Self-consistent quasiparticle model results for ultrarelativistic electron-positron thermodynamic plasma

Vishnu M. Bannur

Department of Physics, University of Calicut, Kerala-673 635, India

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Relativistic plasma with two charge species and radiation at thermodynamic equilibrium is a general system of interest in astrophysics and high-energy physics. We develop a self-consistent quasiparticle model for such a system to take account of the collective behavior of plasma, and thermodynamic properties are derived. It is applied to the ultrarelativistic electron-positron plasma and compared with previous results.

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I. INTRODUCTION

A typical electrodynamic plasma of interest in astrophysics [1] consists of electrons, positrons, and photons (e, e^+, γ) at thermodynamic equilibrium. Since it is a plasma medium, individual particle properties are modified by the collective effects of plasma. One way to take account of this effect is to use a quasiparticle model. Just as in the Debye theory of specific heats or the theory of liquid helium, etc., the thermal properties of the medium may be viewed as a result of thermal excitations or quasiparticles, like plasmons and dressed photons, as a result of the quantization of plasma waves and electromagnetic waves in plasma. The standard procedure is to obtain the classical dispersion relations for plasma or electromagnetic waves in plasma and on their quantization, we get quasi-particles, namely quasi-fermions (corresponding to e, e^+) and quasi-bosons like dressed photons. We study the statistical mechanics and thermodynamics of such a system of quasi-particles. One such study was attempted by Medvedev [1] which we modify and correct it to get the present model.

In passing, it can be noted that another ultrarelativistic system which can be analyzed in a similar fashion is the quark-gluon plasma (QGP). QGP is made up of quarks and gluons [2], governed by a strong interaction called quantum chromodynamics (QCD). It is similar to the (e, e^+, γ) system with electrons (positrons) replaced by quarks (antiquarks) and photons by gluons. This kind of study for QGP was first attempted by Peshier *et al.* [3], and later it was modified (with corrections) and studied by others [4–6]. What is presented here for the electron-positron-γ plasma is similar in spirit to that QGP work.

II. QUASIPARTICLE MODEL OF PLASMA

Both systems we discussed are highly relativistic, and hence we develop here a quasiparticle model for such ultrarelativistic systems. We assume that the collective excitation of plasma leads to a system of noninteracting quasifermions and quasibosons, obeying Fermi and Bose statistics, respectively. Following the standard statistical mechanics [7], the density of quasiparticles may be written as

$$n = \frac{1}{V}\sum_k \frac{1}{z^{-1}e^{\beta\epsilon_k} \mp 1} \rightarrow \frac{g_f}{2\pi^2}\int_0^\infty dk k^2 \frac{1}{z^{-1}e^{\beta\epsilon_k} \mp 1}, \quad (1)$$

where z is the fugacity and $\mp$ refers to bosons and fermions. g_f is the degeneracy associated with the internal degrees of freedom. Here ϵ_k is the energy of a quasiparticle which may be obtained from the classical approximate dispersion relation as

$$\epsilon_k = \hbar\sqrt{k^2c^2 + \omega_p^2} \quad (2)$$

for bosons and

$$\epsilon_k = \hbar\sqrt{k^2c^2 + \omega_p^2} \quad (3)$$

for fermions. These forms of energy-momentum relations are widely used in quasiparticle models of QGP [3,4] with ω_p^2 replaced by temperature-dependent masses, which they obtain from finite-temperature field theory calculations in an ideal thermal bath. The general expressions for ϵ_k are very complicated even at the high-momentum limit and so, following Medvedev [1], we approximate them to the above simpler equations with an error of about 3%. Here c is the speed of light and $\hbar$ is Planck-constant. The ultrarelativistic plasma frequency (i.e., in the approximation that the thermal energy is far greater than the species rest energy) is ω_p, given by

$$\omega_p^2 = \frac{8\pi e^2 n_e c^2}{3T} \equiv a\frac{n_e}{T}, \quad (4)$$

for the (e, e^+, γ) system [Eq. (3) of [1]]. As usual, e is the charge and n_e the electron density which is also equal to the positron density for a system with chemical potential equal to zero. In quasiparticle models, by definition, the electron density is same as that of quasielectrons. T is the temperature of the system.

III. (e, e^+, γ) SYSTEM

Let us consider a (e, e^+, γ) system with chemical potential zero, or $z=1$, and the density of quasielectrons is

$$n_e = \frac{g_e}{2\pi^2}\int_0^\infty dk k^2 \frac{1}{e^{\beta\hbar\sqrt{k^2c^2 + an_e/T}} + 1}, \quad (5)$$

which may be rewritten as

$$n_e = \frac{g_e}{2\pi^2}\left(\frac{T}{\hbar c}\right)^3 \int_0^\infty dx x^2 \frac{1}{e^{\sqrt{x^2 + a\hbar^2 n_e/T^3}} + 1}. \quad (6)$$

The density of positrons is same as that of electrons. Similarly, the density of quasiphotons may be written as

TABLE I. Various thermodynamic quantities of the (e, e^+, γ) system from our model with plasma and without plasma.

Medium	f_e^2	$n_e[(T/\hbar c)^3]$	$n_\gamma[(T/\hbar c)^3]$	$\varepsilon_e[T^4/(\hbar c)^3]$	$\varepsilon_\gamma[T^4/(\hbar c)^3]$
Plasma	1.79925	0.18230	0.24164	0.57527	0.65705
Vacuum	1.80309	0.18254	0.24339	0.57526	0.65744

$$n_\gamma = \frac{g_\gamma}{2\pi^2}\left(\frac{T}{\hbar c}\right)^3 \int_0^\infty dx x^2 \frac{1}{e^{\sqrt{x^2+a\hbar^2 n_e/T^3}} - 1}, \tag{7}$$

where the fugacity is 1 for photons. These equations need to be solved self-consistently because the value of n_e which is to be determined is inside the integral (through ω_p). Redefining the variables, the final equation to be solved self-consistently is, after expanding the denominator and making the well-known change of variables, $x/(\bar{a}f_e)=\sinh(t)$, and employing the well-known integral representation for the modified Bessel function K_n, one obtains

$$f_e^2 = \int_0^\infty dx x^2 \frac{1}{e^{\sqrt{x^2+\bar{a}^2 f_e^2}} + 1} = \bar{a}^2 f_e^2 \sum_{l=1}^{\infty} \frac{(-1)^{(l-1)}}{l} K_2(\bar{a} l f_e), \tag{8}$$

where

$$\bar{a}^2 \equiv \frac{4 g_e}{3\pi}\alpha$$

and

$$f_e^2 \equiv \frac{2\pi^2(\hbar c)^3}{g_e}\frac{n_e}{T^3}. \tag{9}$$

α is the usual fine-structure constant ($e^2/\hbar c$). (The $l=1$ term is the usual relativistic Maxwellian normalization factor.) Once we know n_e or f_e^2, we can obtain the photon density from the relation

$$n_\gamma = \frac{g_\gamma}{2\pi^2}\left(\frac{T}{\hbar c}\right)^3 \bar{a}^2 f_e^2 \sum_{l=1}^{\infty} \frac{1}{l} K_2(\bar{a} l f_e), \tag{10}$$

which follows from Eq. (7). Similarly, the energy densities are given by

$$\varepsilon_e = \frac{g_e}{2\pi^2}\frac{T^4}{(\hbar c)^3}\int_0^\infty dx x^2 \frac{\sqrt{x^2+\bar{a}^2 f_e^2}}{e^{\sqrt{x^2+\bar{a}^2 f_e^2}} + 1} \tag{11}$$

or

$$\varepsilon_e = \frac{g_e}{2\pi^2}\frac{T^4}{(\hbar c)^3}\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l^4}[(\bar{a}f_e l)^3 K_1(\bar{a}f_e l) + 3(\bar{a}f_e l)^2 K_2(\bar{a}f_e l)], \tag{12}$$

in terms of modified Bessel functions K_1, K_2, for electrons, and

$$\varepsilon_\gamma = \frac{g_\gamma}{2\pi^2}\frac{T^4}{(\hbar c)^3}\int_0^\infty dx x^2 \frac{\sqrt{x^2+\bar{a}^2 f_e^2}}{e^{\sqrt{x^2+\bar{a}^2 f_e^2}} - 1} \tag{13}$$

or

$$\varepsilon_\gamma = \frac{g_\gamma}{2\pi^2}\frac{T^4}{(\hbar c)^3}\sum_{l=1}^{\infty}\frac{1}{l^4}[(\bar{a}f_e l)^3 K_1(\bar{a}f_e l) + 3(\bar{a}f_e l)^2 K_2(\bar{a}f_e l)] \tag{14}$$

for photons. The blackbody Planck's distribution may be easily read from Eq. (13) as

$$d\varepsilon_\gamma(x) = \frac{g_\gamma}{2\pi^2}\frac{T^4}{(\hbar c)^3} x^2 \frac{\sqrt{x^2 - \bar{a}^2 f_e^2}}{e^x - 1} dx, \tag{15}$$

where $x \equiv \hbar\omega/T$. Thus Planck's distribution is modified for the nonzero value of $\bar{a}$ due to plasma with a cutoff frequency related to the plasma frequency.

On taking the limit $a \to 0$ in Eqs. (6) and (7), we get

$$n_e = 2\eta(3)\frac{g_e}{2\pi^2}\left(\frac{T}{\hbar c}\right)^3$$

and

$$n_\gamma = 2\zeta(3)\frac{g_\gamma}{2\pi^2}\left(\frac{T}{\hbar c}\right)^3,$$

where $\zeta(3)$ is a Riemann zeta function and has a value approximately 1.202 and $\eta(3)$ is the sum of the alternating-sign reciprocal cubes of the integers, being related to $\zeta(3)$ by $\eta(3)=(1-2^{1-3})\zeta(3)=\frac{3}{4}\zeta(3)$. Hence n_e and n_γ are related as $n_e=\frac{3}{4}n_\gamma$ as expected for ideal gas. Note that in the earlier calculations [1], the starting point of the formalism was $n_e=\frac{7}{8}n_\gamma$, which is not necessarily correct, since it is an ideal gas relation which need not be true for the nonideal system under consideration. In the method of this work one does not require such *ad hoc* relations between n_e and n_γ; instead, both of them are calculated self-consistently by solving the coupled integral equations (6) and (7).

Other thermodynamic functions, like pressure, may be obtained from the thermodynamic relation

$$\varepsilon = T\frac{\partial P}{\partial T} - P, \tag{16}$$

which on integration gives $P=\frac{1}{3}\varepsilon$, the same as that of ideal relativistic gas. Again this result also differs from Ref. [1] where their expression for pressure is not valid for massive particles. It is interesting to look at the expression for plasma frequency from Eq. (4), together with the second equation defining f_e, Eq. (9). Approximating the value of n_e in the

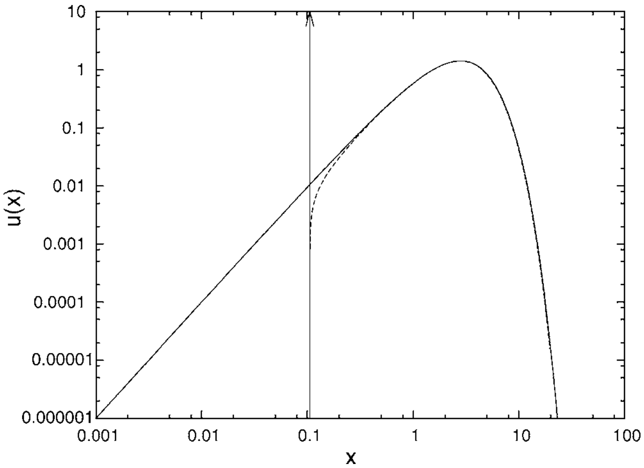

FIG. 1. Normalized Planck's distribution $u(x)$ with plasma (solid line) and without plasma (dashed line) as a function of $x \equiv \hbar\omega/T$. The vertical line with arrow is at the cutoff frequency.

limit as the value of a tends to zero [given just after Eq. (15)] one arrives at the first estimate of the plasma frequency, given on the right-hand side of Eq. (17),

$$\omega_p^2 = \frac{4}{3\pi} g_e \alpha f_e^2 \left(\frac{T}{\hbar}\right)^2 = \frac{2 g_e \alpha \zeta(3)}{\pi}\left(\frac{T}{\hbar}\right)^2 \approx \left(\frac{0.106T}{\hbar}\right)^2, \tag{17}$$

where we took $f_e^2 \approx 2\eta(3) = \frac{3}{2}\zeta(3)$, the value obtained for the limit of a tending to zero. If we express ω_p^2, Eq. (17), in natural units [$\hbar=1$, $c=1$, $\alpha=e^2/(4\pi)$], it reduces to $\approx \frac{\zeta(3)}{\pi^2} e^2 T^2$. It is very close to $\frac{1}{9}e^2T^2$, obtained from the finite-temperature field theory calculations [8], but note that it is not exactly equal because of our self-consistent calculations.

IV. RESULTS AND CONCLUSIONS

For the (e, e^+, γ) system we recalculated various thermodynamical quantities, reported in Ref. [1], using our model with proper corrections. The departure of these quantities from that of an ideal system is too small to be noticed visually, and they are tabulated in Table I. In Table I the row indicated "Vacuum" presents the zeroth order or the approximation of negligible rest energy for the dressed fermions and bosons—i.e., ignoring the second term in the square roots of Eqs. (6), (7), (11), and (13) for (respectively) n_e (and f_e), n_γ, ϵ_e, and ϵ_γ. The row marked "Plasma" shows the result of iteration using the results of Eqs. (8), (10), (12), and (14) for f_e (and n_e), n_γ, ϵ_e, and ϵ_γ. Planck's distribution in plasma is plotted in Fig. 1 along with Planck's distribution in the absence of plasma, qualitatively similar results as in Ref. [1] with a cutoff in the distribution, etc. But note that the numerical values differ and our values listed in Table I with plasma are smaller than without plasma. Just the opposite tendency was seen in Ref. [1], which we believe is due to the somewhat *ad hoc* assumptions used.

In conclusion, we have formulated a self-consistent quasiparticle model to describe the thermodynamics (TD) of relativistic plasma, like (e, e^+, γ). The basic idea is that because of the collective behavior of plasma, the TD of such a system may be obtained by studying the TD of quasiparticles which are thermally excited quanta of plasma and electromagnetic waves in plasma. This is equivalent to a system of bosons and fermions with mass proportional to plasma frequency. The plasma frequency depends on the number density, a TD quantity, which we want to find out and hence a self-consistent problem to be solved. [The temperature effect on the mass for the ultrarelativistic case is that given explicitly in Eq. (4) and needs no further calculation.] For the (e, e^+, γ) system, studied earlier by Medvedev [1], the results obtained here are only slightly different. The reason for the fact that the small deviations from the zeroth-order approximations are of the opposite sign to those of Medvedev is still to be investigated.

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