Adaptive-feedback control algorithm

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(Received 22 November 2005; published 2 June 2006)

This paper is motivated by giving the detailed proofs and some interesting remarks on the results the author obtained in a series of papers [Phys. Rev. Lett. **93**, 214101 (2004); Phys. Rev. E **71**, 037203 (2005); **69**, 067201 (2004)], where an adaptive-feedback algorithm was proposed to effectively stabilize and synchronize chaotic systems. This note proves in detail the strictness of this algorithm from the viewpoint of mathematics, and gives some interesting remarks for its potential applications to chaos control & synchronization. In addition, a significant comment on synchronization-based parameter estimation is given, which shows some techniques proposed in literature less strict and ineffective in some cases.

DOI: 10.1103/PhysRevE.73.066204 PACS number(s): 05.45.Gg

I. INTRODUCTION

Since some pioneer works were given by Ott, Grebogi, and Yorke (OGY) [1], and Pecora and Carroll [2], chaos control and synchronization has become an active subject in the field of nonlinear science due to its potential applications to various disciplines, see the recent reviews [3,4]. A focused problem in chaos control and synchronization is how to design a physically applicable controller to stabilize or synchronize a chaotic system. In Refs. [5–7], the author proposed a simple adaptive-feedback controller to effectively stabilize and synchronize chaotic dynamics governed by the ordinary differential equations (ODEs).

However, since the letters and brief reports [5–7] were presented, some comments have been given to suspect the strictness of this adaptive-feedback algorithm, and meanwhile some private communications have been given to inquire about some crucial deductions in the results. Therefore, one of the motivations of this paper is to complement the detailed proofs for the results obtained in Refs. [5–7]. In the meanwhile, some interesting remarks related to the adaptive-feedback algorithm are given, e.g., applications to control of nonlinear oscillators, synchronization of delayed systems and analysis of time series. Especially, we give a comment on the technique of synchronization-based parameter estimation. We point out that those techniques based on the Lyapunov stability theorem in literature are less strict, so that in some cases the unknown parameters cannot be definitely estimated. Moreover, we stress that the chaotic behavior is necessary to realize such techniques of parameter estimation.

II. ADAPTIVE-FEEDBACK CONTROLLER

Based on the consideration of stabilizing the so-called near-nonhyperbolic systems, which the OGY-type methods fail to control, a control algorithm combining feedback and adaptive control was proposed in Ref. [5].

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Consider an n-dimensional system governed by ODE,

$$\dot{x} = f(x), \tag{1}$$

where $x=(x_1,x_2,\ldots,x_n)\in \mathrm{R}^n$, $f(x)=(f_1(x),f_2(x),\ldots,f_n(x)):\mathrm{R}^n\to\mathrm{R}^n$ is a nonlinear vector function. Without loss of the generality we suppose that $x^*=(x_1^*,x_2^*,\ldots,x_n^*)$ is a fixed point of (1), and the vector function $f(x)$ satisfies

$$|f_i(x)-f_i(y)| \le l \max_j |x_j - y_j|, \quad \forall\, x,y \in R^n, \quad i=1,2,\ldots,n, \tag{2}$$

where $l>0$ is a constant. We call (2) as the uniform Lipschitz condition, which is very loose. Actually, for stabilizing the system to x^* it is sufficient to replace y by x^* in (2).

For system (1), we design the following adaptive-feedback controller:

$$\dot{x} = f(x) - k(x-x^*), \tag{3a}$$

$$\dot{k} = \gamma(x-x^*)^2, \tag{3b}$$

where $kx=(k_1x_1,k_2x_2,\ldots,k_nx_n)$, $\gamma x^2=(\gamma_1x_1^2,\gamma_2x_2^2,\ldots,\gamma_nx_n^2)$, and $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$ is an arbitrary positive constant vector. Note that in comparison with that in Ref. [5] a little adjustment has been given to the controller, which guarantees the feedback strength k positive. We have the following results for system (3).

Theorem 1: The bounded orbits starting from any initial values of system (3), $[x(t),k(t)]$, converge to (x^*,k_0) as $t\to\infty$, where k_0 is a constant vector depending on the initial values. Namely, the adaptive-feedback controller stabilizes the orbits to the fixed point x^*.

Before giving the proof, we introduce an important lemma, i.e., the well-known Lasalle invariance principle [8].

Lemma 2: Consider the n-dimensional vector differential equation

$$\dot{x} = X(x). \tag{4}$$

Let $V(x)$ be a scalar function with continuous first partials for all $x\in R^n$. Assume that

(1) $V(x)\ge 0$ for all $x\in R^n$,

(2) $\dot{V}(x)\equiv \nabla V\cdot X \le 0$ for all $x\in R^n$.

Let E be the set of all points where $\dot{V}(x)=0$, and let M be the largest invariant set of (4) contained in E (a set M is said to be invariant if each solution starting in M remains in M for all t). Then every solution of (4) bounded for $t \geq 0$ approaches M as $t \to \infty$.

Proof of Theorem 1: For the $2n$-dimensional system (3), we construct the following scalar function:

$$V = \frac{1}{2}\sum_{i=1}^{n}(x_i - x_i^*)^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\gamma_i}(k_i - L)^2, \quad (5)$$

where L is a constant bigger than nl, i.e., $L > nl$. Obviously, $V \geq 0$, for all $(x,k) \in R^{2n}$. By differentiating the function V along the trajectories of system (3), we obtain

$$\dot{V} = \sum_{i=1}^{n}(x_i - x_i^*)[f_i(x) - k_i(x_i - x_i^*)] + \sum_{i=1}^{n}(k_i - L)(x_i - x_i^*)^2$$
$$= \sum_{i=1}^{n}(x_i - x_i^*)f_i(x) - L\sum_{i=1}^{n}(x_i - x_i^*)^2. \quad (6)$$

Note that if

$$\sum_{i=1}^{n}(x_i - x_i^*)f_i(x) \leq 0, \quad (7)$$

then the condition $\dot{V} \leq 0$ naturally holds for all $(x,k) \in R^{2n}$. Otherwise, we have

$$\sum_{i=1}^{n}(x_i - x_i^*)f_i(x) = \left|\sum_{i=1}^{n}(x_i - x_i^*)f_i(x)\right| \leq \sum_{i=1}^{n}|x_i - x_i^*||f_i(x)|. \quad (8)$$

Then using the condition (2) above (for the sake of simplicity we assume $|x_1 - x_1^*| \equiv \max_j |x_j - x_j^*|$), we have

$$\sum_{i=1}^{n}(x_i - x_i^*)f_i(x) \leq l|x_1 - x_1^*|\sum_{i=1}^{n}|x_i - x_i^*| \leq nl(x_1 - x_1^*)^2$$
$$\leq nl\sum_{i=1}^{n}(x_i - x_i^*)^2. \quad (9)$$

Substituting (9) into (6) gives for all $(x,k) \in R^{2n}$,

$$\dot{V} \leq (nl - L)\sum_{i=1}^{n}(x_i - x_i^*)^2 \leq 0 \quad (10)$$

because of the condition $L > nl$. Namely, for system (3), the constructed scalar function V satisfies the conditions (1) and (2) in Lemma 2. In the other side, from the above deduction it is easy to find that the set E as in Lemma 2 is given by

$$E = \{(x,k) \in R^{2n} : \dot{V} = 0\} = \{(x,k) \in R^{2n} : x = x^*\}.$$

Moreover, in conjunction with system (3) the largest invariant set M contained in E is

$$M = \{(x,k) \in R^{2n} : x = x^*, k = k_0\},$$

where k_0 is an arbitrary constant vector in R^n. Then Theorem 1 follows from Lemma 2, where k_0 is a constant vector depending on the initial values (and the parameter γ as well).

Besides those remarks given in Ref. [5], we will give some new remarks on this adaptive-feedback control algorithm.

Remark 1: Just as we pointed out in Ref. [5], it is not necessary for some particular models to use all the variables of system as feedback signals. In particular, for those systems with nonhyperbolic chaotic attractor such an adaptive-feedback control added to only partial variables is sufficient to stabilize chaotic orbits. We speculate that the minimal number of the needed control variables is just the number of positive Lyapunov exponents of the considered system. Numerous numerical examples including the Sprott's collection of the simplest chaotic systems [9] show this conjecture likely right. As examples, we consider Hindmarsh-Rose model and FitzHugh-Rinzel model. In comparison with two control terms used in Ref. [5], one control term should be enough to stabilize chaotic orbits of the Hindmarsh-Rose model and FitzHugh-Rinzel model. Adding the adaptive-feedback controller to the first variable of Hindmarsh-Rose model and FitzHugh-Rinzel model, respectively, which represents the membrane potential of neuron, we obtain the controlled Hindmarsh-Rose system

$$\begin{aligned} \dot{x} &= y - x^3 + 5.0505x^2 - 5.5025x - z - k_1 x, \\ \dot{y} &= -5x^2 + 6.835x - y, \\ \dot{z} &= 0.0012(-z + 4x), \end{aligned} \quad (11a)$$

$$\dot{k}_1 = 0.01x, \quad (11b)$$

and the controlled FitzHugh-Rinzel system

$$\begin{aligned} \dot{x} &= 1.7832x - \frac{1}{3}x^3 - 0.885x^2 - y + z - k_1 x, \\ \dot{y} &= 0.08(x - 0.8y), \\ \dot{z} &= 0.0001(-x - z), \end{aligned} \quad (12a)$$

$$\dot{k}_1 = 0.01x. \quad (12b)$$

Note that for the sake of simplicity the corresponding fixed points have been transformed to the origin. Figures 1 and 2 show that the adaptive control is successful although the transient time is very long. In both numerical simulations, the initial values are set as $[x(0), y(0), z(0), k_1(0)] = (-0.5, -0.3, 0.1, 0)$.

Remark 2: The bound of solution is very important for this adaptive-feedback algorithm. Actually, it is because the Lasalle invariance principle is only applicable for those solutions bounded for all $t \geq 0$, see Lemma 2. However, if the considered solution is chaotic, then it is naturally bounded. Especially if the system possesses a bounded chaotic invariant set with global attraction, any solution of the system is naturally bounded for all $t \geq 0$.

Remark 3: Although all illustrative examples used in Ref. [5] are those systems with a unique fixed point, the scheme is free to the number of fixed points. Now we use the famous Lorenz model to illustrate its effectiveness in the case of multiequilibrium. Consider the typical Lorenz equations

$$\dot{x} = -10x + 10y,$$

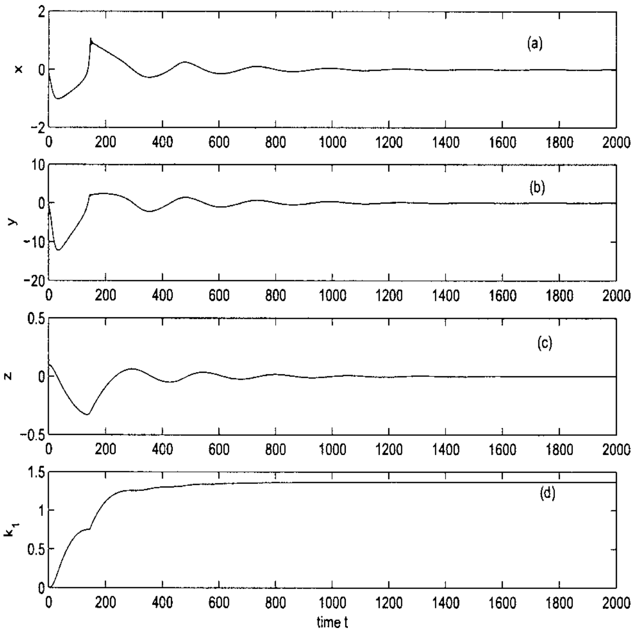

FIG. 1. (a)–(d) show numerically the success in stabilizing Hindmarsh–Rose model to $(0,0,0)$ by adding the adaptive control to the variable x in (11).

$$\dot{y} = 28x - y - xz,$$

$$\dot{z} = xy - \tfrac{8}{3}z \tag{13}$$

with three fixed points

$$X_1 = (0,0,0), \quad X_2 = (6\sqrt{2},6\sqrt{2},27), \quad X_3 = (-6\sqrt{2},-6\sqrt{2},27). \tag{14}$$

To stabilize the Lorenz system to the fixed points above, adding adaptive-feedback controller to the second variable, i.e., y, gives

$$\begin{aligned} \dot{x} &= -10x + 10y, \\ \dot{y} &= 28x - y - xz - k_2(y - y^*), \\ \dot{z} &= xy - \frac{8}{3}z, \end{aligned} \tag{15a}$$

$$\dot{k}_2 = 0.1(y - y^*)^2, \tag{15b}$$

where y^* is the second component of the considered fixed point. Namely, let $y^*=0$, $6\sqrt{2}$, and $-6\sqrt{2}$, respectively, then chaotic orbits will be stabilized to X_1, X_2, and X_3 in (14). Numerical results are shown in Figs. 3–5, where the initial values are $(0.2,0.1,0.01,0)$.

Remark 4: When the considered system is nonautonomous, i.e., the vector function $f(x)$ in (1) explicitly contains the time t, say

$$\dot{x} = f(x,t),$$

the proposed adaptive-feedback algorithm is still effective if $f(x,t)$ is bounded for t. The proof of this conclusion is easy, and hence here we only give an illustrative example. We consider a simple planar pendulum whose base is subject to a vertical, periodic excitation given by $0.1 \sin t$ (such motion will appear chaotic in general). Its motion is governed by equation

$$\ddot{x} + \sin x(1 - 0.1 \sin t) = 0, \tag{16}$$

equivalent with

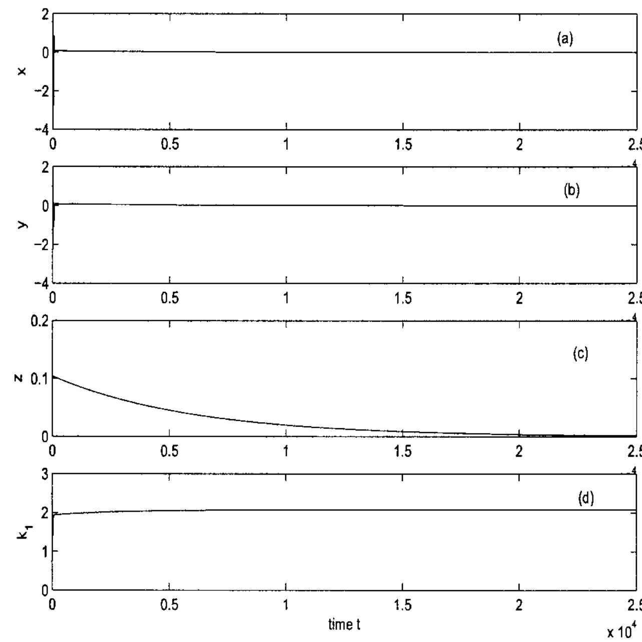

FIG. 2. (a)–(d) show numerical results of stabilizing the chaotic FitzHugh–Rinzel neuron model by the adaptive control algorithm in (12).

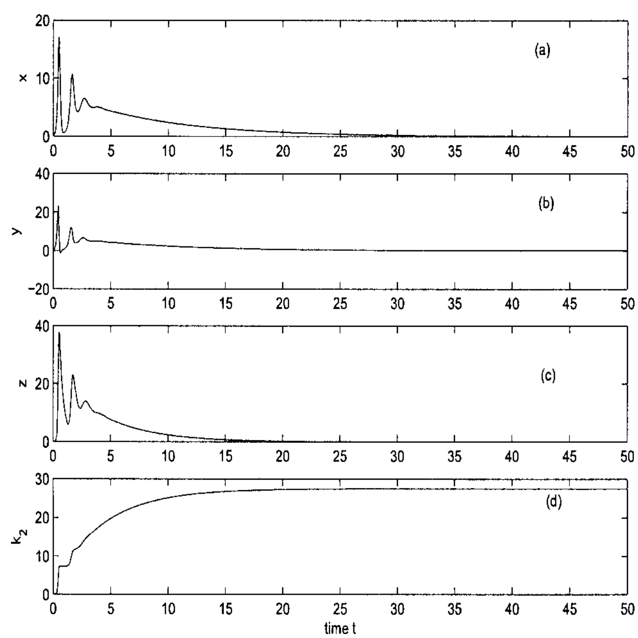

FIG. 3. (a)–(d) show the Lorenz equations are stabilized to the fixed point $X_1=(0,0,0)$ by the adaptive scheme (15) with $y^*=0$.

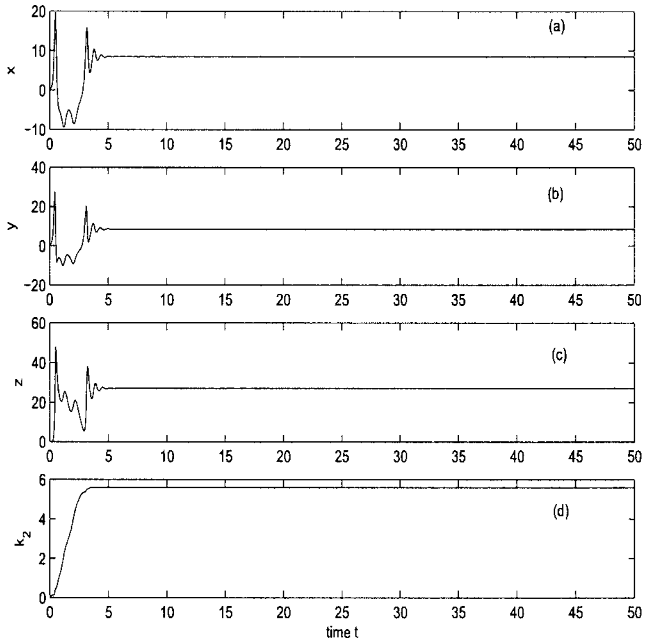

FIG. 4. (a)–(d) show the Lorenz equations are stabilized to the fixed point $X_2=(6\sqrt{2},6\sqrt{2},27)$ by the adaptive scheme (15) with $y^*=6\sqrt{2}$.

$$\dot{x}=y,$$

$$\dot{y}=-\sin x(1-0.1\sin t). \tag{17}$$

To stabilize the system, we add an adaptive damping term to the model, and obtain the following system:

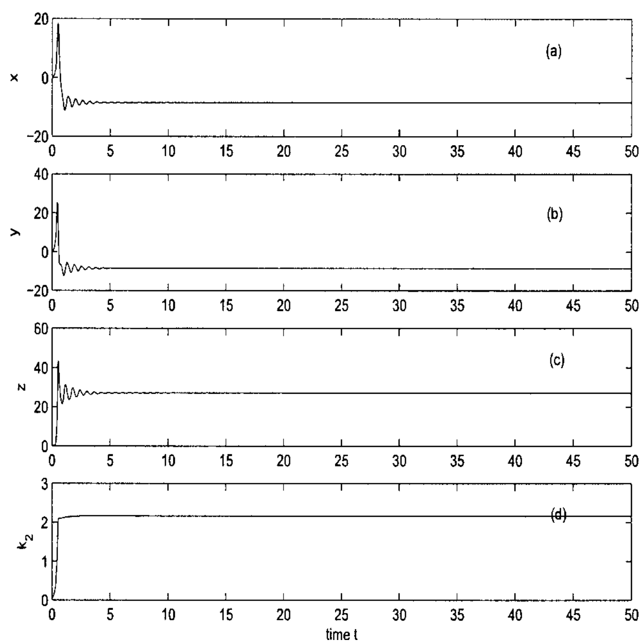

FIG. 5. (a)–(d) show the Lorenz equations are stabilized to the fixed point $X_3=(-6\sqrt{2},-6\sqrt{2},27)$ by the adaptive scheme (15) with $y^*=-6\sqrt{2}$.

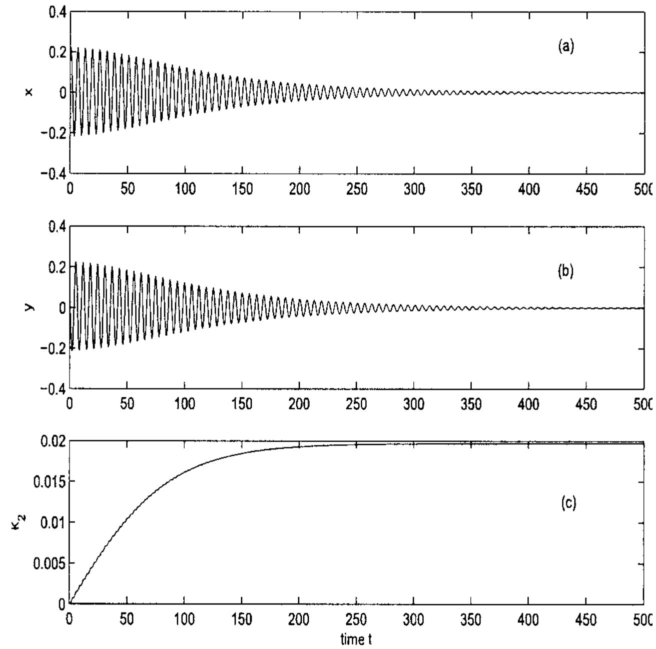

FIG. 6. (a)–(c) show the nonautonomous model of a simply pendulum with periodic excitation is stabilized by adding the adaptive damping in (18).

$$\dot{x}=y,\quad \dot{y}=-\sin x(1-0.1\sin t)-k_2y, \tag{18a}$$

$$\dot{k}_2=0.01y, \tag{18b}$$

where k_2 represents damping. Numerical results in Fig. 6 show that the system is stabilized to the fixed point $(0,0)$ by such a damping control, where the initial values are $(0.2,0.1,0)$. The example implies that the proposed adaptive-feedback algorithm has potential applications to control nonlinear oscillators.

III. ADAPTIVE SYNCHRONIZATION WITH APPLICATION

One of the central questions concerned with chaos synchronization is: Given two arbitrary identical chaotic systems, how to design a physically available coupling scheme to strictly produce stable identical synchronization motion. Instead of the numerical methods used in literature, e.g., based on the computation of the conditional Lyapunov exponents, the author used the idea of the adaptive-feedback algorithm above to give a more reasonable solution to this question, see Ref. [6].

Letting the system in the form of (1) as a drive system, we construct the following drive-response system by unidirectionally coupling:

$$\dot{x}=f(x), \tag{19a}$$

$$\dot{y}=f(y)-k(y-x), \tag{19b}$$

$$\dot{k} = \gamma(y - x)^2, \tag{19c}$$

where $k(y-x)=(k_1e_1,k_2e_2,\ldots,k_ne_n)$ is the added adaptive-feedback term with $e_i=(y_i-x_i)$, $i=1,2,\ldots,n$ denoting the synchronization error, and $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$ is an arbitrary positive constant vector. To guarantee synchronization between x and y we must prove that the invariant manifold of synchronization in (19), $x=y$, is globally attractive (or stable), which is just a conclusion that the following theorem describes.

Theorem 3: Suppose that the uniform Lipschitz condition (2) holds, then the bounded solutions starting from arbitrary initial values of (19) possess asymptotic behavior: $y-x\to 0$ and $k\to k_0$ as $t\to\infty$, where k_0 is a constant vector depending on the initial values.

The proof of this theorem is similar to that of Theorem 1, and the only difference is to construct the following scalar function V for the $3n$-dimensional system (19):

$$V = \frac{1}{2}\sum_{i=1}^{n} e_i^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\gamma_i}(k_i - L)^2,$$

where the constant $L>nl$. Therefore the proof is left out here. In addition, those remarks on Theorem 1 are also applicable to Theorem 3. Here we would stress this adaptive algorithm may be directly extended to the delayed systems, say in the form of

$$\dot{x} = f[x(t-\tau)],$$

where $\tau>0$ denotes the delay. By introducing the function

$$V = \frac{1}{2}\sum_{i=1}^{n} e_i^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\gamma_i}(k_i - L)^2 + \frac{1}{2}\sum_{i=1}^{n}\int_t^{t+\tau}\{f_i[y(s-\tau)] - f_i[x(s-\tau)]\}^2 ds,$$

where the constant $L>\frac{1}{2}(1+nl^2)$, one may prove easily that such two delayed systems will synchronize under the adaptive scheme as (19). Further, one may consider similarly the case of time-varying delay.

Next we will discuss how to apply such a simple adaptive-feedback synchronization algorithm to time series analysis. Time series analysis is one of interesting applications of chaotic synchronization. Assuming that the number of independent variables and the structure of underlying dynamical equations for a chaotic system are available, in Ref. [7] we addressed how to use adaptive chaos synchronization to dynamically estimate all model parameters of the experimental system. We consider an n-dimensional (experimental) chaotic system (e.g., with a bounded chaotic invariant set) in the form of

$$\dot{x} = F(x,p), \tag{20}$$

where $x=(x_1,x_2,\ldots,x_n)\in \mathrm{R}^n$, $F(x,p)=[F_1(x,p),F_2(x,p),\ldots,F_n(x,p)]$ is a nonlinear vector function with

$$F_i(x,p) = c_i(x) + \sum_{j=1}^{m} p_{ij}f_{ij}(x), \quad i = 1,2,\ldots,n. \tag{21}$$

Here $c_i(x), f_{ij}(x)$ are some nonlinear functions, and $p=(p_{ij})\in \mathrm{R}^{nm}$ are nm unknown parameters to be estimated. We still assume the vector field (20) satisfies the uniform Lipschitz condition, i.e., there exists a constant $l>0$ such that

$$|F_i(x,p) - F_i(y,p)| \le l \max_j |x_j - y_j|, \quad \forall\, x,y \in R^n, \quad i = 1,2,\ldots,n. \tag{22}$$

To estimate the unknown parameters p from the time series, we introduce the receiver system

$$\dot{y} = F(y,q) - k(y - x), \tag{23a}$$

$$\dot{q}_{ij} = -\delta_{ij}e_if_{ij}(y), \quad i = 1,2,\ldots,n, \quad j = 1,2,\ldots,m, \tag{23b}$$

$$\dot{k} = \gamma(y - x)^2, \tag{23c}$$

where $\delta_{ij}>0$ are arbitrary constants, and the other notations are the same as those in (19). We have the following conclusion.

Theorem 4: The solutions starting from arbitrary initial values of (20) and (23) possess asymptotic behavior: $y-x\to 0$ and $q\to p$ as $t\to\infty$. Namely, the unknown parameters p may be dynamically estimated from q in the receiver system.

Proof: For the $(3n+nm)$-dimensional system consisting of (20) and (23), we introduce the scalar function

$$V(x,y,q,k) = \frac{1}{2}\sum_{i=1}^{n} e_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{\delta_{ij}}(q_{ij} - p_{ij})^2 + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\gamma_i}(k_i - L)^2, \tag{24}$$

where $L>nl$. Differentiating it along the system gives

$$\begin{aligned}\dot{V}(x,y,q,k) &= \sum_{i=1}^{n} e_i[F_i(y,q) - F_i(x,p) - k_ie_i] - \sum_{i=1}^{n}\sum_{j=1}^{m} e_i(q_{ij} - p_{ij})f_{ij}(y) + \sum_{i=1}^{n}(k_i - L)e_i^2 \\ &= \sum_{i=1}^{n} e_i[F_i(y,q) - F_i(x,p)] - \sum_{i=1}^{n}\sum_{j=1}^{m} e_i(q_{ij} - p_{ij})f_{ij}(y) - L\sum_{i=1}^{n} e_i^2.\end{aligned} \tag{25}$$

Use of (21) we have

$$\sum_{i=1}^{n} e_i[F_i(y,q) - F_i(x,p)] - \sum_{i=1}^{n}\sum_{j=1}^{m} e_i(q_{ij} - p_{ij})f_{ij}(y)$$
$$= \sum_{i=1}^{n} e_i\left(c_i(y) + \sum_{j=1}^{m} q_{ij}f_{ij}(y) - F_i(x,p)\right)$$
$$- \sum_{i=1}^{n}\sum_{j=1}^{m} e_i q_{ij} f_{ij}(y) + \sum_{i=1}^{n}\sum_{j=1}^{m} e_i p_{ij} f_{ij}(y)$$
$$= - \sum_{i=1}^{n} e_i F_i(x,p) + \sum_{i=1}^{n} e_i\left(c_i(y) + \sum_{j=1}^{m} p_{ij}f_{ij}(y)\right)$$
$$= \sum_{i=1}^{n} e_i[F_i(y,p) - F_i(x,p)]. \quad (26)$$

Substituting (26) into (25) and applying condition (22), we obtain

$$\dot{V}(x,y,q,k) = \sum_{i=1}^{n} e_i[F_i(y,p) - F_i(x,p)] - L\sum_{i=1}^{n} e_i^2$$
$$\leq (nl - L)\sum_{i=1}^{n} e_i^2 \leq 0. \quad (27)$$

From the above deduction it is easy to find that the set E such that $\dot{V}=0$ is given by

$$E = \{(x,y,q,k) \in R^{3n+nm}:\dot{V} = 0\} = \{(x,y,q,k) \in R^{3n+nm}:x = y\}. \quad (28)$$

In the other side, thanks to chaotic behavior of the considered system (i.e., all solutions of the system approach a chaotic invariant set), the largest invariant set M contained in E is

$$M = \{(x,y,q,k) \in R^{3n+nm}:x = y, q = p, k = k_0\}, \quad (29)$$

where k_0 is an arbitrary constant vector in R^n. Then Theorem 4 follows from Lemma 2.

For this result, we give the following remarks.

Remark 5: Note from the above proof that Eq. (9) in Ref. [7] is deduced not by simply neglecting the term $-\Sigma_{i=1}^{n}\Sigma_{j=1}^{m}(q_{ij}-p_{ij})f_{ij}(y)$, but by combining the first two terms there. Meanwhile, note that the set M defined in (29) is a unique invariant set contained in the set E due to the chaotic characteristic of system. So $q(t)$ in the receiver system can approximate definitely the unknown parameters p as $t \to \infty$.

Remark 6: According to the results in Ref. [6], it is not necessary for the adaptive synchronization of some particular chaotic models to use all the variables of system as feedback signals. So one may estimate the unknown parameters through the time series of the partial variables. For example, for the Lorenz system

$$\dot{x} = p_1(y - x),$$
$$\dot{y} = p_2 x - xz - y,$$
$$\dot{z} = xy - p_3 z, \quad (30)$$

it is sufficient to estimate the parameter when the time series of the second variable y are available. This is important for secure communication using parameter modulation. For example, when p_2 in the Lorenz system (30) is modulated by a digital information signal, the sent message may be retrieved with a good quality when the time series of y are available.

Remark 7: Note from (21) that the unknown parameters are linear. The proposed method, however, may be directly applied to the case of nonlinearity. For example, we replace (21) by

$$F_i(x,p) = c_i(x) + \sum_{j=1}^{m} S_{ij}(p_{ij})f_{ij}(x), \quad i = 1,2, \ldots ,n, \quad (31)$$

where S_{ij} are differentiable nonlinear functions. Then as long as replacing (23b) by

$$\dot{q}_{ij} = - \delta_{ij}\left(\frac{dS_{ij}(q_{ij})}{dq_{ij}}\right)^{-1} e_i f_{ij}(y), \quad i = 1,2, \ldots ,n,$$
$$j = 1,2, \ldots ,m, \quad (32)$$

the unknown parameters p_{ij} may be estimated similarly. The proof is similar to that of Theorem 4, but it is necessary to replace the scalar function V in (24) by

$$V(x,y,q,k) = \frac{1}{2}\sum_{i=1}^{n} e_i^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{1}{\delta_{ij}}[S_{ij}(q_{ij}) - S_{ij}(p_{ij})]^2$$
$$+ \frac{1}{2}\sum_{i=1}^{n} \frac{1}{\gamma_i}(k_i - L)^2. \quad (33)$$

In conjunction with Theorem 4 and Remark 7, the proposed method gives an analytical treatment on synchronization-based parameter estimation. So it supplies a resolution to the puzzlement in the pioneer work [10], where the author gave such a statement: "In general, however, an analytical treatment of the problem is not possible and a practical approach for deriving the differential equations of the parameters is desirable."

IV. COMMENT ON SYNCHRONIZATION-BASED PARAMETER ESTIMATION

In conclusion, the results obtained in Refs. [5–7] have been proved in detail. However, some investigations still remain. For example, how to extend this adaptive idea to the discrete systems. Actually, in the course of trying to deduce the discrete version of this adaptive algorithm, we found that the difficulty went beyond that we had expected.

We would stress that we prove these conclusions with the well-known Lasalle invariance principle, i.e., Lemma 2, not Lyapunov asymptotic stability theorem (although they look similar). Actually, the obtained results cannot be rigorously deduced by the Lyapunov asymptotic stability theorem. Especially, the technique of identifying parameter based on synchronization is guaranteed effective only by Lasalle invariance principle. To my best knowledge, however, in the literature on synchronization-based parameter estimation (including the simple example discussed in the pioneer work given by Parlitz in Ref. [10]) the Lyapunov asymptotic stability theorem was used to prove the corresponding techniques effective. Actually, the proof like that is problematic. To show this point, we first state the Lyapunov asymptotic stability theorem (for the sake of simplicity here we give only the case of the global asymptotic stability).

Lyapunov theorem: Consider the n-dimensional vector differential equation

$$\dot{x} = X(x). \tag{34}$$

Suppose $x=0$ is its fixed point. Let $V(x)$ be a scalar function with continuous first partials for all $x \in \mathrm{R}^n$. Assume that

(i) $V(0)=0$ and $V(x)>0$ for all $x \neq 0$,

(ii) $\dot{V}(x) \equiv \nabla V \cdot X < 0$ for all $x \neq 0$ and $\dot{V}(0)=0$.

Then all solutions of (34) approach 0 as $t \to \infty$, i.e., the equilibrium solution $x=0$ is globally asymptotically stable.

Note that one of the obvious differences between this theorem and Lemma 2 is: Lyapunov theorem requires that $\dot{V}(x)$ is strictly negative, and vanishes only at the considered fixed point. In addition, it is easy to find that if $x=0$ is globally asymptotically stable, then it must be a unique fixed point of (34).

Next we will show how problematic synchronization-based parameter estimation in literature (where the Lyapunov asymptotic stability theorem was used) is. To do it, we use that example considered in Ref. [10].

Consider a model based on the Lorenz system,

$$\dot{x}_1 = \sigma(x_2 - x_1),$$

$$\dot{x}_2 = p_1 x_1 - p_2 x_2 - x_1 x_3 + p_3,$$

$$\dot{x}_3 = x_1 x_2 - b x_3, \tag{35}$$

where $\sigma=10$, $b=\frac{8}{3}$, p_i, $i=1,2,3$ are unknown parameters with $p_2>0$. To estimate the unknown parameters, such a receiver system is introduced in Ref. [10],

$$\dot{y}_1 = \sigma(x_2 - y_1), \tag{36a}$$

$$\dot{y}_2 = q_1 y_1 - q_2 y_2 - y_1 y_3 + q_3,$$

$$\dot{y}_3 = y_1 y_2 - b y_3,$$

with the parameter update law

$$\dot{q}_1 = (x_2 - y_2) y_1,$$

$$\dot{q}_2 = -(x_2 - y_2) y_2,$$

$$\dot{q}_3 = x_2 - y_2. \tag{36b}$$

By investigating the dynamics of the difference $e=y-x$ and $f=q-p$, it is easy to find $e_1 \to 0$, i.e., $y_1 \to x_1$ as $t \to \infty$. Then in Ref. [10] replacing x_1 by y_1, the remaining equations govern the dynamics of the difference $(e_2, e_3, f_1, f_2, f_3)$ [see Eq. (10) in Ref. [10]],

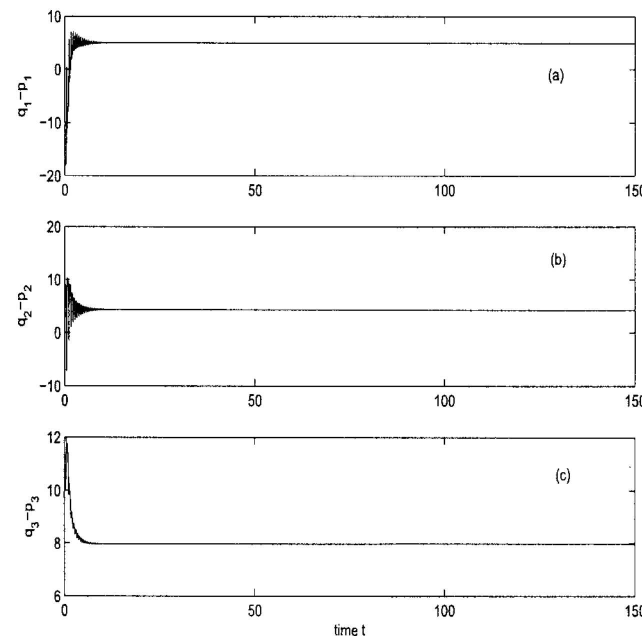

FIG. 7. (a)–(c) show that the parameters p of system (35) with $\sigma=10$, $b=5$ cannot be estimated through q under the scheme in (36), where the initial values are $(x,y,q)=(0.1,0,1,0.1,-0.1,0.1,0,10,10,10)$, $p_1=28$, $p_2=1$, and $p_3=0$.

$$\dot{e}_2 = y_1 f_1 - y_2 f_2 - p_2 e_2 - y_1 e_3 + f_3,$$

$$\dot{e}_3 = y_1 e_2 - b e_3,$$

$$\dot{f}_1 = -e_2 y_1,$$

$$\dot{f}_2 = e_2 y_2,$$

$$\dot{f}_3 = -e_2. \tag{37}$$

For this system, the Lyapunov function, $L=e_2^2+e_3^2+f_1^2+f_2^2+f_3^2$, was introduced in Ref. [10], and $\frac{1}{2}\dot{L}=-p_2 e_2^2 - b e_3^2 \le 0$ due to $p_2>0$. Then the author claimed that since the derivative of the Lyapunov function is strictly negative, the receiver system (36) converges globally to the parameters p in system (35) and synchronizes, i.e., the unknown parameters p may be estimated through q.

However, note that $\frac{1}{2}\dot{L}=-p_2 e_2^2 - \frac{8}{3} e_3^2$ is not strictly negative, but vanishes in the set $\{(e_2, e_3, f_1, f_2, f_3) \in R^5 : e_2=0, e_3=0, \forall f_i\}$. Therefore, $f \to 0$ (i.e., $q \to p$) as $t \to \infty$ cannot be strictly guaranteed by the Lyapunov theorem. Actually, it is easy to find that system (37) probably possesses the other fixed points besides $(0,0,0,0,0)$. Since the fixed points of (37) are determined by $e_2=e_3=0$ and the formula

$$y_1 f_1 - y_2 f_2 + f_3 = 0, \tag{38}$$

if y_1 and y_2 are constants (or approach constants as $t \to \infty$), then the existence of more than one fixed points is sure for system (37). In other words, its fixed point $(0,0,0,0,0)$ would not be globally stable (at most locally stable). In this case, the asymptotic behaviors of the difference

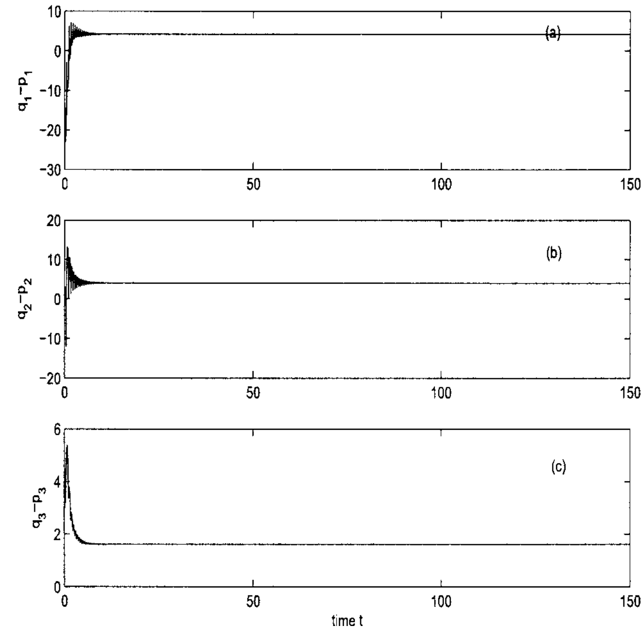

FIG. 8. (a)–(c) show the results of numerically solving systems (35) and (36) with $\sigma=10$, $b=5$ and initial values $(x,y,q)=(0.1,0,1,0.1,-0.1,0.1,0,5,4,3)$. In comparison with those in Fig. 7, the results indicate that the converged values of q depend on the corresponding initial values.

(e_2,e_3,f_1,f_2,f_3) depend on the initial values. Instead, if y_1 and y_2 are chaotic (their values look like random and arbitrary), then the unique solution of the algebra equation (38) is $f_1=f_2=f_3=0$. So in this case (0,0,0,0,0) is a unique fixed point in system (37), and naturally p may be estimated through q. This is just the reason why the technique used in Ref. [10] is effective with those parameters.

Now we show numerically this analysis by altering the system parameter, say b. For example, we only replace $b=\frac{8}{3}$ by $b=5$ while the other parameters are the same as those in Ref. [10], i.e., $\sigma=10, p_1=28, p_2=1, p_3=0$. Numerical results in Figs. 7 and 8 show accordant with the above analysis. In Fig. 7 the initial values are set as $(x,y,q)=(0.1,0.1,0.1,-0.1,0.1,0,10,10,10)$ (i.e., same as those in Ref. [10]), but $(0.1,0.1,0.1,-0.1,0.1,0,5,4,3)$ in Fig. 8. Neither of the two cases q can converge to the target p precisely. Moreover, the converged values with two different initial values are different. In addition, it is easy to check numerically that the converged values indeed satisfy equation (38), and meanwhile y_1 and y_2 converge. Similarly, suitably adjusting the parameter σ, one will find the similar problems in this example.

Similarly, in many papers on synchronization-based estimation parameter the authors introduced a Lyapunov function, say $L=\frac{1}{2}\Sigma_{i=1}^{n}e_i^2+\frac{1}{2}\Sigma_{j=1}^{m}(q_j-p_j)^2$ [here $q(t)$ will be used to estimate the unknown parameters p, which satisfies a certain update law], and concluded that due to the Lyapunov stability theorem $q(t)$ can precisely estimate the target p if $\dot{L}$ is negative, say $\dot{L}=-\Sigma_{i=1}^{n}e_i^2$. From the above discussion, however, the proof like that is actually problematic. Just as we pointed out in Ref. [11], in this case using the Lasalle invariance principle rather than the Lyapunov stability theorem may guarantee the effectiveness of the proposed method. Moreover, the chaotic behavior of the system is necessary for a successful estimation.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (Grants Nos. 10572080; 10432010), Shanghai Rising-Star Program (Grant No. 05QMX1422), Dawn Project of the Education Committee of Shanghai Municipality (Grants Nos. 05SG41; 04YQHB089), and the Alexander von Humboldt Foundation.

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