. 1051117 and 1020374, ECOS-CONICYT collaboration program, and ring program ACT 15 of Programa Bicentenario.

APPENDIX A: STOCHASTIC NORMAL-FORM THEORY

We have stated in the text that noise is always resonant in the reduction to normal forms in the neighborhood of an instability. The discussion will follow closely the previous works. If we have a dynamical system

$$\partial_t \underline{U}(t) = \underline{F}(\underline{U}(t),\{\vec{\lambda}\}) + \sqrt{\eta}\underline{H}(t,\underline{U}(t)), \quad \text{(A1)}$$

where

$$\underline{U}(t) = \sum_{\alpha=1}^{N} U_\alpha(t)\underline{e}_\alpha,$$

$$\underline{F}(\underline{U}(t),\{\vec{\lambda}\}) = \sum_{\alpha=1}^{N} F_\alpha(\underline{U}(t),\{\vec{\lambda}\})\underline{e}_\alpha,$$

$$\underline{H}(t,\underline{U}(t)) = \sum_{\alpha=1}^{N} H_\alpha(t,\underline{U}(t))\underline{e}_\alpha$$

is a noise term, $\{\vec{\lambda}\}=\{\lambda_1,\ldots,\lambda_q\}$ is a set of control parameters, and $\sqrt{\eta}$ is a parameter measuring the intensity of the noise. Let E be the linear space of dimension N spanned by the basis vectors $(\underline{e}_1,\ldots,\underline{e}_N)$ and $\underline{U}^{(0)}(\{\vec{\lambda}\})$ a fixed point of Eq. (A1) for $\vec{\lambda}\in\Omega$, where Ω is a domain in the space of parameters $\{\vec{\lambda}\}=\{\lambda_1,\ldots,\lambda_q\}$. If we linearize the deterministic part $\partial_t\underline{U}=\underline{F}$ of Eq. (A1) around $\underline{U}^{(0)}(\{\vec{\lambda}\})$ putting $\underline{U}=\underline{U}^{(0)}\times(\{\vec{\lambda}\})+\underline{V}$, we find a linear equation for $\underline{V}$ of the form

$$\partial_t \underline{V} = \hat{\Lambda}(\{\vec{\lambda}\})\underline{V}. \quad \text{(A2)}$$

The solution $\underline{U}^{(0)}(\{\vec{\lambda}\})$ is stable if all the eigenvalues of the operator $\hat{\Lambda}(\{\vec{\lambda}\})$ have negative real parts. If we move $\vec{\lambda}$ in Ω and arrive at a critical point $\vec{\lambda}^{(c)}$ in the space of parameters ($\vec{\lambda}^{(c)}\in\bar{\Omega}$, $\bar{\Omega}$ the closure of Ω) where the elements of a set of eigenvalues $(\sigma_1,\ldots,\sigma_m)$ have zero real parts, while the rest of the eigenvalues $(\sigma_{m+j}, j\geq 1)$have negative real parts such that $\mathrm{Re}|\sigma_j|\geq d, j\geq m+1$, where d is a fixed quantity, then the solution $\underline{U}^{(0)}(\{\vec{\lambda}\})$ has lost its stability and the operator $\hat{\Lambda}(\{\vec{\lambda}\})\equiv\hat{L}^{(0)}$ has two invariant subspaces, the critical subspace $E^{(0)}$ and the stable space $E^{(-)}$. Let $(\underline{\chi}^{(1)},\ldots,\underline{\chi}^{(n)},\underline{\chi}^{(n+1)},\ldots,\underline{\chi}^{(N)})$ be the Jordan basis of E associated with the operator $\hat{L}^{(0)}$.One has $\hat{L}^{(0)}\underline{\chi}^{(\alpha)}=\Sigma_{\beta=1}^{N}L_{\beta\alpha}^{(0)}\underline{\chi}^{(\beta)}$, $\alpha=1,2,\ldots,N$, where $L_{\beta\alpha}^{(0)}$ is a Jordan matrix. The critical subspace $E^{(0)}$ is spanned by $(\underline{\chi}^{(1)},\ldots,\underline{\chi}^{(n)})$ and $E^{(-)}$ by $(\underline{\chi}^{(n+1)},\ldots,\underline{\chi}^{(N)})$. In the $\hat{L}^{(0)}$-invariant subspace $E^{(0)}$the operator $\hat{L}^{(0)}$ has all its eigenvalues with zero real part $(\sigma_1,\ldots,\sigma_m)$, $m\leq n$, and $\hat{L}^{(0)}\underline{\chi}^{(\alpha)}=\Sigma_{\beta=1}^{n}J_{\beta\alpha}\underline{\chi}^{(\beta)}$, $\alpha=1,2,\ldots,n$, where $J_{\alpha\beta}$ is an nxn Jordan matrix, and $J_{\alpha\beta}=L_{\beta\alpha}^{(0)}$, $1\leq\alpha,\beta\leq n$. If $\underline{X}=\Sigma_{\alpha=1}^{N}X_\alpha\underline{\chi}^{(\alpha)}\in E$,we define the operators $P_{(0)}$ projecting on $E^{(0)}$ and $P_{(-)}$ projecting on $E^{(-)}$ by

$$P_{(0)}\underline{X} = \sum_{\alpha=1}^{n} X_\alpha\underline{\chi}^{(\alpha)}, \quad P_{(-)}\underline{X} = \sum_{\alpha=n+1}^{N} X_\alpha\underline{\chi}^{(\alpha)}. \quad \text{(A3)}$$

If we are in a neighborhood of the critical point $\vec{\lambda}^{(c)}$, we put $\vec{\lambda}=\vec{\lambda}^{(c)}+\vec{\delta\lambda}$, $\underline{U}(t)=\underline{U}^{(0)}(\{\vec{\lambda}^{(c)}\})+\underline{V}$, and then Eq. (A1) (without the noise term) is written as

$$\partial_t\underline{V} = [\hat{L}^{(0)}\underline{V} + \underline{N}^{(0)}(\underline{V})] + [\underline{D} + \hat{L}^{(1)}\underline{V} + \underline{N}^{(1)}(\underline{V})], \quad \text{(A4)}$$

where the terms in the first set of square brackets on the right-hand side are of order zero in $\{\vec{\delta\lambda}\}$ and in $\sqrt{\eta}$, and the terms in the second set of square brackets of order 1 (or more) in the unfolding parameters $\{\vec{\delta\lambda}\}$ and zero in $\sqrt{\eta}$. If $\underline{D}=0$, where $\underline{D}$ is a constant vector belonging to E, the fixed point $\underline{U}^{(0)}(\{\vec{\lambda}^{(c)}\})$ is persistent in a neighborhood of $\{\vec{\lambda}^{(c)}\}$, $\hat{L}^{(j)}\underline{V}$ are linearterms in $\underline{V}$, and $\underline{N}^{(j)}(\underline{V})$ are nonlinear in $\underline{V}$ $(j=1,2)$. In order to construct the normal form of Eq. (A4) we make the ansatz

$$\underline{V} = [\underline{U}^{[1,0]}(\vec{A}) + \underline{U}^{[2,0]}(\vec{A}) + \cdots] + [\underline{U}^{[0,1]}(\vec{A}) + \underline{U}^{[1,1]}(\vec{A}) + \cdots], \quad \text{(A5)}$$

where $\underline{(X)}^{[n_1,n_2]}$ stands for the part of $\underline{X}$ which is of polynomial order n_1 in $\vec{A}=(A_1,\ldots,A_n)$, order n_2 in the unfolding parameters $\{\vec{\delta\lambda}\}$, and zero in $\sqrt{\eta}$. In Eq. (A5) we have $\underline{U}^{[1,0]}(\vec{A})=\Sigma_{\alpha=1}^{n}A_\alpha\underline{\chi}^{(\alpha)}$ and the normal form is the " simplest" equation involving only the critical variables $\vec{A}=(A_1,\ldots,A_n)$. As shown and discussed in detail in [42].The ansatz (A5) leads to self-contained equations for $\vec{A}$ (the normal form) which are

$$\partial_t A_\alpha = [f_\alpha^{[1,0]}(\vec{A}) + f_\alpha^{[2,0]}(\vec{A}) + \cdots] + [f_\alpha^{[0,1]}(\vec{A}) + f_\alpha^{[1,1]}(\vec{A}) + \cdots], \quad \text{(A6)}$$

where $f_\alpha^{[1,0]}(\vec{A})=J_{\alpha\beta}A_\beta$. Adding the noise term in Eq. (A1) to Eq. (A4) we obtain

$$\partial_t\underline{V} = [\hat{L}^{(0)}\underline{V} + \underline{N}^{(0)}(\underline{V})] + [\underline{D} + \hat{L}^{(1)}\underline{V} + \underline{N}^{(1)}(\underline{V})] + [\underline{\tilde{G}}(t) + \hat{S}(t)\underline{V} + \underline{N}^{(2)}(t,\underline{V})], \quad \text{(A7)}$$

where $\underline{\tilde{G}}(t)$ is an additive noise, $\hat{S}(t)\underline{V}$ a linear multiplicative noise, etc., and the added terms are of order $\sqrt{\eta}$. In order to obtain the stochastic unfolding we change the ansatz (A5) adding in the right-hand side a third set of terms

$$\underline{V} = [\underline{U}^{[1,0]}(\vec{A}) + \underline{U}^{[2,0]}(\vec{A}) + \cdots] + [\underline{U}^{[0,1]}(\vec{A}) + \underline{U}^{[1,1]}(\vec{A}) + \cdots] + [\underline{U}^{[0,0,1]}(t) + \underline{U}^{[1,0,1]}(t,\vec{A}) + \cdots], \quad \text{(A8)}$$

where the notation $\underline{(X)}^{[n_1,n_2,n_3]}$ stands for the part of $\underline{X}$ which is of polynomial order n_1 in $\vec{A}$, n_2 in $\{\vec{\delta\lambda}\}$, and n_3 in $\sqrt{\eta}$. The normal form (A6) will be now $(\alpha=1,2,\ldots,n)$

$$\partial_t A_\alpha = [f_\alpha^{[1,0]}(\vec{A}) + f_\alpha^{[2,0]}(\vec{A}) + \cdots] + [f_\alpha^{[0,1]}(\vec{A}) + f_\alpha^{[1,1]}(\vec{A}) + \cdots] + [\tilde{f}_\alpha^{[0,0,1]}(t) + \tilde{f}_\alpha^{[1,0,1]}(t,\vec{A}) + \cdots], \tag{A9}$$

We remark that the ansatz (A7) can be derived from a general stochastic nonlinear change of variables; i.e., the coefficients of the change of variables are now stochastic processes [43] in a similar way to what is done for the unfolding of the parameters or for the unfolding due to the addition of periodic perturbations to the original equation [44].

The problem now is to calculate the stochastic terms of the ansatz (A9)—i.e.,

$$\underline{U}^{[j,0,1]}(t,\vec{A}) = \sum_{\substack{\beta=1,\ldots,N\\ \alpha_i=1,\ldots,n}} U_{\alpha_1\cdots\alpha_j;\beta}(t) A^{\alpha_1}\ldots A^{\alpha_j}\underline{\chi}^{(\beta)}, \tag{A10}$$

and the new random terms

$$\tilde{f}_\alpha^{[j,0,1]}(t,\vec{A}) = \sum_{\alpha_i=1}^{n} \tilde{f}_{\alpha;\alpha_1\cdots\alpha_j}(t) A^{\alpha_1}\cdots A^{\alpha_j} \tag{A11}$$

in the stochastic normal form. We call $\underline{\Xi}$ and $\underline{\Phi}$ the left- and right-hand sides of Eq. (A7). Then we have

$$\underline{\Xi}^{[0,0,1]} \equiv (\partial_t \underline{V})^{[0,0,1]} = (\partial_t A_\alpha)^{[0,0,1]}\frac{\partial \underline{U}^{[1,0]}(\vec{A})}{\partial A_a} + \hat{\partial}_t \underline{U}^{[0,0,1]}(t) = \tilde{f}_\alpha^{[0,0,1]}(t)\underline{\chi}^{(\alpha)} + \hat{\partial}_t \underline{U}^{[0,0,1]}(t), \tag{A12}$$

where $\hat{\partial}_t$ stands for the time derivative with respect to the explicit time dependence—i.e., the one in the random functions and not the one in $\{A_\alpha(t)\}$—and

$$\underline{\Phi}^{[0,0,1]} = \hat{L}^{(0)}\underline{U}^{[0,0,1]}(t) + \underline{\tilde{G}}(t). \tag{A13}$$

Then $\underline{\Xi}^{[0,0,1]}=\underline{\Phi}^{[0,0,1]}$ gives [putting $\underline{\tilde{G}}(t)=\sqrt{\eta}\underline{G}(t)$, $\tilde{f}_\alpha^{[j,0,1]} =\sqrt{\eta} f_\alpha^{[j,0,1]}$] gives

$$(\hat{\partial}_t - \hat{L}^{(0)})\underline{U}^{[0,0,1]}(t) = \sqrt{\eta}\underline{G}(t) - \sum_{\alpha=1}^{n}\sqrt{\eta} f_\alpha^{[0,0,1]}(t)\underline{\chi}^{(\alpha)}. \tag{A14}$$

Projecting with $\hat{P}_0$ ($\hat{L}_0^{(0)}=\hat{P}_0\hat{L}^{(0)}\hat{P}_{(0)}$) we obtain

$$(\hat{\partial}_t - \hat{L}_0^{(0)})\hat{P}_0\underline{U}^{[0,0,1]}(t) = \hat{P}_0\sqrt{\eta}\underline{G}(t) - \sum_{\alpha=1}^{n}\sqrt{\eta} f_\alpha^{[0,0,1]}(t)\underline{\chi}^{(\alpha)}. \tag{A15}$$

Since $\hat{L}_0^{(0)}$has eigenvalues with zero real part, the linear stochastic differential equation (A13) has no stationary solution. One has $\underline{G}(t)=\Sigma_{\alpha=1}^N G_\alpha(t)\underline{\chi}^{(\alpha)}$, $P_0\underline{G}(t)=\Sigma_{\alpha=1}^n G_\alpha(t)\underline{\chi}^{(\alpha)}$, and if we assume that the random functions $\{G_\alpha(t), \alpha=1,\ldots,n\}$ are δ-correlated white noises with zero mean and correlations $\langle G_\alpha(t)G_\beta(t')\rangle=\delta^{\alpha\beta}\delta(t-t')$, then Eq. (A13) is a Markov process whose conditional probability density $P[\hat{P}_0\underline{U}^{[0,0,1]}(t) =\underline{X}|\hat{P}_0\underline{U}^{[0,0,1]}(0)=\underline{X}^{(0)}]$ has no limit when $t\to\infty$, and then one has no stationary probability and no stationary state. We solve then Eq. (A15) choosing $\hat{P}_0\underline{U}^{[0,0,1]}(t)=0$ and consequently

$$f_\alpha^{[0,0,1]}(t) = G_\alpha(t), \quad \alpha=1,2,\ldots,n. \tag{A16}$$

This equation tells us that additive noise is always resonant as stated in the text. We can go to higher-order polynomial orders in $\vec{A}$ and the conclusion will be the same. Let us calculate the terms in the linear stochastic unfolding; for this we need the projection $P_-\underline{U}^{[0,0,1]}(t)$ on the stable subspace E_-. We project Eq. (A14) with P_- to obtain

$$\hat{\partial}_t - \hat{L}_-^{(0)}\hat{P}_-\underline{U}^{[0,0,1]}(t) = P_-\sqrt{\eta}\underline{G}(t); \tag{A17}$$

Since all the eigenvalues of $\hat{L}_-^{(0)}\equiv P_-\hat{L}P_-$ have negative real parts, this linear equation has can be solved for $\hat{P}_-\underline{U}^{[0,0,1]}(t)$ and we can go to the next order in $\sqrt{\eta}$. One has

$$\underline{\Xi}^{[1,0,1]} \equiv (\partial_t\underline{V})^{[1,0,1]} = (\partial_t A_\alpha)^{[1,0,1]}\frac{\partial\underline{U}^{[1,0]}(\vec{A})}{\partial A_\alpha} + \hat{\partial}_t\underline{U}^{[1,0,1]}(t,\vec{A}) + f_\alpha^{[1,0]}\frac{\partial\underline{U}^{[1,0,1]}(\vec{A})}{\partial A_\alpha} + (\partial_t A_\alpha)^{[0,0,1]}\frac{\partial\underline{U}^{[2,0]}}{\partial A_\alpha}, \tag{A18}$$

$$\underline{\Phi}^{[1,0,1]} = \hat{L}^{(0)}\underline{U}^{[1,0,1]}(t,\vec{A}) + \underline{N}_2^{(0)}(\underline{V}=\underline{U}^{[1,0]}(\vec{A}),\underline{V}=\underline{U}^{[0,0,1]}(t)) + \underline{N}_2^{(0)}(\underline{V}=\underline{U}^{[0,0,1]}(t),\underline{V}=\underline{U}^{[1,0]}(\vec{A})) + \hat{S}(t)\underline{U}^{[1,0]}(\vec{A}). \tag{A19}$$

Then $\underline{\Xi}^{[1,0,1]}=\underline{\Phi}^{[1,0,1]}$ gives ($f_\alpha^{[1,0]}=J_{\alpha\beta}A^\beta$)

$$\left(\hat{\partial}_t + J_{\alpha\beta}A^\beta\frac{\partial}{\partial A^\alpha} - \hat{L}^{(0)}\right)\underline{U}^{[1,0,1]}(t,\vec{A}) = \underline{I}^{[1,0,1]}(t,\vec{A}) - \sum_{\alpha=1}^{n} f_\alpha^{[1,0,1]}(t,\vec{A})\underline{\chi}^{(\alpha)}, \tag{A20}$$

with $\underline{I}^{[1,0,1]}(t,\vec{A})=\Sigma_{\substack{\beta=1,\ldots,n\\ \alpha=1,\ldots,N}} I_{\beta\alpha}^{[1,0,1]}(t)A_\beta\underline{\chi}^{(\alpha)}$. The operator on the right-hand side is $\hat{\partial}_t+\Pi(\hat{L}^{(0)})$ where $\Pi(\hat{L}^{(0)})\equiv J_{\alpha\beta}A^\beta\frac{\partial}{\partial A^\alpha} -\hat{L}^{(0)}$ is the homological operator and $\underline{I}^{[1,0,1]}(t,\vec{A})$ can be read directly from Eqs. (A18) and (A19). Projecting Eq. (A20) with $\hat{P}_0$we obtain

$$\left(\hat{\partial}_t + J_{\alpha\beta}A^\beta\frac{\partial}{\partial A^\alpha} - \hat{L}_0^{(0)}\right)\hat{P}_0\underline{U}^{[1,0,1]}(t,\vec{A}) = \hat{P}_0\underline{I}^{[1,0,1]}(t,\vec{A}) - \sum_{\alpha=1}^{n} f_\alpha^{[1,0,1]}(t,\vec{A})\underline{\chi}^{(\alpha)}. \tag{A21}$$

Then, since $\left(J_{\alpha\beta}A^\beta\frac{\partial}{\partial A^\alpha} - \hat{L}_0^{(0)}\right)$ has eigenvalues with zero real part, we have the same situation as in Eq. (A15) and we must solve (A21) putting $\widehat{P}_0\underline{U}^{[1,0,1]}(t,\vec{A})=0$ and then

$$f_\alpha^{[1,0,1]}(t,\vec{A}) = \sum_{\substack{\beta=1,\ldots,n\\ \alpha=1,\ldots,n}} I_{\beta\alpha}^{[1,0,1]}(t)A_\beta. \quad (A22)$$

This equation is the analog of Eq. (A16) for linear multiplicative noise, and once again we find that we have to keep all the noise terms. For a complete discussion of the stochastic center manifold see [37].

APPENDIX B: DERIVATION OF THE EQUATION FOR THE CORE OF THE FRONT

We call M the noise term in Eq. (12), where the product of the function of the stochastic process $y_0(\tau)$ with the white noise $\zeta(y,\tau)$ is interpreted in the Stratonovich sense. We discretize the variable y as $\{y_j=\Delta j, j\in Z\}$, i.e., $y_j-y_{j-1}=\Delta$, and then the discrete form of the noise correlation is $\langle\zeta(y_l,\tau)\zeta(y_{l'},\tau')\rangle=\delta(y_l-y_{l'})\delta(\tau-\tau')=\frac{\delta_{l,l'}}{\Delta}\delta(\tau-\tau')$. We define $\tilde{\zeta}_l(\tau)\equiv\Delta^{1/2}\zeta(y_l,\tau)$, which has correlation $\langle\tilde{\zeta}_l(\tau)\tilde{\zeta}_l'(\tau')\rangle=\delta_{l,l'}\delta(\tau-\tau')$. With this, M can be written in the form

$$M=\sqrt{\tilde{\eta}}\Delta\sum_l a_l(y_0(\tau))\frac{\tilde{\zeta}_l(\tau)}{\sqrt{\Delta}}, \quad (B1)$$

with

$$a_l(y_0(\tau))\equiv(-1)R_{0y}(y_l-y_0(\tau))\cos\left(\frac{qy_l}{\alpha\sqrt{|\epsilon|}}\right), \quad (B2)$$

and Eq. (12) takes the form

$$\dot{y}_0(\tau)=A(y_0(\tau))+M\equiv A(y_0(\tau))+\sqrt{\tilde{\eta}\Delta}\sum_l a_l(y_0(\tau))\tilde{\zeta}_l(\tau). \quad (B3)$$

If we discretize the time τ as $\tau_j=\beta j$, $j\in Z$, $\beta=\tau_j-\tau_{j-1}$, we can write Eq. (B3) for the variables $y_0(\tau_j)=y_{0,j}$. Defining $\Delta y_{0,j}=y_{0,j}-y_{0,j-1}$,

$$\Delta y_{0,j}=\beta A(y_{0,j-1})+\sqrt{\tilde{\eta}\Delta}\sum_l a_l\left(y_{0,j-1}+\frac{1}{2}\Delta y_{0,j}\right)\Delta w_{lj}, \quad (B4)$$

where we have discretized $a_l(y_0(\tau))$ in the midpoint of the interval $[t_{j-1},t_j]$ due to the Stratonovich prescription. In Eq. (B4) one has $\Delta w_{lj}\equiv w_{lj}-w_{lj-1}$, $w_{lj}=w_l(\tau_j)$, where $\{w_l(\tau), l\in Z\}$ are independent Wiener processes defined by $dw_l(\tau)=\tilde{\zeta}_l(\tau)d\tau$, and consequently $\Delta w_{lj}\Delta w_{l'j}=\beta\delta_{l,l'}$. Since $\Delta y_{0,j}$ is of order $\beta^{1/2}$, we obtain up to order β the expression

$$\Delta y_{0,j}=\beta A(y_{0,j-1})+\frac{\sqrt{\tilde{\eta}\Delta}}{2}\sum_l a_l'(y_{0,j-1})\Delta y_{0,j}\Delta w_{lj} + \sqrt{\tilde{\eta}\Delta}\sum_l a_l(y_{0,j-1})\Delta w_{lj}, \quad (B5)$$

where the prime denotes derivative with respect to the argument. Using the fact that Δw_{lj} is of order $\beta^{1/2}$, the dominant term in the latter equation is

$$\Delta y_{0,j}=\sqrt{\tilde{\eta}\Delta}\sum_l a_l(y_{0,j-1})\Delta w_{lj}. \quad (B6)$$

Replacing expression (B6) in the right hand side of equation (B5) and using $\Delta w_{lj}\Delta w_{l'j}=\beta\delta_{l,l'}$ we obtain

$$\Delta y_{0,j}=\beta\left[A(y_{0,j-1})+\frac{\tilde{\eta}\Delta}{2}\sum_l a_l'(y_{0,j-1})a_l(y_{0,j-1})\right] + \sqrt{\tilde{\eta}\Delta}\sum_l a_l(y_{0,j-1})\Delta w_{lj}. \quad (B7)$$

The stochastic differential equation for $y_0(\tau)$ is now written in the prepoint discretization (Ito prescription). The noise term is a sum of independent white noises $\sqrt{\tilde{\eta}\Delta}\Sigma_l a_l(y_{0,j-1})\Delta w_{lj}$ which can be replaced by a white noise $\sqrt{\tilde{\eta}\tilde{\Omega}}\zeta(\tau)$ with

$$\tilde{\Omega}=\Delta\sum_l a_l(y_0(\tau))^2 = \frac{1}{2}\int_{-\infty}^{\infty}dyR_{0y}(y-y_0(\tau))^2\left[1+\cos\left(\frac{2qy}{\alpha\sqrt{|\epsilon|}}\right)\right],$$

and $\zeta(\tau)$ is a white noise of zero mean and correlation $\langle\zeta(\tau)\zeta(\tau')\rangle=\delta(\tau-\tau')$. The contribution of the cosine in the integral gives an exponentially small contribution of order $O(e^{-c2q/|\epsilon|^{1/2}})$ with $c=\frac{2\pi}{\sqrt{3}\alpha}$, and consequently we approximate $\tilde{\Omega}$ by

$$\Omega=\frac{1}{2}\int_{-\infty}^{\infty}dyR_{0y}(y-y_0(\tau))^2=1/2a.$$

On the other hand, the second term on the right-hand side of Eq. (B7) is

$$\frac{\Delta}{2}\sum_l a_l(y_{0,j-1})a_l'(y_{0,j-1}) = \frac{1}{4}\int_{-\infty}^{\infty}dy\left[1+\cos\left(\frac{2qy}{\alpha\sqrt{|\epsilon|}}\right)\right]\frac{d}{dy}R_{0y}(y-y_0(\tau))^2. \quad (B8)$$

Since $\int_{-\infty}^{\infty}R_{0y}R_{0yy}=0$, we can finally write the stochastic differential equation for $y_0(\tau)$ in the form

$$\dot{y}_0(\tau)=A(y_0(\tau))+\frac{\tilde{\eta}}{4}\int_{-\infty}^{\infty}dyR_{0y}(y)R_{0yy}(y)\cos\left(\frac{2q(y+y_0(\tau))}{\alpha\sqrt{|\epsilon|}}\right) + \sqrt{\frac{\tilde{\eta}}{2a}}\zeta(\tau). \quad (B9)$$

After doing all the integrals in Eq. (B9), we obtain the following equation:

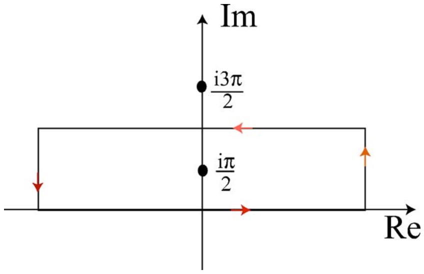

FIG. 11. (Color online) Contour of integration Γ on the complex plane.

$$\dot{y}_0(\tau) = -\frac{3a\delta\sigma}{2} + \mathrm{Re}(K)\cos\left(\frac{2qy_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right) - \mathrm{Im}(K)\sin\left(\frac{2qy_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right) + \mathrm{Re}(J)\cos\left(\frac{4qy_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right) - \mathrm{Im}(J)\sin\left(\frac{4qy_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right) - \frac{\tilde{\eta}}{4}\left[\mathrm{Re}(S)\cos\left(\frac{2qy_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right)\right] - \mathrm{Im}(S)\sin\left(\frac{2qy_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right) + \frac{\sqrt{\tilde{\eta}}}{4}\zeta(\tau), \tag{B10}$$

where $S=-\frac{iq}{\alpha\sqrt{|\epsilon|}}\frac{9}{64}I$, I and K are defined in expressions (14) in the text, and J is defined by

$$J \equiv \frac{a}{10}\int_{-\infty}^{\infty} dy R_{0y}(y)R_0(y)^5 e^{4qy_0(\tau)/\alpha\sqrt{|\epsilon|}}.$$

We show below that $K\sim O(e^{-c2q/\sqrt{|\epsilon|}})$, $I\sim O(e^{-c2q/\sqrt{|\epsilon|}})$, and $J\sim O(e^{-c4q/\sqrt{|\epsilon|}})$; consequently, we can neglect in Eq. (B10) the terms proportional to J. We obtain

$$\dot{y}_0(\tau) = -\frac{3a\delta\sigma}{2} + e^{-c2q/\sqrt{|\epsilon|}}\left[K_1\cos\left(\frac{2qy_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right) + K_2\sin\left(\frac{2qy_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right)\right] + \frac{\sqrt{\tilde{\eta}}}{4}\zeta(\tau), \tag{B11}$$

where K_1 and K_2 are of order 1 and defined in the text. Finally Eq. (B11) coincides with Eq. (13) in the text. We obtain now the estimates for the oscillatory integrals K, I, and J. We define $\tilde{I}$ (μ is a positive number),

$$\tilde{I} \equiv \int_{-\infty}^{\infty} dy \frac{e^{-\gamma y}}{(1+e^{-\nu y})^n} e^{i\omega y/\mu}.$$

Using the contour Γ in Fig. 11, we consider the integral in the complex plane I. We call I_L the integral over Γ_1:

$$I_L = \lim_{L\to\infty}\int_{-L}^{L} dx \frac{e^{-\gamma x}}{(1+e^{-\nu x})^n} e^{i\omega x/\mu} = \tilde{I}.$$

The explicit calculation of the integral over the contour Γ gives, in the limit $L\to\infty$, the result

$$I = (1 - e^{-2\omega\pi/\mu\nu})I_L. \tag{B12}$$

The residue theorem gives $I=2\pi i\alpha^{(0)}$, where $\alpha^{(0)} \approx e^{-\omega\pi/\mu\nu}P(\frac{i\omega}{\mu})$ is the residue in the pole $z=i\pi/\nu$ and $P(i\omega/\mu)$ is a polynomial with maximum degree $(2n-3)$. We obtain then for $\tilde{I}$ the final result

$$\tilde{I} = e^{-\omega\pi/\mu\nu}P\left(\frac{i\omega}{\mu}\right) + O(e^{-2\omega\pi/\mu\nu}). \tag{B13}$$

APPENDIX C: PHASE OF THE CORE OF THE FRONT REMAINS BOUNDED

We shall study here the stochastic process $R_0(y-y_0(\tau))^2\Theta_{1y}(y,y_0(\tau))$, which involves the derivative of the phase $\Theta_1(y,y_0(\tau))$, and prove that the mean value of $\langle R_0^2\Theta_{1y}\rangle$ is bounded for all values of τ. The value of $R_0^2\Theta_{1y}$ is given by Eq. (10) in the text, and we can see there that we have in the last term an undefined product of a function of $y_0(\tau)$ with the noise $\zeta(y,\tau)$. As explained before we have to interpret this product with the Stratonovich prescription. We write this noise term as $\frac{\sqrt{\tilde{\eta}}}{4}Q$, with

$$Q = \int_{-\infty}^{y} dy' R_0(y'-y_0(\tau))\sin\left(\frac{qy'}{\alpha\sqrt{|\epsilon|}}\right)\zeta(y',\tau). \tag{C1}$$

We proceed as in Appendix B and we use the same notation. The discretized version of Eq. (C1) involves using the midpoint discretization for the τ dependence in $y_0(\tau)$ due to the Stratonovich prescription. Then $R_0(y-y_{0,j-1}(\tau)+\frac{1}{2}\Delta y_{0,j}) = R_0(y-y_{0,j-1}(\tau))+\frac{1}{2}\Delta y_{0,j})+\frac{1}{2}\Delta y_{0,j}R_0'(y-y_{0,j-1}(\tau))$, and one has

$$Q = \Delta^{1/2}\sum_l R_0(y_l-y_{0,j-1})\sin\left(\frac{qy_l}{\alpha\sqrt{|\epsilon|}}\right)\tilde{\zeta}_{l,j}\Big|_{\gamma_1(0)} + \Delta^{1/2}\sum_l \frac{1}{2}R_{0y}(y_l-y_{0,j-1})\sin\left(\frac{qy_l}{\alpha\sqrt{|\epsilon|}}\right)\Delta y_{0,j}\frac{\Delta w_{lj}}{\beta}, \tag{C2}$$

where γ_0 stands for the prepoint discretization. Replacing $\Delta y_{0,j}$ using Eq. (B6) and the correlation of the Wiener processes, we obtain $Q=Q_1+Q_2$ with

$$Q_1 = -\frac{\sqrt{\tilde{\eta}}}{2}\int_{-\infty}^{y} dy' R_{0y}(y'-y_0(\tau))^2\sin\left(\frac{2qy'}{\alpha\sqrt{|\epsilon|}}\right), \tag{C3}$$

$$Q_2 = \int_{-\infty}^{y} dy' R_0(y'-y_0(\tau))\sin\left(\frac{qy'}{\alpha\sqrt{|\epsilon|}}\right)\zeta(y',\tau)\Big|_{\gamma_1(0)}. \tag{C4}$$

We replace now in expression (6) for $R_0^2\Theta_{1y}$ the last term $\frac{\sqrt{\tilde{\eta}}}{a}Q$ using Eqs. (C3) and (C4) and we perform in all the integrals (except in Q_2) the change of variables $y=y'-y_0(\tau)$. The final result is

$$R_0(y-y_0(\tau))^2\Theta_{1y}(y,y_0(\tau)) = -\frac{1}{16\sigma}\Bigg[\mathrm{Im}\Bigg[S^{(1)}(y-y_0(\tau)]\cos\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right)+\mathrm{Re}\Bigg[S^{(1)}(y-y_0(\tau))\sin\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right)\Bigg]$$
$$+\frac{27}{128}\mathrm{Im}S^{(2)}(y-y_0(\tau))\cos\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right)+\mathrm{Re}\Bigg[S^{(2)}(y-y_0(\tau))\sin\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right)\Bigg]$$
$$+\frac{27}{240}\Bigg[\mathrm{Im}\,S^{(3)}(y-y_0(\tau))\cos\left(4q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right)+\mathrm{Re}\,S^{(3)}(y-y_0(\tau))\sin\left(4q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right)\Bigg]$$
$$-\frac{9\tilde{\eta}}{256a}\left(\mathrm{Im}\,I(y-y_0(\tau))\cos\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right)+\mathrm{Re}\Bigg[I(y-y_0(\tau))\sin\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}\right)\Bigg]\right.$$
$$\left.+\frac{\tilde{\eta}^{1/2}}{a}\int_{-\infty}^{y}dy' R_0(y'-y_0(\tau))\sin\left(2q\frac{y}{\alpha\sqrt{|\epsilon|}}\right)\zeta y',\tau\right)\Bigg|_{\gamma_0}, \tag{C5}$$

where $(j=1,2)$

$$S^{(j)}(y-y_0(\tau))\int_{-\infty}^{y-y_0(\tau)}dy'\frac{1}{(1+e^{-\sqrt{(3/4)}y'})^{1+j}}e^{i2qy'/\alpha|\epsilon|^{1/2}},$$

$$S^{(3)}(y-y_0(\tau))=\int_{-\infty}^{y-y_0(\tau)}dy'\frac{1}{(1+e^{-\sqrt{(3/4)}y'})^{3}}e^{i4qy'/\alpha|\epsilon|^{1/2}},$$

$$I(y-y_0(\tau))=\int_{-\infty}^{y-y_0(\tau)}dy'\frac{e^{-2\sqrt{(3/4)}y'}1}{(1+e^{-\sqrt{(3/4)}y'})^{1+j}}e^{i2qy'/\alpha|\epsilon|^{1/2}}. \tag{C6}$$

We remark that $I(y-y_0(\tau))$ is bounded and moreover (see Appendix A). Expression (C5) can be written in the form

$$R_0(y-y_0(\tau))^2\Theta_{1y}(y,y_0(\tau))$$
$$=-\frac{1}{16\sigma}|S^{(1)}(y-y_0(\tau))|\cos\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}-\varphi^{(1)}\right)$$
$$+\frac{27}{128}|S^{(2)}(y-y_0(\tau))|\cos\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}-\varphi^{(2)}\right)$$
$$+\frac{27}{240}|S^{(3)}(y-y_0(\tau))|\cos\left(4q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}-\varphi^{(3)}\right)$$
$$-\frac{9\tilde{\eta}}{256a}|I(y-y_0(\tau))|\cos\left(2q\frac{y_0(\tau)}{\alpha\sqrt{|\epsilon|}}-\varphi\right)$$
$$+\frac{\tilde{\eta}^{1/2}}{a}\int_{-\infty}^{y}dy' R_0(y'-y_0(\tau))\sin\left(\frac{2qy'}{\alpha\sqrt{|\epsilon|}}\right)\zeta(y',\tau)\Big|_{\gamma_0} \tag{C7}$$

where, for $j=1,2,3$,

$$\cos(\varphi^{(j)})=\frac{\mathrm{Im}\{S^{(j)}[y-y_0(\tau)]\}}{|S^{(1)}[y-y_0(\tau)]|}, \tag{C8}$$

$$\sin(\varphi^{(j)})=\frac{\mathrm{Re}\{S^{(j)}[y-y_0(\tau)]\}}{|S^{(1)}[y-y_0(\tau)]|}; \tag{C9}$$

and the same formulas can be written for $\{\cos(\varphi),\sin(\varphi)\}$ replacing $S^{(j)}[y-y_0(\tau)]$ by $I[y-y_0(\tau)]$. We prove now that the functions $S^{(j)}[y-y_0(\tau)]$ are bounded. These integrals are of the type

where $n'=(m'+n'+1)/2$ is always an entire number, $\Omega'=(m'-n'-1)q/\alpha$, $\mu=\sqrt{\epsilon}$, and the integrals $\tilde{S}$ comes from a nonresonant term (see Eq. (4) in the text) of the form $c_{m'n'}A^{m'}\bar{A}^{n'}e^{i(m'-n'-1)q/\alpha}$. The functions $S^{(1)}[y-y_0(\tau)]$, $S^{(2)}[y-y_0(\tau)]$, and $S^{(3)}[y-y_0(\tau)]$ correspond to $(m'=3,n'=0,\Omega'=2q/\alpha)$, $(m'=4,n'=1,\Omega'=2q/\alpha)$, and $(m'=5,n'=0,\Omega'=4q/\alpha)$, respectively. Rescaling $\tilde{S}\{[y-y_0(\tau)]=\frac{4}{\sqrt{3}}S[\tilde{y}-\tilde{y}_0(\tau)]=\sqrt{\frac{3}{4}}[y-y_0(\tau)]\}$, we have

$$S(y-y_0(\tau))=\int_{-\infty}^{\widetilde{y-y_0(\tau)}}dy'\frac{e^{i\Omega y'/\mu}}{(1+e^{-2y'})^n}, \tag{C11}$$

where $\Omega=\frac{4}{\sqrt{3}}\Omega'$. To calculate S we use the contour Γ of Fig. 2.

Using the theorem of residues,

$$J=\int_\Gamma dz\frac{e^{i\Omega z/\mu}}{(1+e^{-2z})^n}=2\pi\alpha^{(0)}, \tag{C12}$$

where $\alpha^{(0)}\sim e^{-\Omega\pi/2\mu}$ is the residue at the pole $z=i\pi/2$ of the integrand. The sum of the integrals over Γ_1 and Γ_2 gives $S(y-y_0(\tau))(1-e^{-\Omega\pi/2\mu})$, and the integral over Γ_2 has the value $ie^{i\Omega(y-y_0(\tau))}M$, with

$$M=\int_0^\pi dy\frac{e^{\Omega y/\mu}}{(1+e^{-2iy-2[\widetilde{y-y_0(\tau)}]})^n}. \tag{C13}$$

A short calculation gives, for the modulus of M, the bound

$$|M| \leq \int_0^{\pi} dy \frac{e^{i\Omega y/\mu}}{(1+e^{-2[\tilde{y}-\widetilde{y_0(\tau)}]})^n} = \frac{\mu}{2}\frac{1-e^{-\Omega\pi/\mu}}{(1+e^{-2[\tilde{y}-\widetilde{y_0(\tau)}]})^n}. \tag{C14}$$

We have then from Eq. (C10)

$$\tilde{S}(\tilde{y}-\tilde{y}_0(\tau)) = \frac{2\pi i\alpha^{(0)}}{1-e^{-\Omega\pi/\mu}} - i\frac{e^{i\Omega(\widetilde{\tilde{y}-y_0(\tau)})}}{1-e^{-\Omega\pi/\mu}}M. \tag{C15}$$

Since M is bounded by $O(\mu=\sqrt{\epsilon})$ according to Eq. (C14), we conclude from Eq. (C15) that it is bounded by a quantity of $O(\sqrt{\epsilon})$ for al values of $\tilde{y}$. If we take the mean value of $\langle R_0^2\Theta_{1y}\rangle$ in Eq. (C5), the last term vanishes due to the $\gamma_1(0)$ discretization and all the other terms are bounded due to Eq. (C15) and the bound $O(e^{-c2q/\sqrt{|\varepsilon|}})$ [6,45].

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