

Chaotic observer-based synchronization under information constraints

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Limitations of observer-based synchronization systems under information constraints (limited information capacity of the coupling channel) are evaluated. We give theoretical analysis for multidimensional drive-response systems represented in the Lurie form (linear part plus nonlinearity depending only on measurable outputs). It is shown that the upper bound of the limit synchronization error (LSE) is proportional to the upper bound of the transmission error. As a consequence, the upper and lower bounds of LSE are proportional to the maximum rate of the coupling signal and inversely proportional to the information transmission rate (channel capacity). Optimality of the binary coding for coders with one-step memory is established. The results are applied to synchronization of two chaotic Chua systems coupled via a channel with limited capacity.

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I. INTRODUCTION

Chaotic synchronization has attracted the attention of researchers since the 1980s [1–4] and is still an area of active research [5–9]. Recently information-theoretic concepts were applied to analyze and quantify synchronization [10–14]. In [11,12] mutual information measures were introduced for evaluating the degree of chaotic synchronization. In [10,13] the methods of symbolic dynamics were used to relate synchronization precision to capacity of the information channel and to the entropy of the drive system. Baptista and Kurths [14] introduced the concept of a chaotic channel as a medium formed by a network of chaotic systems that enables information from a source to pass from one system (transmitter) to another system (receiver). They characterized a chaotic channel by the mutual information (difference between the sum of the positive Lyapunov exponents corresponding to the synchronization manifold and the sum of positive exponents corresponding to the transverse manifold). However, in existing papers limit possibilities for the precision of controlled synchronization have not been analyzed.

Recently the limitations of control under constraints imposed by a finite capacity information channel have been investigated in detail in the control theoretic literature, see [15–20], and references therein. It was shown that stabilization under information constraints is possible if and only if the capacity of the information channel exceeds the entropy production of the system at the equilibrium [17–19]. In [21,22] a general statement was proposed, claiming that the difference between the entropies of the open loop and the closed loop systems cannot exceed the information introduced by the controller, including the transmission rate of the information channel. However, results of the previous works

on control system analysis under information constraints do not apply to synchronization systems since in a synchronization problem trajectories in the phase space converge to a set (a manifold) rather than to a point, i.e., the problem cannot be reduced to simple stabilization.

In this paper we establish limit possibilities of observer-based synchronization systems under information constraints. Observer-based synchronization systems are used when only one phase variable is available for measurement and coupling. Such systems are well studied without information constraints [23–25]. Here we present a theoretical analysis for n -dimensional drive-response systems represented in the so-called Lurie form (linear part plus nonlinearity, depending only on measurable outputs). It is shown that the upper bound of the limit synchronization error (LSE) is proportional to the upper bound of the transmission error. As a consequence, the upper and lower bounds of LSE are proportional to the maximum rate of the coupling signal and inversely proportional to the information transmission rate (channel capacity). Optimality of the binary coding for coders with one-step memory is established.

Note also that it was claimed in some papers, see, e.g. [10], that if the capacity of the channel is larger than the Kolmogorov-Sinai entropy of the driving system, then the synchronization error can be made arbitrarily small. Such a claim is based upon the noisy channel theorem of Shannon information theory stating that, if the source entropy is smaller than the channel capacity, then the data generated by the source can be transmitted over the channel with negligible probability of error. However, according to the Shannon theory [26] in order to transmit data with a sufficiently small error a sufficiently long codeword and a long transmission time is needed. During such a long time an unstable chaotic trajectory may go far from its predicted value and synchronization may fail. Therefore analysis of the system precision under information constraints requires more subtle arguments which are provided in this paper for synchronization systems based on Lyapunov functions and coding analysis. Such an argument provides a motivation for development

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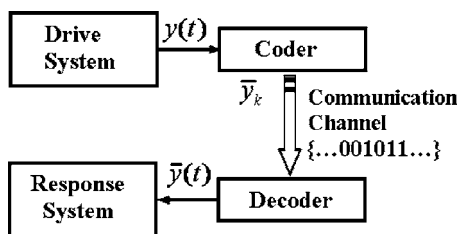


FIG. 1. Block diagram for drive-response synchronization using a discrete communication channel.

new Shannon-type theorems taking into account the system dynamics [15–18,20].

II. DESCRIPTION OF OBSERVED-BASED SYNCHRONIZATION SYSTEM

A block diagram for implementing drive-response synchronization of two unidirectionally coupled oscillators via a discrete communication channel is shown in Fig. 1. To simplify exposition we will consider coupled systems in the so-called Lurie form; right-hand sides are split into a linear part and a nonlinearity vector depending only on the measured output. Then the drive (master, entraining) system is modelled as follows:

$$\dot{x} = Ax + \varphi(y), \quad y = Cx, \quad (1)$$

where x is an n -dimensional (column) vector of state variables, y is the scalar output (coupling) variable, A is an ($n \times n$) matrix, C is a $1 \times n$ (row) matrix, $\varphi(y)$ is a continuous nonlinearity. We assume that all the trajectories of the system (1) belong to a bounded set Ω (e.g., attractor of a chaotic system). Such an assumption is typical for chaotic systems.

The response (slave, entrained) system is described as a nonlinear observer

$$\dot{\hat{x}} = A\hat{x} + \varphi(y) + K(y - \hat{y}), \quad \hat{y} = C\hat{x}, \quad (2)$$

where K is the vector of the observer parameters (gain). Apparently, the dynamics of the state error vector $e(t) = x(t) - \hat{x}(t)$ is described by a linear equation

$$\dot{e} = A_K e, \quad y = Cx, \quad (3)$$

where $A_K = A - KC$.

As is known from control theory, e.g. [26,27], if the pair (A, C) is observable, i.e., if $\text{rank}[C^T, A^T C^T, \dots, (A^T)^{n-1} C^T] = n$, then there exists K providing the matrix A_K with any given eigenvalues. Particularly, all eigenvalues of A_K can have negative real parts, i.e., the system (3) can be made asymptotically stable and $e(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore, in the absence of measurement and transmission errors the synchronization error decays to zero.

Now let us take into account transmission errors. We assume that the observation signal $y(t)$ is coded with symbols from a finite alphabet at discrete sampling time instants $t_k = kT_s$, $k=0, 1, 2, \dots$, where T_s is the sampling time. Let the coded symbol $\bar{y}_k = \bar{y}(t_k)$ be transmitted over a digital communication channel with a finite capacity. To simplify the analy-

sis, we assume that the observations are not corrupted by observation noise; transmissions delay and transmission channel distortions may be neglected. Therefore, the discrete communication channel with the sampling period T_s is considered, but it is assumed that the coded symbols are available at the receiver side at the same sampling instant $t_k = kT_s$, as they are generated by the coder.

Assume that *zero-order extrapolation* is used to convert the digital sequence \bar{y}_k to the continuous-time input of the response system $\bar{y}(t)$, namely, that $\bar{y}(t) = \bar{y}_k$ as $kT_s \leq t < (k+1)T_s$. Then the *transmission error* is defined as follows:

$$\delta_y(t) = y(t) - \bar{y}(t). \quad (4)$$

In the presence of the transmission error, Eq. (3) takes the form

$$\dot{e} = A_K e + \varphi(y) - \varphi(y + \delta_y(t)) - K\delta_y(t). \quad (5)$$

Our goal is to evaluate limitations imposed on the synchronization precision by limited transmission rate. To this end introduce an upper bound of the limit synchronization error $Q = \sup \lim_{t \rightarrow \infty} \|e(t)\|$, where $e(t)$ is from (5), $\|\cdot\|$ denotes the Euclidian norm of a vector, and the supremum is taken over all admissible transmission errors. In the next two sections we describe encoding and decoding procedures and evaluate the set of admissible transmission errors $\delta_y(t)$ for the optimal choice of coder parameters. It will be shown that $\delta_y(t)$ is bounded and does not tend to zero.

III. CODING PROCEDURES

At first, consider the memoryless (static) coder with uniform discretization and constant range. For a given real number $M > 0$ and positive integer $\nu \in \mathbb{Z}$ define a uniform scaled coder to be a discretized map $q_{\nu, M}: \mathbb{R} \rightarrow \mathbb{R}$ as follows. Introduce the *range interval* $\mathcal{I} = [-M, M]$ of length $2M$ and the *discretization interval* of length $\delta = 2^{1-\nu}M$ and define the coder function $q_{\nu, M}(y)$ as

$$q_{\nu, M}(y) = \begin{cases} \delta \cdot \langle \delta^{-1} y \rangle, & \text{if } |y| \leq M, \\ M \text{ sign}(y), & \text{otherwise,} \end{cases} \quad (6)$$

where $\langle \cdot \rangle$ denotes round-up to the nearest integer function, $\text{sign}(\cdot)$ is the signum function: $\text{sign}(y) = 1$, if $y \geq 0$, $\text{sign}(y) = -1$, if $y < 0$. Evidently, $|y - q_{\nu, M}(y)| \leq \delta/2$ for all y such that $y: |y| \leq M + \delta/2$ and all values of $q_{\nu, M}(y)$ belong to the range interval \mathcal{I} . Notice that the interval \mathcal{I} is equally split into 2^ν parts. Therefore, the cardinality of the mapping $q_{\nu, M}$ image is equal to $2^\nu + 1$, and each codeword symbol contains $R = \log_2(2^\nu + 1) = \log_2(2M/\delta + 1)$ bits of information. Thus, the discretized output of the considered coder is found as $\bar{y} = q_{\nu, M}(y)$. We assume that the coder and decoder make decisions based on the same information.

In a number of papers more sophisticated encoding schemes have been proposed and analyzed, see [17,28–30], for example. The underlying idea for coders of this kind is to reduce the range parameter M , replacing the symmetric range interval \mathcal{I} by the interval \mathcal{Y}_{k+1} , covering some area around the predicted value for the $(k+1)$ th observation y_{k+1} ,

$y_{k+1} \in \mathcal{Y}_{k+1}$. If the length of \mathcal{Y}_{k+1} is small compared with the full range of possible measured output values y , then there is an opportunity to reduce the range parameter M and, consequently, to decrease the coding interval δ preserving the bit-rate of transmission. To realize this scheme, memory should be introduced into the coder. Using such a “zooming” strategy it is possible to increase coder accuracy in the steady-state mode, and, at the same time, to prevent coder saturation at the beginning of the process.

In this paper we use a simple version of such a coder having one-step memory and time-based zooming. To describe it we introduce the sequence of *central numbers* c_k , $k=0,1,2,\dots$ with initial condition $c_0=0$. At step k the coder compares the current measured output y_k with the number c_k , forming the deviation signal $\partial y_k = y_k - c_k$. Then this signal is discretized with a given ν and $M = M_k$ according to (6). The output signal

$$\bar{\partial}y_k = q_{\nu, M_k}(\partial y_k) \quad (7)$$

is represented as an R -bit information symbol from the coding alphabet and transmitted over the communication channel to the decoder. Then the central number c_{k+1} and the range parameter M_k are renewed based on the available information about the driving system dynamics. We use the following update algorithms:

$$c_{k+1} = c_k + \bar{\partial}y_k, \quad c_0 = 0, \quad k = 0, 1, \dots, \quad (8)$$

$$M_k = (M_0 - M_\infty)\rho^k + M_\infty, \quad k = 0, 1, \dots, \quad (9)$$

where $0 < \rho \leq 1$ is the decay parameter, M_∞ stands for the limit value of M_k . The initial value M_0 should be large enough to capture all the region of possible initial values of y_0 .

Equations (6), (7), and (9) describe the coder algorithm. The same algorithm is realized by the decoder. Namely, the decoder calculates the variables \tilde{c}_k, \tilde{M}_k based on received codeword flow similarly to c_k, M_k .

IV. CODER OPTIMIZATION

We now find a relation between the transmission rate and the achievable accuracy of the coder-decoder pair, assuming that the growth rate of $y(t)$ is uniformly bounded. Obviously, the exact bound L_y for the rate of $y(t)$ is $L_y = \sup_{x \in \Omega} |C\dot{x}|$, where \dot{x} is from (1). To analyze the coder-decoder accuracy, evaluate the upper bound $\Delta = \sup_t |\delta_y(t)|$ of the transmission error $\delta_y(t) = y(t) - \bar{y}(t)$. Consider the sampling interval $[t_k, t_{k+1}]$. It is clear that $|\delta_y(t_k)|$ does not exceed $\delta/2$. Additionally, the error may increase from t_k to t_{k+1} due to a change of $y(t)$ by a value not exceeding $\sup_{t_k < t < t_{k+1}} |y(t) - y(t_k)| \leq \int_{t_k}^{t_{k+1}} |\dot{y}(\tau)| d\tau \leq \int_{t_k}^{t_{k+1}} L_y d\tau = L_y T_s$. Therefore the total transmission error for each interval $[t_k, t_{k+1}]$ satisfies the inequality,

$$|\delta_y(t)| \leq \delta/2 + L_y T_s. \quad (10)$$

Inequality (10) shows that in order to meet the inequality $|\delta_y(t)| \leq \Delta$ for all t , the sampling interval T_s should satisfy condition

$$T_s < \Delta/L_y. \quad (11)$$

Furthermore, if the condition (11) holds, the given bound for the coding error will be guaranteed if the coding interval δ is appropriately chosen, namely, $\delta < 2\Delta - 2L_y T_s$. It provides the lower bound for the transmission bit-per-step rate R ; its value should not be less than $\log_2\left(\frac{M}{\Delta - L_y T_s} + 1\right)$. Therefore, the coder with range $2M$, coding interval δ and sampling period T_s ensures the total transmission error Δ if (11) holds and the transmission rate satisfies inequality

$$R \geq \log_2\left(\frac{M}{\Delta - L_y T_s} + 1\right). \quad (12)$$

It follows from (11) and (12) that if T_s is sufficiently small and R is sufficiently large, then an arbitrarily small value of Δ can be assured.

Let us now optimize the coder parameters to achieve the minimum bound for the error Δ . As seen from (12), reduction in the coder range $2M$ results in a reduction in the transmission rate R and channel capacity R^* . On the other hand, to prevent coder saturation, M should not be less than $\sup_{k \in \mathbb{Z}} |\delta_y(t_k)| - \delta/2 = \Delta - \delta/2$. Taking into account that $\delta = 2^{1-\nu}M$, we arrive at the following formula for the minimal admissible range:

$$M = \frac{2^\nu}{2^\nu + 1} \Delta. \quad (13)$$

Remark 1: At the initial stage of the system evolution the error $|\delta_y|$ may exceed the bound Δ , because the initial value $y(0)$ is not known. This leads to the transient mode of the system behavior. The zooming strategy may be efficient at this stage, providing the following recipe for the choice of coder parameters: $M_0 = M_y$, $M_\infty = 2^\nu \Delta / (2^\nu + 1)$, where $M_y = \sup_{x_0 \in \Omega} |y(t)|$.

Now optimize the coder w.r.t. T_s . Consider the steady-state mode when $|y(t) - c_k| \leq \Delta$ for each time interval $t \in [t_k, t_{k+1}]$. Let M be found from (13). Introduce a real number ε as $\varepsilon = L_y T_s / \Delta$, (evidently, $0 < \varepsilon < 1$) and rewrite the lower bound R^* for R in the form

$$R^* = \log_2\left(\frac{2^\nu}{(2^\nu + 1)(1 - \varepsilon)} + 1\right). \quad (14)$$

Defining the *bit-per-second* rate $\bar{R} = R/T_s$ and its lower bound \bar{R}^* we have from (14):

$$\bar{R}^* = \frac{L_y}{\varepsilon \Delta} \log_2\left(\frac{2^\nu}{(2^\nu + 1)(1 - \varepsilon)} + 1\right). \quad (15)$$

Now the optimization of the coder is reduced to the following minimization problem. Find $(\varepsilon^*, \nu^*) = \arg \min_{\substack{\varepsilon \in (0,1) \\ \nu \in \mathbb{Z}}} \bar{R}(\varepsilon, \nu)$. Since the right-hand side of (15) is strictly growing in ν , the optimal value of ν is $\nu^* = 0$. This means that the *binary* coding scheme gives the optimal transmission rate $R^* = 1$ bit per step, which yields $M^* = \Delta/2$ as an optimal value for M , and the *signum* function as an optimal coder function, $\bar{y} = \frac{\Delta}{2} \text{sign } y$.

For the optimal value of ν we have $\bar{R}^* = (L_y/\Delta)r(\varepsilon)$, where

$$r(\varepsilon) = \frac{1}{\varepsilon} \log_2 \left(\frac{1}{2(1-\varepsilon)} + 1 \right). \quad (16)$$

Let $r^* = \min_{0 < \varepsilon < 1} r(\varepsilon)$. It is easy to see that this minimum exists and satisfies the transcendental equation $dr(\varepsilon)/d\varepsilon = 0$. Numerical one-dimensional minimization yields $r^* = r(\varepsilon^*) \approx 1.688$, where $\varepsilon^* \approx 0.5923$.

Therefore, the optimal sampling time T_s^* is

$$T_s^* = \varepsilon^* \frac{\Delta}{L_y}. \quad (17)$$

Then the minimal channel bit rate $\bar{R}^* = 1/T_s^*$ is

$$\bar{R}^* = r^* \frac{L_y}{\Delta}, \quad (18)$$

and this bound is tight for the considered class of coders. The relation (18) can be rewritten as

$$\bar{R}\Delta \geq r^* L_y, \quad (19)$$

playing the role of an uncertainty relation between the propagation rate of information and the transmission error.

V. EVALUATION OF SYNCHRONIZATION ERROR

Now let us evaluate the total guaranteed synchronization error $Q = \sup \lim_{t \rightarrow \infty} \|e(t)\|$, where \sup is taken over the set of transmission errors $\delta_y(t)$ not exceeding the level Δ in absolute value. The ratio $C_e = Q/\Delta$ (the relative error) can be interpreted as the norm of the transformation from the input function $\delta_y(\cdot)$ to the output function $e(\cdot)$ generated by the system (5). Owing to nonlinearity of Eq. (5) evaluation of the norm C_e is nontrivial and it even may be infinite for rapidly growing nonlinearities $\varphi(y)$. To obtain a reasonable upper bound for C_e we assume that the nonlinearity is Lipschitz continuous along all the trajectories of the drive system (1). More precisely, we assume existence of some positive real number $L_\varphi > 0$ such that

$$\|\varphi(y) - \varphi(y + \delta)\| \leq L_\varphi |\delta|$$

for all $y = Cx$, $x \in \Omega$, where Ω is a set containing all the trajectories of the drive system (1), starting from the set of initial conditions Ω_0 , $|\delta| \leq \Delta$.

The error Eq. (5) can be represented as

$$\dot{e} = A_K e + \xi(t), \quad (20)$$

where $\|\xi(t)\| \leq (L_\varphi + \|K\|)\Delta$, i.e., the problem is reduced to a standard problem of linear system theory. Choose K such that A_K is a Hurwitz (stable) matrix and choose a positive-definite matrix $P = P^T > 0$ satisfying the modified Lyapunov inequality

$$PA_K + A_K^T P \leq -\mu P, \quad (21)$$

for some $\mu > 0$. Note that the solutions of (21) exist if and only if $\mu > \mu_*$, where $\mu_* = -\max \operatorname{Re} \lambda_i(A)$ is a stability degree of matrix A . After simple algebra we obtain the differential inequality for the function $V(t) = e(t)^T P e(t)$,

$$\dot{V} \leq -\mu V + e^T P \xi(t) \leq -\mu V + \sqrt{V} \cdot \sqrt{\xi^T P \xi}.$$

Since $\dot{V} < 0$ within the set $\sqrt{V} > \mu^{-1} \sup_{t \in \mathbb{R}} \sqrt{\xi(t)^T P \xi(t)}$, the value of $\overline{\lim}_{t \rightarrow \infty} \sup V(t)$ cannot exceed $\Delta^2 (L_\varphi + \|K\|)^2 \lambda_{\max}(P) / \mu^2$. In view of positivity of P , $\lambda_{\min}(P) \|e(t)\|^2 \leq V(t)$, where $\lambda_{\min}(P)$, $\lambda_{\max}(P)$ are minimum and maximum eigenvalues of P , respectively. Hence

$$\overline{\lim}_{t \rightarrow \infty} \|e(t)\| \leq C_e^+ \Delta, \quad (22)$$

where

$$C_e^+ = \sqrt{\frac{\lambda_{\max}(P) L_\varphi + \|K\|}{\lambda_{\min}(P) \mu}}. \quad (23)$$

The inequality (22) shows that the total synchronization error is proportional to the upper bound of transmission error Δ . Taking into account the relation (18) for optimal transmission rate, the synchronization error can be estimated as follows:

$$\overline{\lim}_{t \rightarrow \infty} \|e(t)\| \leq C_e^+ r^* L_y / \bar{R}, \quad (24)$$

i.e., it can be made arbitrarily small for sufficiently large transmission rate R .

Similar estimates for synchronization errors in coupled systems were obtained in several papers [31–35]. However, in [31–35] either existence of Lyapunov functions, i.e., stability of uncoupled systems is required, or a partial stability (stability of the synchronization manifold) is provided by a strong coupling playing the role of state feedback in the error system. In the observer-based scheme studied in this paper it is assumed that only output feedback is admitted. Then the partial stability conditions are provided by the linear observer theory. In addition the final result (24) is presented in terms of transmission rate.

One can pose the following problem: choose an optimal gain vector K and the matrix P providing the minimum value of C_e . However an analytical solution is difficult to obtain in view of the system nonlinearity. For the fixed K the value C_e^+ can be minimized numerically by means of solving Lyapunov inequality (21) for P for every μ from a finite grid in the range $0 < \mu < \mu^*$. To solve (21) one of a number of existing software packages can be used, e.g. YALMIP [36]. An alternative approach is to evaluate upper and lower bounds for C_e based on worst case inputs $\delta_y(t)$. Such a problem is similar to the energy control problem for systems with dissipation [37,38] and C_e can be interpreted as *excitability index* of the system. Employing the lower bound for excitability index for passive systems [37,38] we conclude that if the gain vector K is chosen to ensure strict passivity of the system (5) then the lower bound for C_e is positive, i.e.,

$$\sup_{|\delta_y(t)| \leq \Delta} \overline{\lim}_{t \rightarrow \infty} \|e(t)\| \geq C_e^- \Delta. \quad (25)$$

Thus, for finite channel capacity the guaranteed synchronization error is not reduced to zero being of the same order of magnitude as the transmission error.

Remark 2: The limits of synchronization error may be different if a more sophisticated coder is used, e.g., a first-

order coder with linear extrapolation of the signal or an n th order coder with predictive model of the drive system. For example, if a full order observer is admitted at the transmitter side and there are n channels for simultaneous transmission of the n -dimensional vector $\hat{x}(t_k)$ of estimates of the drive system state, then the coder can calculate the best estimate $\hat{x}(t_{k+1})$ and choose $c_{k+1} = C\hat{x}(t_{k+1})$. In this case the prediction error for a binary coder will be determined by the divergence rate of neighboring trajectories, i.e., relation (10) should be replaced by $\frac{\delta}{2} \exp(hT_s) < \Delta$, where $h > 0$ is the upper Lyapunov exponent of the chaotic drive system. This yields the bound $T_s < \ln(2/\|C\|/C_e^+)/h$, where C_e^+ is from (23) instead of the bound (11). For the transmission rate it gives the necessary condition $R^* > h/\ln(2/\|C\|/C_e^+)$ instead of the lower bound $R^* > L_y/\Delta$ following from (11). If the condition $R^* > h/\ln(2/\|C\|/C_e^+)$ holds, then the upper bound for transmission error Δ will decrease at each sampling interval $[t_k, t_{k+1})$ in $R^* > h/\ln(2/\|C\|/C_e^+)$ times and, therefore, will converge to zero exponentially.

Remark 3: It worth mentioning that relations (22) and (25) give an overestimate for the synchronization error, because it provides its upper bound for the worst case of the input signal $\delta_y(t)$. Also the estimate of the mean square value of the synchronization error may be used. There is a significant body of work in which the quantization error signal $\delta_y(t)$ is modelled as an extra additive white noise. This assumption, typical for the digital filtering theory, is reasonable if the quantizer resolution is high [39–41], but it must be considered revision for the cases of low number of quantization levels [42]. Some comparison between the theoretical and numerical results is given below in Sec. VI.

Remark 4: It is assumed in the present work, that the coupled systems are in the Lurie form. It can be seen that the proposed theory can be applied to observer-based synchronization of chaotic nonlinear systems not presented in the Lurie form. For example the drive system may be described by the following equations:

$$\dot{x} = A(y)x + B\varphi(y), \quad y = Cx, \quad (26)$$

where $x \in \mathbb{R}^n$ is a vector of state variables, y is the scalar output, $A(y)$ is an $(n \times n)$ -matrix, B is an $(n \times 1)$ -vector, C is $(1 \times n)$ -matrix, $\varphi(y)$ is a continuous nonlinear function.

Following [23,43–46] let us choose the response (observer) system model as

$$\dot{\hat{x}} = A(y)\hat{x} + B\varphi(y) + K(\bar{y}(t) - \hat{y}(t)), \quad \hat{y} = C\hat{x}, \quad (27)$$

where \bar{y} is the drive system output, transmitted over the communication channel, \hat{x} is the estimate of x , K is an $(n \times 1)$ vector of design parameters.

Taking into account the transmission error (4), one obtains the following equation for the synchronization error $e(t) = x(t) - \hat{x}(t)$,

$$\dot{e} = (A(y) - KC)e - K\delta_y(t). \quad (28)$$

Assume that there exists vector K such that the following strengthened stability condition (*Demidovich condition* [47,48]) for the matrix $A_K(y) = A(y) + KC$ holds:

$$\lambda_i(A_K(y) + A_K(y)^T) \leq -\varepsilon < 0, \quad i = 1, 2, \dots, n$$

for some $\varepsilon > 0$, where $\lambda_i(A)$, $i = 1, 2, \dots, n$ are eigenvalues of the matrix A . Then the upper bounds (22) and (23) for the synchronization error remain true.

VI. EXAMPLE SYNCHRONIZATION OF CHAOTIC CHUA SYSTEMS

Let us apply the above results to synchronization of two chaotic Chua systems coupled via a channel with limited capacity.

A. System equations

Consider the chaotic Chua system model,

$$\begin{cases} \dot{x}_1 = p(-x_1 + \varphi(y) + x_2), & t \geq 0, \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -qx_2, \end{cases}$$

$$y(t) = x_1(t), \quad (29)$$

where $y(t)$ is the sensor output (to be transmitted over the communication channel), p, q are known plant model parameters, $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ is the plant state vector, the initial condition vector $x_0 = x(0)$ is assumed to be unknown, $\varphi(y)$ is a piecewise-linear function, having the following form:

$$\varphi(y) = m_0 y + |x + 1| - |x - 1| + 0.5(m_1 - m_0)(|x + 1| - |x - 1|), \quad (30)$$

where m_0, m_1 are given plant parameters.

B. Observer design

To obtain estimates $\hat{x}(t)$ of the current state $x(t)$ of the system (29), the special case of a continuous time observer (5) is designed as follows:

$$\begin{cases} \dot{\hat{x}}_1 = p(-\hat{x}_1 + \varphi(y) + \hat{x}_2) + k_1 \varepsilon(t), \\ \dot{\hat{x}}_2 = \hat{x}_1 - \hat{x}_2 + \hat{x}_3 + k_2 \varepsilon(t), \\ \dot{\hat{x}}_3 = -q\hat{x}_2 + k_3 \varepsilon(t), \\ \varepsilon(t) = \bar{y}(t) - \hat{y}(t), \end{cases}$$

$$\hat{y}(t) = \hat{x}_1(t), \quad \hat{x}(0) = \hat{x}_0, \quad (31)$$

where k_1, k_2, k_3 are observer parameters, forming the 3×1 observer matrix gain $K = [k_1, k_2, k_3]^T$.

Subtracting (31) from (29) yields

$$\begin{cases} \dot{e}_1 = p(-e_1 + e_2) + k_1(\delta_y(t) - e_1) + \xi_1(t), \\ \dot{e}_2 = e_1 - e_2 + e_3 + k_2(\delta_y(t) - e_2), \\ \dot{e}_3 = -qe_2 + k_3(\delta_y(t) - e_3), \\ \xi_1(t) = \varphi(y(t) - \delta_y(t)) - \varphi(y(t)). \end{cases} \quad (32)$$

Equation (32) describes the linear time-invariant (LTI) system $\dot{e}(t) = Ae(t)$, $e(0) = x_0 - \hat{x}_0$ with the following matrix A :

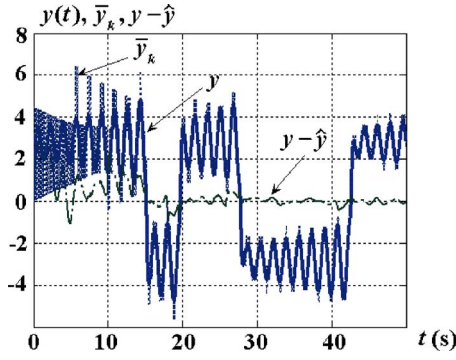


FIG. 2. (Color online) Time histories for $y(t)$ (solid line), \bar{y}_k (dotted line), $y(t)-\hat{y}(t)$ (dashed-dotted line); $\Delta=1$.

$$A = \begin{bmatrix} -p - k_1 & p & 0 \\ 1 - k_2 & -1 & 1 \\ -k_3 & q & 0 \end{bmatrix}. \quad (33)$$

Matrix K should be chosen so that the observer (31) stability conditions are satisfied, i.e., the characteristic polynomial $D_K(s) = \det(sI - A_K)$ is Hurwitz. For the observer (34), the polynomial $D_K(s)$ has the form

$$D_K(s) = s^3 + (1 + p + k_1)s^2 + (-q + pk_2 + k_1)s - pq + k_3p - k_1q. \quad (34)$$

Evidently, we may find the matrix K for any arbitrarily assigned parameters d_1, d_2, d_3 so that the characteristic polynomial $D_K(s) = s^3 + d_1s^2 + d_2s + d_3$. This leads to asymptotic convergence of the synchronization error $e(t)$ to zero with prescribed dynamics in the disturbance-free case.

C. Simulation results

For simulation the following parameter values of the Chua system model (29) were chosen: $p=10.0$, $q=15.6$, $m_0=0.33$, $m_1=0.22$. The system exhibits a chaotic behavior, see $y(t)$ in Fig. 2.

In our simulations parameter Δ has been taken from the set $\Delta = \{0.1, 0.2, 0.5, 0.7, 1\}$. The sampling time T_s for each Δ has been chosen in accordance with (17) for $L_y = 30 \text{ s}^{-1}$. In (9) the initial value M_0 has been taken as $M_0 = 5$, decay parameter $\rho = \exp(-0.1T_s)$, and limit value $M_\infty = M^* = \Delta/2$.

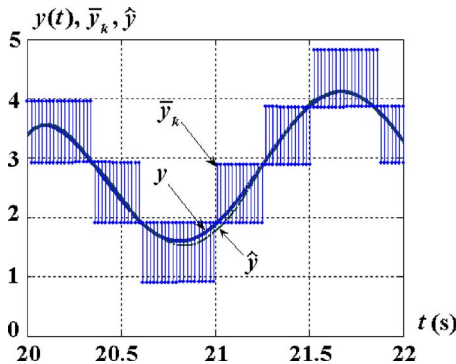


FIG. 3. (Color online) Zooming of Fig. 2 for $t \in [20, 22] \text{ s}$.

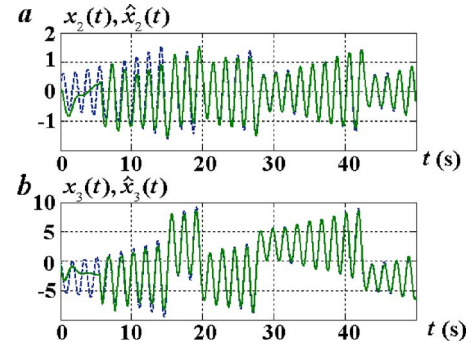


FIG. 4. (Color online) Time histories for $\Delta=1$: (a) $x_2(t)$ (dotted line), $\hat{x}_2(t)$ (solid line); (b) $x_3(t)$ (dotted line), $\hat{x}_3(t)$ (solid line).

To evaluate the minimal synchronization error the optimal observer gain matrix $K^*(\Delta)$ was found numerically for several values of the transmission error Δ . We obtained $K^*(0.1) = [-4.66, 0.50, -4.40]^T$, $K^*(0.5) = [-4.40, 0.46, -4.54]^T$, $K^*(1.0) = [-4.97, 0.46, -4.47]^T$. For comparison the observer design by assigning a *Butterworth* distribution of the observer matrix A_K eigenvalues was performed. For the third order system the Butterworth design provides characteristic polynomial (34) as $D(s) = s^3 + 2\omega_0s^2 + 2\omega_0^2s + \omega_0^3$, where parameter $\omega_0 > 0$ specifies the desired estimation rate. In our example $\omega_0 = 6 \text{ s}^{-1}$ is taken. It provides the observer eigenvalues: $s_1 = -6.0$, $s_{2,3} = -3.0 \pm 5.2i$. The observer feedback gain matrix K is found as $K = K_B = [1.00, 5.54, 4.44]^T$. For simulation the initial condition vectors for the systems (29) and (31) were taken as $x_0 = [0.3, 0.3, 0.3]^T$ and $\hat{x}_0 = [0, 0, 0]^T$.

Simulation results for the coder (6), (7), and (9) and observer with optimally chosen gains for $\Delta=1$ are shown in Figs. 2–4. The sampling interval is $T_s = 0.02 \text{ s}$, which corresponds to the transmission rate $\bar{R} = 50$ bits per second. The following coder parameters were chosen: $M_0 = 5.0$, $M_\infty = 0.5$, $\rho = 0.998$. It is seen that the synchronization process possesses sufficiently fast dynamics even in the presence of information constraints.

It is seen from Fig. 5 that the difference in the limit synchronization error for different rationally chosen observer gains is not significant. Moreover, it is seen from Fig. 6 that

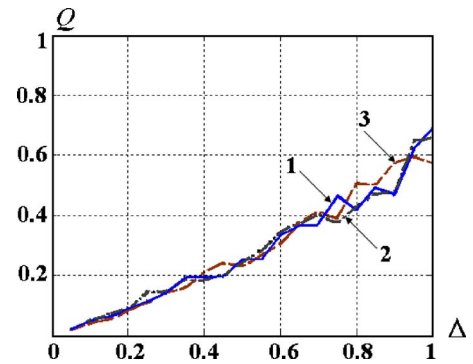


FIG. 5. (Color online) Synchronization error Q vs Δ for different K . 1: $K = K^*(0.1)$ (solid line); 2: $K = K^*(0.5)$ (dashed-dotted line); 3: $K = K^*(1)$ (dashed line).

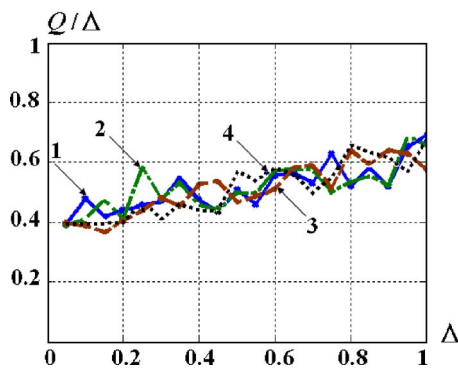


FIG. 6. (Color online) Relative synchronization error Q/Δ vs Δ for different K . 1: $K=K^*(0.1)$ (solid line); 2: $K=K^*(0.5)$ (dashed-dotted line); 4: $K=K_B$ (dotted line).

the relative error does not approach zero for all choices of the observer gains.

Let us consider the theoretical estimate of the relative synchronization error Q/Δ . The upper bound of this error C_e^+ is given by (23).

To find C_e^+ numerically, the following procedure has been employed. Parameter μ in (23) has been considered as a free argument for minimization of the value of C_e^+ . To find the matrix P , satisfying for the given μ Lyapunov inequality (21), the software package YALMIP [36] has been used. It is found that for $K=K_B=[1.0, 5.54, 4.44]^T$ the optimal μ is $\mu^*=4.25$ and (21) is satisfied for

$$P = P(\mu^*) = \begin{bmatrix} 0.544 & -0.9477 & 0.024 \\ -0.9477 & 3.066 & -0.360 \\ 0.024 & -0.360 & 0.133 \end{bmatrix}. \quad (35)$$

The eigenvalues λ of the matrix P are as follows: $\lambda_1(P)=0.05$, $\lambda_2(P)=0.274$, $\lambda_3(P)=3.42$. Hence $\lambda_{\max}(P)=3.412$, $\lambda_{\min}(P)=0.05$, and $C_e^+=18.3$. This value is significantly larger than the one found by simulation (see Fig. 6).

Dependence of the synchronization error Q on the transmission rate \bar{R} is shown in Fig. 7, demonstrating that the synchronization error becomes small for sufficiently large transmission rates.

VII. CONCLUSIONS

We have studied the dependence of the synchronization error in the observer-based synchronization system both analytically and numerically. It is shown that upper and lower bounds for limit synchronization error depend linearly on the transmission error which, in turn, is proportional to the driving signal rate and inversely proportional to the transmission rate. Though these results are obtained for a special type of

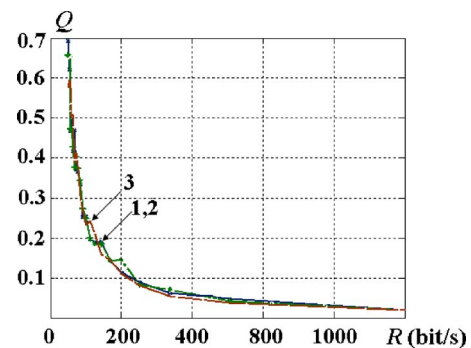


FIG. 7. (Color online) Synchronization error Q vs transmission rate \bar{R} .

coder, it reflects the peculiarity of the synchronization problem as a nonequilibrium dynamical problem. On the contrary, the stabilization problem considered previously in the literature on control under information constraints belongs to a class of equilibrium problems.

Note that though convergence of trajectories to a point is a special case of convergence to a set and the difference is not essential from the dynamical systems point of view, the difference may be significant for the system analysis or synthesis. Indeed, in the case of one-dimensional limit set (convergence to a point or to a reference trajectory) the current system state approaches a known position and its limit behavior is easier to study than in the case when the current state is wandering over a set. Even if the two subsystem states tend to a synchronous mode, each state may tend to infinity or may escape in a finite time. In our analysis we imposed special assumptions allowing to reduce the analysis of the overall nonlinear system (1) and (5) to the analysis of the linear error system (20). It is important, however, that the dimensions of systems (1), (5), and (20) are not equal and their analysis problems are not equivalent in general. Even in the absence of disturbances, when the trajectories of the synchronized system (1) and (5) tend to the n -dimensional subspace of the overall $2n$ -dimensional state space, the behavior of the systems (1) and (5) may be much more complex than that of the linear system (20) converging to a point. A more detailed discussion about controlled synchronization can be found in [8,49].

As an intermediate result we obtained relation (19) playing the role of an uncertainty relation between the transmission rate of information and the transmission error. Future research is aimed at analysis of controlled synchronization and control of chaos problems under information constraints.

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