Virial theorem and dynamical evolution of self-gravitating Brownian particles in an unbounded domain. I. Overdamped models

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(Received 5 May 2005; revised manuscript received 6 December 2005; published 1 June 2006)

We derive the virial theorem appropriate to the generalized Smoluchowski-Poisson (GSP) system describing self-gravitating Brownian particles in an overdamped limit. We extend previous works by considering the case of an unbounded domain and an arbitrary equation of state. We use the virial theorem to study the diffusion (evaporation) of an isothermal Brownian gas above the critical temperature T_c in dimension $d=2$ and show how the effective diffusion coefficient and the Einstein relation are modified by self-gravity. We also study the collapse at $T=T_c$ and show that the central density increases logarithmically with time instead of exponentially in a bounded domain. Finally, for $d>2$, we show that the evaporation of the system is essentially a pure diffusion slightly slowed down by self-gravity. We also study the linear dynamical stability of stationary solutions of the GSP system representing isolated clusters of particles and investigate the influence of the equation of state and of the dimension of space on the dynamical stability of the system.

DOI: 10.1103/PhysRevE.73.066103 PACS number(s): 05.90.+m, 05.40.−a, 47.20.−k, 05.70.−a

I. INTRODUCTION

The theory of Brownian motion is a fundamental problem in statistical mechanics [1]. It has been applied to a diversity of domains of physics such as colloidal suspensions, stellar systems, chemical reaction rate theory, and nuclear dynamics, just to mention a few. In the standard theory developed by Einstein and Smoluchowski, the particles evolve in a *fixed* external potential. However, it is of interest to consider the more general situation in which the potential is generated by the particles themselves. In particular, we shall consider a system of self-gravitating Brownian particles introduced and studied in [2–10]. The statistical equilibrium states are the same as for stellar systems but the dynamics is different. Stellar systems are governed by the Newton-Hamilton equations of motion. By contrast, self-gravitating Brownian particles are subject, in addition to the gravitational force, to a friction force and a stochastic force. Therefore, their dynamics is governed by a system of coupled Langevin equations. This Brownian model could find application in the process of planetesimal formation in the solar nebula since the dust particles (giving rise to the planetesimals through gravitational collapse) are submitted to a friction with the gas and to a random force due to turbulence [11]. More generally, the friction and noise can mimic the interaction of the system with a heat bath (of nongravitational origin) where the temperature measures the strength of the stochastic force. We emphasize that the self-gravitating Brownian gas model describes *dissipative* systems while stellar systems are isolated. Therefore, the proper statistical ensemble for self-gravitating Brownian particles is the canonical ensemble (fixed temperature) while the proper statistical ensemble for stellar systems is the microcanonical ensemble (fixed energy) [9]. This is important because statistical ensembles are generically nonequivalent for systems with long-range interactions like gravity [12] so that the limits of stability differ. In the high-friction limit $\xi\rightarrow+\infty$ (or equivalently for large times $t\gg\xi^{-1}$) and for a large number of particles, $N\rightarrow+\infty$, in a proper thermodynamic limit [9], the evolution of the self-gravitating Brownian gas is described by the Smoluchowski-Poisson (SP) system. Interestingly, equations isomorphic to the Smoluchowski-Poisson system also appear in biology to describe the chemotactic aggregation of bacterial populations in the framework of the Keller-Segel model [13–15]. The analogy between self-gravitating Brownian particles and bacterial populations is developed in [7].

In previous works, we have undertaken a detailed study of the SP system describing the evolution of a self-gravitating Brownian gas with an isothermal equation of state $p=\rho k_B T/m$ [2,3,5,6]. In dimension $d=3$ for box-confined systems, two situations can occur depending on the value of the temperature T (or mass M). For sufficiently high temperatures (small mass), the system tends to an equilibrium state corresponding to a cluster of particles with a moderate density contrast. Below a critical temperature T_c (or above a critical mass M_c), there is no equilibrium state and the system collapses. We have studied the structure and stability of the equilibrium configurations, and we have found self-similar solutions describing the isothermal collapse when no steady state exists. In these studies, the particles were confined within a material box because isothermal systems tend to evaporate and disperse to infinity. In the present paper, we shall relax this assumption and study the evolution of an isothermal Brownian gas in an unbounded domain. We shall also consider more general equations of state that allow self-confined clusters to form. In that case, there is no need to introduce a box to have equilibrium states. This is the case, for example, when the equation of state is polytropic, $p=K\rho^{1+1/n}$ with an index $n<3$ in $d=3$ (ensuring the stability of the solution) [4], but other equations of state can yield

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similar results. We will therefore study the generalized Smoluchowski-Poisson (GSP) system for an arbitrary barotropic equation of state of the form $p=p(\rho)$. The generalized Smoluchowski equation was introduced in [16,17] (see an earlier version in [18] associated with the Fermi-Dirac equation of state). It can be derived from a generalized Kramers equation in a strong friction limit $\xi \to +\infty$ by using a Chapman-Enskog expansion [19] or a hierarchy of moment equations (see Sec. 9 of [20]). The GSP system can also provide a generalized model of chemotactic aggregation when the diffusion coefficient depends on the concentration of the particles, a possibility that is considered in the original Keller-Segel model [14]. For a large class of equations of state, the stationary solutions of the GSP system describe clusters of particles where the density drops to zero at a certain distance R identified with the size of the cluster. These equilibrium states are the same as for barotropic stars described by the Euler-Poisson system in astrophysics [21,22]. They correspond to a balance between pressure and gravity.

One purpose of this paper is to adapt the techniques developed in astrophysics to our problem of self-gravitating Brownian particles described by the GSP system. In particular, we shall derive the appropriate expression of the virial theorem for this system and study the dynamical stability of stationary solutions of the GSP system by a method similar to that developed by Eddington [23] and Ledoux and Walraven [24] for barotropic stars. There are two main differences between our problem and the astrophysical one. First of all, we shall work in a space of arbitrary dimension d so as to account for possible symmetries of the system (reducing the effective dimension to $d=1$ or $d=2$) and also because the mathematical problem has a rich structure as a function of the dimension of space which leads to intriguing results for $d \geq 4$ [25]. Second, although the equilibrium states are the same for self-gravitating Brownian particles and barotropic stars, the dynamical equations are different. The Smoluchowski equation is parabolic while the Euler equations are hyperbolic. Consequently, a stable barotropic star that is slightly perturbed oscillates around its equilibrium position while the density perturbation of a stable cluster of self-gravitating Brownian particles is damped exponentially. We shall derive an approximate analytical expression of the damping rate and obtain a condition of instability when it becomes negative.

The paper is organized as follows. In Sec. II, we discuss the generalized Smoluchowski-Poisson system that will be studied in the following. We show that it describes a gas of Langevin particles in gravitational interaction in a mean-field limit and in an overdamped regime. In Sec. III, we provide the form of the virial theorem appropriate to the GSP system. In Sec. III A we give its general expression, and in Sec. III B we specialize to the case of isothermal systems. In Sec. IV, we consider the diffusion of particles in $d=2$ above the critical temperature T_c and we use the virial theorem to show how the effective diffusion coefficient and the Einstein relation are affected by self-gravity (Sec. IV A). Self-similar solutions describing the evaporation of the system are obtained perturbatively in a limit of large temperature $T \gg T_c$ (Sec. IV B) and close to the critical temperature $T \to T_c^+$ (Sec. IV C). In Sec. IV D, we consider the collapse of the Brownian gas at $T=T_c$ and show that it leads to a Dirac peak for $t \to +\infty$. We show that the central density increases logarithmically with time (contrary to the case of a bounded domain studied in [3] where it increases exponentially rapidly) and that the collapsing core is surrounded by a dilute halo that diffuses at large distances so as to conserve the moment of inertia. In Sec. V, we consider the evaporation of the particles in $d>2$ and determine the universal correction to the ordinary diffusion caused by self-gravity. In Sec. VI, we analyze the dynamical stability of stationary solutions of the GSP system. We find an eigenvalue equation for the damping rate (Sec. VI A) which is the counterpart (in d dimensions) of the Eddington equation of pulsations in astrophysics (Sec. VI B). This equation can be solved by a method similar to that developed by Ledoux for barotropic stars, and this yields an analytic approximate expression for the damping rate (or growth rate) of the perturbation. This provides a criterion of linear dynamical stability and instability (Sec. VI C). In the Appendix we introduce the potential energy tensor in d dimensions. Inertial models of self-gravitating Brownian particles generalizing Eq. (5) will be discussed in our companion paper (paper II) [26].

II. GENERALIZED SMOLUCHOWSKI-POISSON SYSTEM

A system of self-gravitating Brownian particles is described, in a strong-friction limit, by the N-coupled stochastic equations

$$\frac{d\mathbf{r}_i}{dt} = -\mu m^2 \nabla_i U(\mathbf{r}_1, \ldots, \mathbf{r}_N) + \sqrt{2D}\mathbf{R}_i(t), \tag{1}$$

where $\mu=1/m\xi$ is the mobility (ξ is the friction coefficient), D is the diffusion coefficient, $\mathbf{R}$ is a white noise, and $U(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\Sigma_{i<j} u(\mathbf{r}_i-\mathbf{r}_j)$ with $u(\mathbf{r}_i-\mathbf{r}_j)=-G/[(d-2)|\mathbf{r}_i-\mathbf{r}_j|^{(d-2)}]$ denotes the gravitational potential of interaction in d dimensions [$u(\mathbf{r}_i-\mathbf{r}_j)=G \ln|\mathbf{r}_i-\mathbf{r}_j|$ for $d=2$]. For systems with long-range interactions, the mean-field approximation is exact in a properly defined thermodynamic limit $N\to+\infty$ [9]. In that case, the evolution of the density $\rho(\mathbf{r},t)$ of Brownian particles is governed by the Smoluchowski-Poisson system

$$\frac{\partial \rho}{\partial t} = \Delta(D\rho) + \nabla \cdot (\mu m \rho \nabla \Phi), \tag{2}$$

$$\Delta \Phi = S_d G \rho, \tag{3}$$

where $\Phi(\mathbf{r},t)$ is the self-consistent gravitational potential produced by the particles and S_d is the surface of the unit d-dimensional sphere. When D is constant, the stationary state of Eq. (2) is the Boltzmann distribution $\rho=Ae^{-\beta m\Phi}$ where $\beta=1/k_BT$ is the inverse temperature, provided that the diffusion coefficient D and the mobility μ are connected by the Einstein relation $\mu=D\beta$. The Smoluchowski equation can then be written in the familiar form

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{1}{\xi}\left(\frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right) \right]. \tag{4}$$

We shall consider here a more general situation in which $D=D(\rho)$ is an arbitrary function of the density ρ. This situation corresponds to a generalized class of stochastic processes introduced in [16,17]. The corresponding Fokker-Planck equations are associated with a generalized thermodynamical framework. For example, by taking $D(\rho)=K\rho^{1/n}$ [27], one describes a gas of self-gravitating Langevin particles displaying anomalous diffusion [4]. The static properties of this system reproduce that of a gravitational gas described by Tsallis statistics [28]. More generally, we set $D(\rho)=p(\rho)/\xi\rho$ where $p(\rho)$ is an arbitrary barotropic equation of state. In that case, Eq. (2) becomes

$$\frac{\partial\rho}{\partial t}=\nabla\cdot\left[\frac{1}{\xi}(\boldsymbol{\nabla}p+\rho\boldsymbol{\nabla}\Phi)\right], \quad (5)$$

$$\Delta\Phi=S_dG\rho. \quad (6)$$

The generalized Smoluchowski equation (5) has been introduced in [16,17] (see an early version in [18]). It can be derived from a generalized Kramers equation or from a generalized isotropic BGK equation in a strong-friction limit [19,20]. It can be shown that the generalized Smoluchowski-Poisson system (5), (6) decreases the Lyapunov functional

$$F[\rho]=\int\rho\int^{\rho}\frac{p(\rho')}{\rho'^2}d\rho'd\mathbf{r}+\frac{1}{2}\int\rho\Phi d\mathbf{r}, \quad (7)$$

which can be interpreted as a generalized free energy. Indeed, we have $\dot{F}\leqslant 0$ which is the counterpart of the H theorem in the canonical ensemble. The stationary states ($\dot{F}=0$) satisfy the condition of hydrostatic balance:

$$\boldsymbol{\nabla}p+\rho\boldsymbol{\nabla}\Phi=\mathbf{0}. \quad (8)$$

They extremize the free energy (7) at fixed mass. Moreover, it can be shown [16] that the condition of linear dynamical stability with respect to the GSP system ($\delta\rho\sim e^{\lambda t}$ with $\lambda<0$) coincides with the condition of thermodynamical stability in the canonical ensemble (minimum of free energy F at fixed mass).

The generalized Smoluchowski-Poisson system (5), (6) which corresponds to an overdamped evolution will be our object of interest in this paper. Different equations of state have already been considered. For the isothermal equation of state $p=\rho k_BT/m$, we recover the ordinary SP system (4), (3) studied in [2,3,5,6]. The stationary density profile of isothermal systems extends to infinity and contains an infinite mass in $d>2$. In that case, the system must be confined within a box to prevent its evaporation. For the polytropic equation of state $p=K\rho^{\gamma}$ with $\gamma=1+1/n$, we get the nonlinear Smoluchowski-Poisson (NSP) system studied in [4]. For $d>2$, the stationary solutions of the NSP system, describing polytropic distributions, have a compact support if $n<n_5=(d+2)/(d-2)$: the density vanishes at a finite distance R determining the size of the cluster. Only polytropic distributions with index $n<n_3=d/(d-2)$ are stable with respect to the NSP system (canonical ensemble). We have also considered in [7] the fermionic Smoluchowski-Poisson (FSP) system corresponding to an equation of state $p(\rho)$ appropriate to fermions. In the nondegenerate limit it reduces to the isothermal equation of state, and in the completely degenerate limit it becomes equivalent to the equation of state of a polytrope with index $n_{3/2}=d/2$. The main interest of this model is to show *dynamically* how a small-scale regularization (played here by the Pauli exclusion principle) can prevent a blowup of isothermal systems, as predicted by equilibrium statistical mechanics [29]. In the present paper, we shall consider the case of an arbitrary equation of state $p(\rho)$. We shall be particularly interested in equations of state leading to clusters with a compact support. They are probably the most physically relevant since we do not need an artificial box to confine them. For example, a relevant model generalizing the isothermal model would be a *composite model* with an isothermal core and a polytropic halo. This falls into the class of equations of state that we consider here. The models described previously have been studied in parallel by mathematicians who derived rigorous results for the existence of solutions and blowup conditions. We refer to [30,31] and references therein for a connection to the rich mathematical literature on this subject.

III. VIRIAL THEOREM FOR THE GSP SYSTEM

A. General expression

The virial theorem plays a central role in astrophysics [21,22]. It is usually derived from the Newton equations describing a system of particles in gravitational interaction or from the Euler equations describing a self-gravitating barotropic gas (star). At equilibrium, it provides a general relation between the kinetic energy and the potential energy of the system. It has also been used to determine the dynamical stability of stars and their pulsation periods [24]. We shall derive here the form of the virial theorem appropriate to a system of self-gravitating Brownian particles described by the GSP system. More general forms of the virial theorem will be given in paper II starting directly from stochastic, kinetic, or fluid equations.

Multiplying Eq. (5) by x_ix_j and integrating over the entire domain, we get

$$\int\frac{\partial\rho}{\partial t}x_ix_jd\mathbf{r}=\int x_ix_j\frac{\partial}{\partial x_k}\left[\frac{1}{\xi}\left(\frac{\partial p}{\partial x_k}+\rho\frac{\partial\Phi}{\partial x_k}\right)\right]d\mathbf{r}. \quad (9)$$

Introducing the tensor of inertia

$$I_{ij}=\int\rho x_ix_jd\mathbf{r} \quad (10)$$

and integrating the second term by parts twice, we find that

$$\frac{1}{2}\xi\frac{dI_{ij}}{dt}=\frac{2}{d}K\delta_{ij}+W_{ij}, \quad (11)$$

where W_{ij} is the potential energy tensor,

$$W_{ij}=-\int\rho x_i\frac{\partial\Phi}{\partial x_j}d\mathbf{r}, \quad (12)$$

and K is the kinetic energy defined by

$$K = \frac{d}{2}\int p d\mathbf{r}. \tag{13}$$

If the system is confined within a box, we have to account for boundary terms. The first integration by parts in Eq. (9) yields a residual term

$$\int \nabla \cdot [x_i x_j(\nabla p + \rho \nabla \Phi)] d\mathbf{r} = \oint x_i x_j (\nabla p + \rho \nabla \Phi) \cdot d\mathbf{S}, \tag{14}$$

where $d\mathbf{S}$ is the surface element normal to the frontier of the confining box. By virtue of the conservation of mass, the diffusion current in Eq. (5) is always perpendicular to the surface and consequently the term (14) vanishes. The second integration by parts yields

$$-\int \frac{\partial}{\partial x_k}\left[p(x_j\delta_{ki} + x_i\delta_{kj})\right] d\mathbf{r} = -\oint p(x_j\delta_{ki} + x_i\delta_{kj}) dS_k. \tag{15}$$

Therefore, the general form of the virial theorem for the GSP system including boundary terms is

$$\frac{1}{2}\xi\frac{dI_{ij}}{dt} = \frac{2}{d}K\delta_{ij} + W_{ij} - \frac{1}{2}\oint p(x_j dS_i + x_i dS_j). \tag{16}$$

By contracting the indices, we get the scalar virial theorem

$$\frac{1}{2}\xi\frac{dI}{dt} = 2K + W_{ii} - \oint p\mathbf{r} \cdot d\mathbf{S}, \tag{17}$$

where

$$I = \int \rho r^2 d\mathbf{r} \tag{18}$$

is the moment of inertia. If the pressure P is uniform on the edge of the box (this is the case at least for a spherically symmetric system), the boundary term can be simplified as

$$\oint p\mathbf{r} \cdot d\mathbf{S} = P\oint \mathbf{r} \cdot d\mathbf{S} = P\int \nabla \cdot \mathbf{r} d\mathbf{r} = dPV, \tag{19}$$

where V is the volume of the confining box. More generally, we define

$$P = \frac{1}{dV}\oint p\mathbf{r} \cdot d\mathbf{S}. \tag{20}$$

Then, the scalar virial theorem reads

$$\frac{1}{2}\xi\frac{dI}{dt} = 2K + W_{ii} - dPV. \tag{21}$$

Furthermore, it is shown in the Appendix that, for $d \neq 2$, $W_{ii} = (d-2)W$ where W is the potential energy. In that case, we get

$$\frac{1}{2}\xi\frac{dI}{dt} = 2K + (d-2)W - dPV \quad (d \neq 2). \tag{22}$$

For $d=2$, we have $W_{ii} = -GM^2/2$ (see the Appendix) and we obtain

$$\frac{1}{2}\xi\frac{dI}{dt} = 2K - \frac{GM^2}{2} - 2PV \quad (d=2). \tag{23}$$

At equilibrium ($\dot{I}=0$), the virial theorem reduces to

$$2K + W_{ii} = dPV. \tag{24}$$

It should be noted that the moment of inertia (18) depends on the origin of coordinates O. If we consider an origin O' at a distance $\mathbf{a}$ from O, the new moment of inertia is $I' = \int \rho(\mathbf{r} - \mathbf{a})^2 d\mathbf{r} = Ma^2 + I - 2\mathbf{a} \cdot \int \rho \mathbf{r} d\mathbf{r}$. The last term corresponds to the position of the center of mass $\mathbf{R} = (1/M)\int \rho \mathbf{r} d\mathbf{r}$ with respect to O. Now, for overdamped systems, the quantity $\mathbf{R}$ is a constant of motion. Indeed, using Eq. (5), we get

$$\frac{dR_i}{dt} = \int x_i \frac{\partial}{\partial x_j}\left[\frac{1}{\xi}\left(\frac{\partial p}{\partial x_j} + \rho\frac{\partial \Phi}{\partial x_j}\right)\right] d\mathbf{r} = -\int \frac{1}{\xi}\left(\frac{\partial p}{\partial x_i} + \rho\frac{\partial \Phi}{\partial x_i}\right) d\mathbf{r} = 0, \tag{25}$$

where we have used $\int \rho \nabla \Phi d\mathbf{r} = (1/S_d G)\int \Delta\Phi \nabla\Phi d\mathbf{r} = -(1/S_d G)\int \nabla\Phi \Delta\Phi d\mathbf{r} = \mathbf{0}$ resulting from an integration by parts (physically this expresses the fact that the sum of the internal forces acting in the system is equal to zero). Therefore, I and I' just differ by a constant term $Ma^2 - 2M\mathbf{a} \cdot \mathbf{R}$ and the preceding results with $\dot{I}$ do not, indeed, depend on the origin. It will be convenient in the following to take the origin at the center of mass such that $\mathbf{R} = \mathbf{0}$. *For overdamped systems, the center of mass does not move*. Note that if the system collapses into a Dirac peak containing the whole mass (see the sequel), this peak will form (by definition) at the center of mass where $\int \rho \mathbf{r} d\mathbf{r} = \mathbf{0}$.

B. Isothermal systems

We shall now consider the special case of isothermal systems corresponding to the standard SP system (3), (4). For an isothermal equation of state $p = \rho k_B T/m$, the kinetic energy (13) takes the usual form $K = \frac{d}{2}Nk_B T$. In that case, the scalar virial theorem (21) becomes

$$\frac{1}{2}\xi\frac{dI}{dt} = dNk_B T + W_{ii} - dPV. \tag{26}$$

In the absence of gravity ($W_{ii}=0$), the system is homogeneous. At equilibrium ($\dot{I}=0$), we recover the ideal gas law

$$PV = Nk_B T. \tag{27}$$

On the other hand, in dimension $d=2$, the virial theorem (26) becomes

$$\frac{1}{2}\xi\frac{dI}{dt} = 2Nk_B T - \frac{GM^2}{2} - 2PV. \tag{28}$$

Introducing the critical temperature

$$k_B T_c = \frac{GMm}{4}, \tag{29}$$

this can be rewritten

$$\frac{1}{2}\xi\frac{dI}{dt} = 2Nk_B(T - T_c) - 2PV. \tag{30}$$

At equilibrium ($\dot{I}=0$), we obtain the equation of state (see also Appendix B of [4])

$$PV = Nk_B(T - T_c). \tag{31}$$

This is the equation of state of a self-gravitating gas in statistical equilibrium in $d=2$ and in the mean-field limit $N \to +\infty$. This mean-field relation can also be obtained by solving the Boltzmann-Poisson equation and calculating the pressure at the edge of the box [32]. In paper II, the *exact* equation of state (valid for arbitrary value of N) is derived from the stochastic equations of motion of the N-body system. It is of the form of Eq. (31) with a critical temperature given by $k_BT_c = Gm^2(N-1)/4$ [26]. This exact relation can also be obtained directly from the partition function in the canonical ensemble [12].

We can use the virial theorem (30) to obtain some general results about the dynamics of self-gravitating Brownian systems in $d=2$ without explicitly solving the equations of motion. First, consider the case of systems enclosed within a box. Since $P \geq 0$, stationary solutions can exist only for $T \geq T_c$. They have been studied in [3] by solving the Emden equation, and their density profile is known analytically. For $T > T_c$ these solutions are maintained by the box since $P > 0$, implying $\rho(R) > 0$. For $T = T_c$ the density profile is a Dirac peak implying $\rho(R) = 0$ so that $P = 0$. For $T \leq T_c$, there is no steady state and the system undergoes gravitational collapse as studied in [3]. For $T = T_c$ the collapse is self-similar and leads for $t \to +\infty$ to a Dirac peak $\rho(\mathbf{r}) = M\delta(\mathbf{r})$ containing the whole mass. For $T < T_c$ the collapse is not exactly self-similar. In a finite time t_{coll} the system develops a Dirac peak containing a fraction T/T_c of the total mass M, surrounded by a halo whose tail decreases as $\rho \sim r^{-\alpha(t)}$ with $\alpha(t)$ converging extremely slowly to $\alpha = 2$ for $t \to t_{coll}$. Now, from the virial theorem (30), we note that $\dot{I} \leq \epsilon < 0$ for $T < T_c$ so that the moment of inertia tends to its minimum value $I = 0$ in a finite time t_{end}. This corresponds to the formation of a Dirac peak at $\mathbf{r} = \mathbf{0}$ containing the whole mass M. Since this final state is different from the structure obtained at $t = t_{coll}$, this means that the evolution continues in the post-collapse regime $t_{coll} \leq t \leq t_{end}$. In this post-collapse regime, the Dirac peak formed at $t = t_{coll}$ accretes the mass of the surrounding halo until all the mass is at $\mathbf{r} = \mathbf{0}$ at t_{end}. The post-collapse evolution has been studied in $d=3$ in [5] but the situation in $d=2$ is different and more complicated. Note that the analysis of [3] applies to a bounded domain. The case of an unbounded domain ($P=0$ at infinity) is treated specifically in Sec. IV where the evaporation (diffusion) of the system for $T \geq T_c$ is considered.

In dimension $d \neq 2$, the virial theorem for an isothermal Brownian gas reads

$$\frac{1}{2}\xi\frac{dI}{dt} = dNk_BT + (d-2)W - dPV. \tag{32}$$

At equilibrium,

$$dNk_BT + (d-2)W = dPV. \tag{33}$$

Contrary to the case $d=2$, the equation of state cannot be written in a closed form since the potential energy depends on the precise distribution of density in the cluster. We can get an approximate analytical expression of the equation of state if we assume (inconsistently) that the density distribution is uniform and use Eq. (A13) so that

$$W_{ii} = (d-2)W = -\frac{d}{d+2}\frac{GM^2}{R^{d-2}}. \tag{34}$$

In that case,

$$PV = Nk_BT - \lambda\frac{GM^2}{V^{(d-2)/2}}, \tag{35}$$

where $\lambda = \frac{1}{d+2}(S_d/d)^{(d-2)/d}$ is a geometrical constant. To be more precise, we need to take into account the spatial inhomogeneity of the density by solving the Boltzmann-Poisson equation [33].

IV. SELF-GRAVITATING BROWNIAN GAS IN $d=2$ WITH NO BOUNDARY

A. Generalized Einstein relation

In this section, we provide some explicit results concerning the dynamics of the self-gravitating Brownian gas in $d=2$. The case of a bounded domain has been discussed in [3]. Here, we consider the case of an open domain. Since the pressure vanishes at infinity ($P=0$), the virial theorem (30) reduces to

$$\frac{1}{2}\xi\frac{dI}{dt} = 2Nk_B(T - T_c). \tag{36}$$

Defining the mean-squared radius of the cluster through the relation

$$\langle r^2\rangle = \frac{\int \rho r^2 d^2\mathbf{r}}{\int \rho d^2\mathbf{r}} = \frac{I}{M}, \tag{37}$$

we can rewrite the foregoing equation in the form

$$\frac{d\langle r^2\rangle}{dt} = \frac{4k_BT}{\xi m}(1 - T_c/T). \tag{38}$$

Integrating, we get

$$\langle r^2\rangle = \frac{4k_BT}{\xi m}(1 - T_c/T)t + \langle r^2\rangle_0. \tag{39}$$

This relation suggests that an effective diffusion coefficient

$$D(T) = \frac{k_BT}{\xi m}(1 - T_c/T) \tag{40}$$

be introduced so that Eq. (39) can be rewritten

$$\langle r^2\rangle = 4D(T)t + \langle r^2\rangle_0. \tag{41}$$

For $T \gg T_c$ when gravitational effects become negligible, the Smoluchowski equation (4) reduces to a pure diffusion equation and the diffusion coefficient $D(+\infty)=k_BT/\xi m$ is given by the Einstein formula [1]. However, Eq. (40) shows that the diffusion is less and less effective as temperature decreases and gravitational effects come into play. *Therefore, relation* (40) *provides a generalization of the Einstein relation to the case of self-gravitating Brownian particles.* For $T>T_c$, we have a diffusive motion (evaporation) with an effective diffusion coefficient depending linearly on the distance $(T-T_c)$ to the critical temperature. For $T=T_c$, the effective diffusion constant vanishes $D(T_c)=0$ so that the moment of inertia is conserved. We will show below that the collapse solution of [3] in a bounded domain still holds but with a much slower divergence of the central density. Finally, for $T<T_c$ the effective diffusion coefficient is negative, implying a finite-time blowup. In particular, $\langle r^2\rangle=0$ for $t_{end}=\langle r^2\rangle_0/4|D(T)|$ (recall that, with our conventions, $\langle r^2\rangle_0$ is calculated from the center of mass). This corresponds to the formation of a Dirac peak $\rho(\mathbf{r})=M\delta(\mathbf{r})$ at $\mathbf{r}=\mathbf{0}$ containing the whole mass. The solution of [3] probably describes the collapse of the core accurately, but in the absence of a confining box, the collapse is accompanied by an unlimited expansion of the halo. In the post-collapse regime, the virial theorem indicates that all the matter falls at $\mathbf{r}=\mathbf{0}$ in a finite time. Note that the evolution toward the final Dirac peak can be progressive. During the transient regime, the system can form an ensemble of $\tilde{N}$ Dirac peaks which move according to a set of discrete renormalized equations. These Dirac peaks can merge (collapse on each other) so that their number $\tilde{N}(t)$ decreases [and their mass $\tilde{m}(t)$ increases] until a single Dirac peak containing the whole mass M remains at the end. This problem shares some analogies with the decay of the vortex number in two-dimensional (2D) decaying turbulence, and we plan to adapt the technique develops in [34] to this new context.

B. Diffusing profile for $T \gg T_c$

We now wish to study the modified diffusion for $T>T_c$. For convenience, we shall work with dimensionless variables. As shown in previous works [2], this is equivalent to taking $\xi=M=G=k_B=1$. Introducing the integrated density

$$M(r,t) = 2\pi\int_0^r \rho(r',t)r'dr', \tag{42}$$

we can show that the SP system in $d=2$ is equivalent to

$$\frac{\partial M}{\partial t} = T\left(\frac{\partial^2 M}{\partial r^2} - \frac{1}{r}\frac{\partial M}{\partial r}\right) + \frac{M}{r}\frac{\partial M}{\partial r}. \tag{43}$$

We look for a self-similar solution of the form

$$M(r,t) = g\left(\frac{r}{r_0(t)}\right). \tag{44}$$

The corresponding density profile reads

$$\rho(r,t) = \rho_0(t)F\left(\frac{r}{r_0(t)}\right), \tag{45}$$

with

$$\rho_0(t) = \frac{1}{2\pi r_0(t)^2}, \quad F(x) = \frac{g'(x)}{x}. \tag{46}$$

The conservation of mass implies that $g(x)\to 1$ for $x\to+\infty$. Inserting the ansatz (44) into Eq. (43), we obtain

$$-g'(x)xr_0\frac{dr_0}{dt} = T\left(g'' - \frac{1}{x}g'\right) + \frac{1}{x}gg', \tag{47}$$

where $x=r/r_0$. The variables x and t separate provided that we set

$$r_0\frac{dr_0}{dt} = T, \tag{48}$$

so that

$$r_0^2 = 2Tt + \text{const}. \tag{49}$$

Then, the scaling equation is

$$g'' + \left(x - \frac{1}{x}\right)g' + \epsilon\frac{gg'}{x} = 0, \tag{50}$$

where we have defined $\epsilon=1/T$. On the other hand, the mean-squared radius of the cluster is $\langle r^2\rangle=\int_0^{+\infty} r^2 dM(r)$, yielding

$$\langle r^2\rangle = r_0^2\int_0^{+\infty} g'(x)x^2dx. \tag{51}$$

For $\epsilon=0$, corresponding to pure diffusion, Eq. (50) is easily solved with the boundary conditions $g(0)=0$ and $g(+\infty)=1$, and we get

$$g_0(x) = 1 - e^{-x^2/2}, \tag{52}$$

where the subscript 0 indicates that this solution corresponds to $\epsilon=0$. We can then make an expansion in powers of ϵ. Writing $g(x)=g_0(x)+\epsilon h(x)+\cdots$ and collecting terms of order ϵ, we find that h satisfies the equation

$$h'' + \left(x - \frac{1}{x}\right)h' + \frac{g_0 g_0'}{x} = 0. \tag{53}$$

Using the expression (52) of g_0 and setting $H=h'$, we find that

$$H' + \left(x - \frac{1}{x}\right)H = e^{-x^2} - e^{-x^2/2}. \tag{54}$$

This first-order differential equation is readily solved, and we finally obtain

$$h(x) = \int_0^x (e^{-x^2/2} - e^{-y^2/2})\frac{1-e^{-y^2/2}}{y}dy + C\int_0^x ye^{-y^2/2}dy. \tag{55}$$

The constant C is determined by the requirement $h(+\infty)=0$. This yields

$$C = \frac{1}{2}\ln 2. \tag{56}$$

We note that

$$h'(x) = -xe^{-x^2/2}\int_0^x \frac{1-e^{-y^2/2}}{y}dy + Cxe^{-x^2/2}, \tag{57}$$

$$g_0'(x) = xe^{-x^2/2}. \tag{58}$$

From these relations, we can compute the mean-squared radius (51) and check that this returns the exact relation (39) obtained from the virial theorem.

C. Diffusing profile for $T-T_c \ll T_c$

Alternatively, one can look for self-similar diffusing solutions when the temperature is very close to but above $T_c = 1/4$. It proves convenient to introduce the function f defined by

$$g(x) = 2Tf(a(T)x^2/2), \tag{59}$$

where $a(T)$ is such that $f'(0)=1$. We also introduce the variable $u=a(T)x^2/2$. From Eq. (50), we obtain the equivalent simpler equation for f:

$$f''(u) + f'(u)\left(\frac{1}{a(T)} + \frac{f(u)}{u}\right) = 0, \tag{60}$$

where the temperature now appears in the boundary conditions

$$f(0) = 0, \quad f'(0) = 1, \quad f(+\infty) = f_\infty = \frac{1}{2T} < 2. \tag{61}$$

The above boundary conditions impose a unique value for $a(T)$. The maximum value of $f_\infty = 1/2T_c = 2$ is obtained for $a(T) = +\infty$, for which Eq. (60) admits the following zeroth-order exact solution:

$$f_0(u) = \frac{2u}{2+u}, \quad f_0'(u) = \frac{1}{(1+u/2)^2}. \tag{62}$$

Replacing the exact equation (60) for f by

$$f''(u) + f'(u)\left(\frac{1}{a(T)} + \frac{f_0(u)}{u}\right) = 0, \tag{63}$$

we obtain a good approximation for the scaling function:

$$f_1'(u) = \frac{1}{(1+u/2)^2}\exp(-u/a). \tag{64}$$

In addition, from Eq. (60), the exact large-u asymptotic for $f(u)$ is found to be

$$f'(u) \sim \left(\frac{u}{u_\infty}\right)^{-f_\infty}\exp(-u/a), \tag{65}$$

where u_∞ is a constant converging to 2 as f_∞ goes to its maximal value 2—that is, when $a(T) \to +\infty$. This suggests the alternative good approximation for $f'(u)$, which in addition satisfies the correct behavior at infinity:

$$f_2'(u) = \frac{1}{(1+u/f_\infty)^{f_\infty}}\exp(-u/a), \tag{66}$$

where we have chosen $u_\infty = f_\infty$ in order to reproduce the exact result $-f''(0) = 1 + 1/a$ now satisfied by both approximate solutions. To the prize of a lengthly calculation (at least for the second inequality), one can show that

$$f_1'(u) \leq f'(u) \leq f_2'(u), \tag{67}$$

which implies

$$f_1(u) \leq f(u) \leq f_2(u), \tag{68}$$

since these functions all vanish for $u=0$. In practice, f_1 is a very good approximation of f for $u \ll a$, whereas f_2 becomes better (due to the correct power-law correction) for $u \gg a$. Anyway, since we consider the case $T \to T_c$ where f_∞ is very close to 2, these bounds are in fact very stringent. In addition, multiplying Eq. (60) by u and integrating between $u=0$ and $u=\infty$, one obtains

$$\int_0^{+\infty} uf'(u)du = a(T)\frac{f_\infty}{2}(2-f_\infty) = a(T)\frac{T-T_c}{2T^2}, \tag{69}$$

which is equivalent to Eq. (39). Hence, using Eq. (67), and evaluating the above integral in the limit of large $a(T)$, we find the asymptotic result

$$\frac{T-T_c}{T_c} = 2\frac{\ln(a)-1-\gamma}{a} + O\left(\frac{\ln^2 a}{a^2}\right), \tag{70}$$

where $\gamma = 0.577\,216\ldots$ is Euler's constant and the estimate of the error is obtained from the difference between our upper and lower bounds. Inverting this relation and introducing $\varepsilon = (T-T_c)/T_c$, we find, up to leading order,

$$a(T) = 2\frac{\ln(1/\varepsilon)}{\varepsilon} + O\left(\frac{\ln\ln(1/\varepsilon)}{\varepsilon}\right). \tag{71}$$

Coming back to the original notations and expressing the result in terms of the density (which is simply related to f') rather than the integrated density (which is simply related to f), we find that, for a given T very close to T_c and for large time,

$$\rho(r,t) = \frac{a(T)}{2\pi t}f'\left(\frac{a(T)r^2}{t}\right), \tag{72}$$

where a good approximation for f' in this limit is given by f_1' or f_2' obtained above.

It is straightforward to check that by multiplying Eq. (72) by $r^2 2\pi r dr$ and after integration over space, one recovers the exact result of Eq. (39), after using Eq. (69). Note, however, that the extent $R_0(t)$ of the diffusing front grows more slowly than the naive estimate $r_0(t)$. Equation (72) indeed leads to a diffusing front

$$R_0(t) \sim r_0(t)\sqrt{\frac{\varepsilon}{\ln(1/\varepsilon)}} \ll r_0(t). \tag{73}$$

It also grows slightly slower than $r_{rms}(t) = \langle r^2\rangle^{1/2}$ since

$$R_0(t) \sim r_{rms}(t)\sqrt{\frac{1}{2\ln(1/\varepsilon)}}. \tag{74}$$

Numerical simulations of the Smoluchowski-Poisson system showing the evaporation (diffusion) of the system in a large domain (such that boundary effects are negligible) are

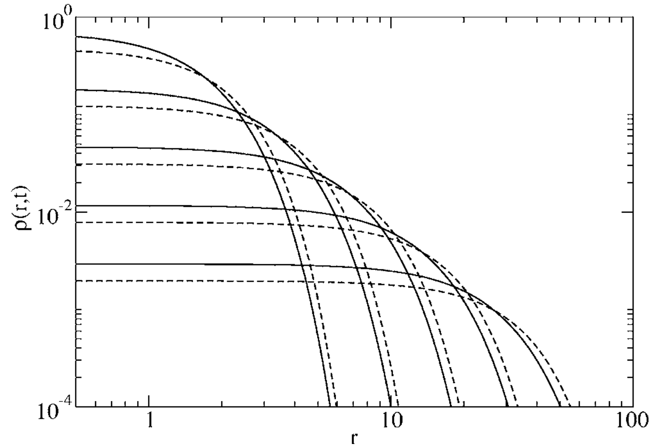

FIG. 1. Evolution of the density profile in $d=2$ for different times $t=4^n$ with $n=0,1,2,3,4$. The initial density profile has been chosen as $\rho(r,t=0)\sim(1-r^2)^3$ and the temperature as $T=1$. The solid lines correspond to the evolution of self-gravitating Brownian particles described by Eq. (43). The dashed lines correspond to a purely diffusive evolution without gravitational drift. We see that the effect of gravitational attraction is to reduce the diffusion. This is a correction of order 1 which leads to a constant shift between the two curves in the log-log plot.

reported in Figs. 1 and 2. The convergence to the self-similar profile is rapid and in excellent agreement with the theory in the asymptotic limits considered.

D. Collapse dynamics for $T=T_c$

We now consider the collapse regime in $d=2$, for $T=T_c=1/4$. If the system is confined in a box of radius R, the density profile is given with great accuracy by [3]

$$M(r,t)=\frac{a(t)r^2}{1+[a(t)-R^{-2}]r^2}, \tag{75}$$

where the central density is related to $a(t)$ by $\rho(0,t)=a(t)/\pi$ and $a(t)$ has been computed exactly for large time in [3]:

$$a(t)=\pi\rho(0,t)\sim\frac{M}{R^2}\exp\left(\frac{5}{2}+\sqrt{\frac{2Mt}{R^2}}\right). \tag{76}$$

We have reintroduced the dimensional parameters—the total mass M and the radius of the box R, (keeping $G=\xi=1$)—to emphasize that this expression does not have a proper limit for $R\to\infty$. However, this expression suggests that in this limit the divergence of the central density should be much weaker than in a finite domain. In an unbounded domain, the profile of Eq. (75) cannot be valid at large distances because it would lead to an infinite moment of inertia while we have shown in Sec. IV A that the moment of inertia is exactly conserved at T_c. In analogy with the previous result just above T_c, we expect that for a scale $\sqrt{b(t)}$ of the order of the diffusive scale $t^{1/2}$, the density profile should decay like a Gaussian. In fact, inserting the ansatz

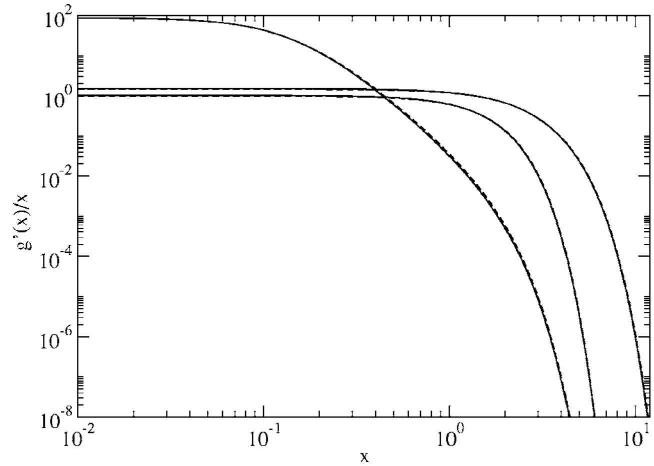

FIG. 2. Scaling profiles of the density for different values of the temperature. The upper curve corresponds to $T=0.26$ close to $T_c=1/4$. The solution of the scaling equation (50) (solid line) is in excellent agreement with the analytical expression (66) (dashed line). The lower curve corresponds to a large value of the temperature $T=10$. The solution of the scaling equation (50) (solid line) is in excellent agreement with the analytical expressions (57) and (58) (dashed line) for $T\gg1$. In particular, the measured value of the scaling profile at the origin $g'(x)/x_{|x=0}=g''(0)=1.035\,74\ldots$ is in good agreement with the expression $1+\frac{1}{T}\ln 2/2+O(T^{-2})$ coming from the theory. Finally, the middle curve corresponds to an intermediate value $T=1$ of the temperature and shows the data collapse for $t=4^{n+1}$ ($n=1,2,3,4$; solid line) of the density profiles obtained by solving the dynamical equation (43) as in Fig. 1. We see that the scaling regime is reached rapidly and that the invariant profile is perfectly described by the scaling equation (50) (dashed line). This figure shows how one goes from a Gaussian profile for $T\gg1$ to a profile generating an algebraic r^{-4} region for $T\to T_c$.

$$M(r,t)=1-\frac{h(r^2/b(t))}{1+a(t)r^2},\quad h(0)=1, \tag{77}$$

into Eq. (43), we find, up to leading order in $b(t)^{-1}$, that $h''+h'=0$. Hence, our zeroth-order starting point will be

$$M_0(r,t)=1-\frac{e^{-r^2/b(t)}}{1+a(t)r^2}, \tag{78}$$

which coincides with Eq. (75) for $r^2\ll b(t)$. In the limit of large $a(t)$ and $b(t)$, the moment of inertia I can be asymptotically computed:

$$I=2\int_0^{+\infty}[1-M(r,t)]rdr\sim\frac{\ln[a(t)b(t)]}{a(t)}, \tag{79}$$

where this result is in fact independent of the precise form of the cutoff function h for length scales of order $\sqrt{b(t)}$. Since I is exactly conserved at $T=T_c$, this provides a relation between $a(t)$ and $b(t)$.

In order to obtain a better estimate of the density scaling function, we write

$$M(r,t) = g(a(t)r^2,t), \quad g'(0) = 1, \tag{80}$$

where the last condition expresses that $a(t)=\rho(0,t)/\pi$. The usual derivatives are understood as derivatives with respect to the spatial variable of g, $u=a(t)r^2$. With this notation, the equation of motion for M given by Eq. (43) becomes

$$\frac{1}{a}\frac{\partial g}{\partial t} + \frac{\dot{a}}{a^2}ug' = ug'' + 2gg', \tag{81}$$

where the overdot expresses a time derivative. We will solve Eq. (81) perturbatively by replacing g in $\frac{\partial g}{\partial t}$ with our result of Eq. (78). On the right-hand side (RHS) we also replace g (but not its derivatives) by its zeroth-order expression in the limit of large b, leading to

$$g'' + g'\left(\frac{2}{1+u} - \frac{\dot{a}}{a^2}\right) = \frac{\dot{L}}{aL^2}\frac{\exp(-u/L)}{1+u}, \tag{82}$$

where $L(t)=a(t)b(t)$. This equation can be solved exactly, leading to

$$g'(u) = \frac{\dot{L}}{ac^2(L)}\frac{e^{-u/L}}{(1+u)^2}\left(1 + c(L)\frac{u}{L}\right), \tag{83}$$

where

$$c(L) = 1 + \frac{L\dot{a}}{a^2}. \tag{84}$$

The condition $g'(0)=1$ leads to

$$\dot{L} = ac^2(L) \approx a. \tag{85}$$

In order to obtain this simple estimate, we have used Eq. (79), which can be written

$$\ln L \approx Ia, \tag{86}$$

which implies $c(L)-1 \sim \frac{L\dot{a}}{a^2} \sim (Ia)^{-1} \ll 1$. Finally, we can solve the system (85), (86), which leads to

$$a(t) = \pi\rho(0,t) \sim \frac{M^2}{I}\ln t, \quad b(t) \sim t,$$
$$L(t) \sim \frac{M^2}{I}t\ln t, \quad c(L) = 1 + \frac{1}{\ln L} + \cdots. \tag{87}$$

As expected, $\sqrt{b(t)}$ coincides with the diffusing scale. We thus find that the divergence of the central density at T_c is strongly dependent on whether the domain is finite or not: in a bounded domain it diverges exponentially rapidly according to Eq. (76) while in an infinite domain it diverges logarithmically according to Eq. (87). Note that as the scale of the confining box R is not anymore an available parameter, the only parameter with the correct dimension of a density is M^2/I. Hence, it is not surprising to see this combination appearing in our final result, Eq. (87).

We also obtain the density profile

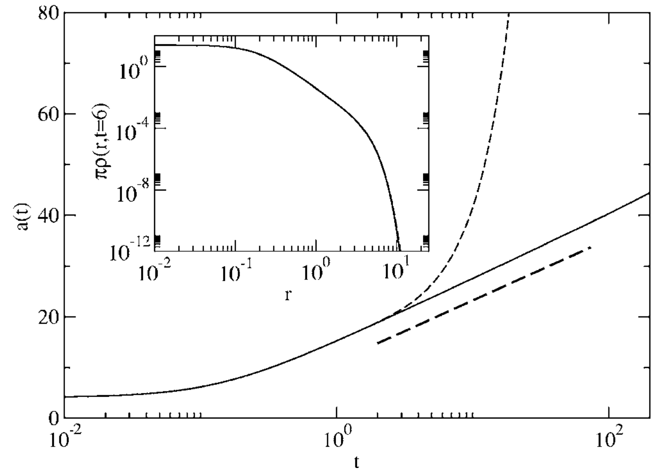

FIG. 3. We plot $a(t)=\pi\rho(0,t)$ as a function of $\ln t$, starting from a profile with $\rho(r,t=0)=\frac{4}{\pi}(1-r^2)^3$, for which $I=1/5$, in an unbounded domain [in fact a box of radius $R=200 \gg \sqrt{b(t)}$]. Although we expect strong corrections to the asymptotic behavior for finite times (of order $\ln\ln t$), we already find that the observed slope is in fact quite close to the prediction $1/I=5$ (dashed straight line). We compare this to the evolution of the same initial profile in a box of radius $R=2$, which asymptotically diverges as $\exp(5/2+\sqrt{t/2})/4$ (dashed curve). In an unbounded domain, the inset compares the actual density profile (solid line) to our result Eq. (88) (dashed line), which appears undistinguishable even for the moderate value of $t=6$. We have taken the actual measured value of $a=24.82$ but have fitted $b \sim t$ ($b=6.84$) and $c=1+1/\ln(ab)+\cdots$ ($c=1.44$) as the expressions given in the text are only valid for very large time.

$$\frac{\pi\rho(r,t)}{a(t)} = g'(u=a(t)x^2,t) = \frac{e^{-u/L}}{(1+u)^2}\left(1 + c(L)\frac{u}{L}\right), \tag{88}$$

where the dependence on $L(t)=a(t)b(t)$ leads to the explicit dependence of the scaling profile on time. Note that the form of $c(L)$ ensures that the density is properly normalized to $M=1$ up to order $1/\ln^2 L$. A more precise expression of $c(L)$ ensuring normalization at every order in $1/\ln L$ and including the first correction in $1/L$ is

$$c(L) = 1 + \frac{1}{\ln L - \gamma - 1} - \frac{1}{L} + O(1/L\ln L). \tag{89}$$

In Fig. 3, we illustrate this slow divergence of the central density and the fact that our expression (88) for the mass profile is extremely accurate even for moderate time.

We note finally that the situation at $T=T_c$ is complex as it lies at the frontier between the evaporation regime ($T>T_c$) and the collapse regime ($T<T_c$). Note that according to the virial theorem (36), there exists a static solution at $T=T_c$ in an unbounded domain. This is confirmed by solving the static SP equation (43) yielding

$$M(r) = \frac{\pi\rho_0 r^2}{1+(\pi\rho_0/M)r^2}, \quad \rho(r) = \frac{\rho_0}{[1+(\pi\rho_0/M)r^2]^2},$$
$$\Phi(r) = \frac{GM}{2}\ln\left(\frac{M}{\pi\rho_0} + r^2\right), \tag{90}$$

where we have adopted the Gauge condition $\Phi - GM \ln r \to 0$ for $r \to +\infty$. This family of solutions is parametrized by the central density ρ_0. The potential energy is $W=(GM^2/4)\times[1-\ln(\pi\rho_0/M)]$ so that the central density is related to the energy $E=K+W$ by

$$E = Nk_B T_c + W = \frac{GM^2}{2} - \frac{GM^2}{4}\ln\left(\frac{\pi\rho_0}{M}\right). \tag{91}$$

In the microcanonical ensemble (fixed E), a statistical equilibrium state of the form (90) exists for any value of the energy (and it has a unique temperature T_c); these states are stable (entropy maxima at fixed mass and energy). In the canonical ensemble (fixed T), static solutions exist only for $T=T_c$. However, since $\rho \sim r^{-4}$ at large distances, the moment of inertia of these steady states diverges logarithmically. Since the moment of inertia is conserved at $T=T_c$ for self-gravitating Brownian particles described by the SP system, this means that these solutions cannot be reached, except the solution with $\rho_0=+\infty$, corresponding to a Dirac peak: $\rho(\mathbf{r})=M\delta(\mathbf{r})$. This is consistent with the picture that we gave previously where *the system forms a Dirac peak at* $\mathbf{r}=\mathbf{0}$ *and ejects a tiny amount of mass at large distances so as to satisfy the moment of inertia constraint*.

V. SELF-GRAVITATING BROWNIAN GAS IN $d>2$ WITH NO BOUNDARY

For $d>2$, the integrated density satisfies

$$\frac{\partial M}{\partial t} = T\left(\frac{\partial^2 M}{\partial r^2} - \frac{d-1}{r}\frac{\partial M}{\partial r}\right) + \frac{M}{r^{d-1}}\frac{\partial M}{\partial r}. \tag{92}$$

Looking for a diffusive scaling solution of the form

$$M(r,t) = g\left(\frac{r}{r_0(t)}\right), \tag{93}$$

with $r_0(t)=\sqrt{2Tt}$, we find by simple power counting that the last term of Eq. (92) associated with gravity becomes negligible for large time. Hence, starting from a localized cluster of total mass $M=1$, we expect that, for large time,

$$\langle r^2\rangle \sim 2dTt, \tag{94}$$

as for a pure diffusion. This also implies that the free energy diverges as

$$F(t) \equiv \frac{1}{2}\int \rho\Phi d\mathbf{r} + T\int \rho\ln\rho d\mathbf{r} \sim_{+\infty} -\frac{d}{2}T\ln t \to -\infty \tag{95}$$

due to diffusion (evaporation). The absence of a minimum of free energy for an unbounded isothermal system is also clear from general considerations (see Appendix B of [3]). We note that for $d>2$, there also exists a self-similar solution of the SP system describing gravitational collapse [2,3]. For box-confined systems, there exists a critical temperature T_c (depending on the box radius) so that the system converges in general[1] to an equilibrium state for $T>T_c$ and collapses for $T<T_c$ [2,3]. In the absence of boundary, it is not clear how the system will choose its evolution (collapse or evaporation). The two evolutions are possible (see Appendix B of [3]), and the choice probably depends on the initial conditions and on a notion of basin of attraction. By contrast, in $d=2$ the critical temperature T_c is independent of the size of the domain (and still exists for an unbounded domain) so that the system evaporates for $T>T_c$ and collapses for $T<T_c$ (see Appendix B of [3]).

In this subsection, we focus on the evaporation and we show that for $2\leqslant d\leqslant 4$ there exists a universal correction to the above leading diffusive behavior, due to the gravitational dragging force. Integrating by parts the integral definition of $\langle r^2\rangle$, we observe that

$$\langle r^2\rangle = 2\int_0^{+\infty}[1-M(r,t)]r dr. \tag{96}$$

Multiplying Eq. (92) by $2r$, we find the virial theorem

$$\frac{d\langle r^2\rangle}{dt} = 2dT - 2\int_0^{+\infty} M(r,t)M'(r,t)r^{2-d}dr. \tag{97}$$

In $d=2$, the last integral is equal to 1, which reproduces the exact diffusion coefficient $D(T)=4(T-T_c)$, with $T_c=1/4$. For $d\geqslant 2$, Eq. (92) can be solved exactly by neglecting the gravitational term, leading to the freely diffusive scaling solution

$$g(x) = g_0(x) = S_d\int_0^x u^{d-1}\exp(-u^2/2)du. \tag{98}$$

Inserting this result into Eq. (97), we find

$$\frac{d\langle r^2\rangle}{dt} = 2dT - (8Tt)^{1-d/2}/\Gamma(d/2). \tag{99}$$

After integration, we find, for $2\leqslant d<4$,

$$\langle r^2\rangle = 2dTt - c_d T^{1-d/2}t^{2-d/2}, \tag{100}$$

up to a nonuniversal additive constant, which depends on the initial conditions. We have defined the d-dependent universal constant

$$c_d = 2\frac{8^{1-d/2}}{(4-d)\Gamma(d/2)}. \tag{101}$$

In particular, for $d=3$,

$$\langle r^2\rangle = 6Tt - \left(\frac{2}{\pi}\right)^{1/2}T^{-1/2}t^{1/2}. \tag{102}$$

For $d=4$, we find

[1] This may not always be the case because the equilibrium states are only *local* minima of the free energy (metastable). Depending on the shape of the initial condition, the system can collapse (forming ultimately a Dirac peak with infinite free energy) instead of reaching an equilibrium state; see the numerical confirmation in [2]. However, the local free-energy minima are highly robust in the thermodynamic limit $N\to+\infty$ and are physically relevant; see [35]. We believe that their basin of attraction is huge [2].

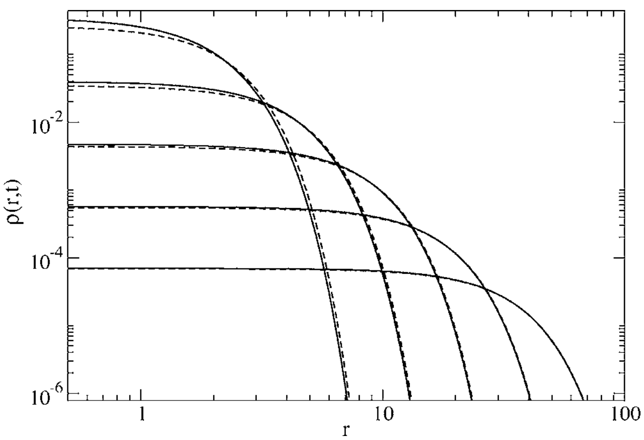

FIG. 4. Evolution of the density profile in $d=3$ for different times $t=4^n$ with $n=0,1,2,3,4$ (compare with Fig. 1 for $d=2$). The initial density profile has been chosen as $\rho(r,t=0)\sim(1-r^2)^3$, and the temperature is $T=1$. The solid lines correspond to the case of self-gravitating Brownian particles described by Eq. (92). The dashed lines correspond to a purely diffusive evolution without gravitational drift. We see that the effect of gravitational attraction is to reduce the diffusion for small times. However, this is just a correction which vanishes for $t\to+\infty$ so that the curves rapidly become indistinguishable.

$$\langle r^2\rangle = 8Tt - \frac{1}{8T}\ln t, \tag{103}$$

while for $d>4$, self-gravitation only contributes for a constant to the diffusion, which adds to the nonuniversal constant originating from the nonuniversal initial conditions. Hence, we have found that for d between the two critical dimensions for diffusion, $d=2$ and $d=4$, self-gravitation does not modify the bare diffusion constant, but is still responsible for a subleading growing negative drag, given by Eqs. (100)–(103). This is illustrated in Fig. 4 which compares the evolution of self-gravitating Brownian particles in $d=3$ with a pure diffusion.

Finally, by solving perturbatively Eq. (92), we find that the purely diffusive front has a small universal correction in $2<d<4$:

$$M(r,t) = g_0\left(\frac{r}{r_0(t)}\right) + T^{-1}r_0(t)^{-(d-2)}g_1\left(\frac{r}{r_0(t)}\right), \tag{104}$$

where g_1 satisfies the linear equation of second degree,

$$g_1'' + \left(x - \frac{d-1}{x}\right)g_1' + (d-2)g_1 = -\frac{g_0 g_0'}{x^{d-1}}, \tag{105}$$

along with the boundary condition $g_1(0)=g_1(+\infty)=0$. Note that after plugging the form of Eq. (104) into Eq. (96), one recovers the correct power-law correction of Eq. (100). In $d=3$, we can find both solutions of the homogeneous differential equation (which can be generally expressed in terms of hypergeometric functions). The simplest of these solutions is actually $(x^2+1)\exp(-x^2/2)$, and the second one can be obtained by standard methods [36]. Then, the general solution of the inhomogeneous equation can be obtained with the Lagrange method of "variation of constant," involving the Wronskian. This permits an exact solution for g_1 involving cumbersome integrals. We do not present this expression here as it is not particularly enlightening.

Finally, for the sake of completeness, we briefly discuss the situation for $d<2$. The case of box-confined systems has been treated in [3] where it is found that an equilibrium state exists for all values of temperature. Since the density decreases rapidly with the distance [see Eq. (23) of [3]], the box is not really necessary and equilibrium states also exist for $d<2$ with no boundary. For example, in $d=1$, the static solution of the SP system (92) is

$$M(r) = M\tanh(r/H), \quad \rho(r) = \frac{\rho_0}{\cosh^2(r/H)}, \quad H = \left(\frac{k_B T}{Gm\rho_0}\right)^{1/2}, \tag{106}$$

where the central density and the scale length are determined by the temperature according to

$$\rho_0 = \frac{GM^2 m}{4k_B T}, \quad H = \frac{2k_B T}{GMm}, \quad \rho_0 H = \frac{M}{2}. \tag{107}$$

For $T=0$, the density tends to a Dirac peak: $\rho(\mathbf{r})=M\delta(\mathbf{r})$. The time evolution of the system at $T=0$ is described in [3], Appendix E. On the other hand, according to the virial theorem in $d=1$, we have $2K-W=0$ so that the total energy $E=K+W$ of the configuration (106) is

$$E = \frac{3}{2}Nk_B T. \tag{108}$$

VI. DYNAMICAL STABILITY OF THE GSP SYSTEM

For an isothermal equation of state $p=\rho k_B T/m$, no equilibrium state can exist in an unbounded domain for $d\geqslant 2$. However, for other equations of state such as the polytropic one $p=K\rho^\gamma$, self-confined clusters can exist. The present section is concerned with the dynamical stability of these solutions.

A. Eigenvalue equation

We consider the linear dynamical stability of a stationary solution of the GSP system satisfying the hydrostatic balance (8). Let ρ and Φ refer to a stationary solution of Eqs. (5) and (6) and consider a small perturbation $\delta\rho$ that conserves mass. We restrict ourselves to spherically symmetric perturbations (nonspherically symmetric perturbations do not induce new instabilities for a nonrotating system). Writing $\delta\rho\sim e^{\lambda t}$ and expanding Eq. (5) to first order, we find that

$$\lambda\,\delta\rho = \frac{1}{r^{d-1}}\frac{d}{dr}\left[\frac{r^{d-1}}{\xi}\left(\frac{d\delta p}{dr} + \delta\rho\frac{d\Phi}{dr} + \rho\frac{d\delta\Phi}{dr}\right)\right]. \tag{109}$$

It is convenient to introduce the notation

$$\delta\rho = \frac{1}{S_d r^{d-1}}\frac{dq}{dr}. \tag{110}$$

Physically, q represents the mass perturbation $q(r)\equiv\delta M(r)=\int_0^r S_d r'^{d-1}\delta\rho(r')dr'$ within the sphere of radius r. It satisfies therefore the boundary conditions $q(0)=q(R)=0$ [the condition $q(R)=0$ is clear for a box-confined system; it is also valid for a self-confined system with $\rho(R)=0$ to prevent the singularity of the velocity perturbation δu in Eq. (118)]. Substituting Eq. (110) into Eq. (109) and integrating, we obtain

$$\frac{\lambda\xi}{r^{d-1}}q=\frac{d}{dr}\left(\frac{p'(\rho)}{r^{d-1}}\frac{dq}{dr}\right)+\frac{1}{r^{d-1}}\frac{dq}{dr}\frac{d\Phi}{dr}+S_d\rho\frac{d\delta\Phi}{dr}, \tag{111}$$

where we have used $q(0)=0$ to eliminate the constant of integration. Using the condition of hydrostatic equilibrium, $dp/dr+\rho d\Phi/dr=0$, and the Gauss theorem in perturbed form, $d\delta\Phi/dr=Gq/r^{d-1}$, we can rewrite Eq. (111) as

$$\frac{\lambda\xi}{S_d\rho r^{d-1}}q=\frac{1}{S_d\rho}\frac{d}{dr}\left(\frac{p'(\rho)}{r^{d-1}}\frac{dq}{dr}\right)-\frac{1}{S_d\rho^2}\frac{1}{r^{d-1}}\frac{dq}{dr}\frac{dp}{dr}+\frac{Gq}{r^{d-1}} \tag{112}$$

or, alternatively,

$$\frac{d}{dr}\left(\frac{p'(\rho)}{S_d\rho r^{d-1}}\frac{dq}{dr}\right)+\frac{Gq}{r^{d-1}}=\frac{\lambda\xi}{S_d\rho r^{d-1}}q, \tag{113}$$

with $q(0)=q(R)=0$. This eigenvalue equation determines the growth rate λ (real) of the perturbation. For $\lambda<0$ the stationary solution is stable and the perturbation is damped exponentially. For $\lambda>0$ the stationary solution is unstable and the perturbation grows exponentially rapidly. Equation (113) has been studied in [2–4] for isothermal and polytropic distributions. This form is particularly convenient in the case of box-confined systems. However, in the case of self-confined systems (e.g., complete polytropes) it is more convenient to transform it into an equation for the radial displacement as in the next section.

B. Sturm-Liouville problem

We first note that the generalized Smoluchowski equation (5) can be written in the form of a continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0, \tag{114}$$

with a velocity field given by

$$\mathbf{u}=-\frac{1}{\xi\rho}(\nabla p+\rho\nabla\Phi). \tag{115}$$

For small perturbations, we have

$$\frac{\partial\delta\rho}{\partial t}+\nabla\cdot(\rho\delta\mathbf{u})=0, \tag{116}$$

where we have used the fact that $\mathbf{u}=\mathbf{0}$ at equilibrium. For a spherically symmetric distribution, the foregoing equation can be rewritten

$$\lambda\delta\rho+\frac{1}{r^{d-1}}\frac{d}{dr}(r^{d-1}\rho\delta u)=0. \tag{117}$$

Introducing the variable $q(r)$ defined previously and integrating the resulting equation, we find that

$$\delta u=-\frac{\lambda q}{S_d\rho r^{d-1}}. \tag{118}$$

We introduce the function

$$\xi(r)\equiv-\frac{\delta u}{\lambda r}=\frac{q}{S_d\rho r^d}. \tag{119}$$

Physically, $\xi\sim\delta r/r$ denotes the radial displacement δr of a particle with respect to its equilibrium position r. In the sequel, we shall denote the friction coefficient by $\chi=1/\xi$ in order to avoid the confusion with the radial displacement $\xi(r)$ which is customarily denoted by the same Greek letter. Substituting $q=S_d\rho r^d\xi$ in Eq. (113), we get

$$\frac{d}{dr}\left(p'(\rho)\left[r\frac{d\xi}{dr}+\frac{1}{\rho}\frac{d\rho}{dr}r\xi+d\xi\right]\right)+S_dG\rho r\xi=\frac{\lambda}{\chi}r\xi. \tag{120}$$

We now define the function

$$\gamma(r)=\frac{d\ln p}{d\ln\rho}=\frac{\rho}{p}p'(\rho). \tag{121}$$

For a polytropic equation of state $p=K\rho^\gamma$, we have $\gamma(r)=\gamma=\text{const.}$, and for an isothermal equation of state $p=\rho k_BT/m$, we have $\gamma=1$. Therefore, $\gamma(r)$ is a generalization of the polytropic index to an arbitrary equation of state $p(\rho)$. After some algebra, Eq. (120) can be rewritten

$$\frac{d}{dr}\left(p\gamma r^{d+1}\frac{d\xi}{dr}\right)+\left[\rho r^d\frac{d}{dr}\left(\frac{p\gamma r}{\rho^2}\frac{d\rho}{dr}\right)+d\rho r^d\frac{d}{dr}\left(\frac{p}{\rho}\gamma\right)+S_dG\rho^2r^{d+1}\right]\xi=\rho\frac{\lambda}{\chi}r^{d+1}\xi. \tag{122}$$

Now, combining the Poisson equation (6) with the condition of hydrostatic balance $dp/dr+\rho d\Phi/dr=0$, we get

$$\frac{1}{r^{d-1}}\frac{d}{dr}\left(\frac{r^{d-1}}{\rho}\frac{dp}{dr}\right)=-S_dG\rho, \tag{123}$$

which is the fundamental equation of hydrostatic equilibrium [21]. Using this relation to simplify the term in brackets in Eq. (122), and after some calculations, we can finally put the eigenvalue equation into the form

$$\frac{d}{dr}\left(p\gamma r^{d+1}\frac{d\xi}{dr}\right)+r^d\frac{d}{dr}([d\gamma+2-2d]p)\xi=\rho\frac{\lambda}{\chi}r^{d+1}\xi. \tag{124}$$

It must be supplemented by the boundary conditions

$$\delta r=0\quad\text{at } r=0, \tag{125}$$

$$Dp=\lambda\gamma p\left(d\xi+r\frac{d\xi}{dr}\right)=0\quad\text{at } r=R, \tag{126}$$

where $Dp/Dt=\partial\delta p/\partial t+\delta u dp/dr$ is the Lagrangian derivative of the pressure [for box-confined models, we have $\delta r=0$ at $r=R$ instead of Eq. (126)]. Equation (124) with Eqs. (125) and (126) determines the growth rate λ of a small perturbation around a stationary solution of the GSP system in d dimensions. This is the counterpart of the Eddington [23] equation of pulsations for barotropic stars. The standard Eddington equation of pulsations is obtained by taking $d=3$ and by making the substitution $\lambda/\chi \leftrightarrow \lambda^2$. Due to this correspondence, we can generalize the methods developed in astrophysics to study the dynamical stability of self-gravitating Brownian particles in d dimensions. In particular, Eq. (124) has the form of a Sturm-Liouville problem [37]. Therefore the system has countably infinite number of eigenvalues λ_n labeled by an integer n. All these eigenvalues are real and can be ordered as $\lambda_0 > \lambda_1 > \cdots > \lambda_n > \cdots$ with $\lambda_n \to -\infty$ as $n\to\infty$. The eigenfunction $\xi_n(r)$ corresponding to the eigenvalue λ_n has n nodes in the interval $[0,R]$, and the normalized eigenfunctions ξ_n form a complete set and can be taken to be orthonormal with the weighting function ρr^{d+1} in the interval $[0,R]$.

C. Simple criterion for dynamical stability

We first extend the stability criterion given in [22]. Let R be the radius of the system at which the density and pressure vanish. If we multiply Eq. (124) by ξ and integrate between 0 and R, we obtain

$$\frac{\lambda}{\chi}\int_0^R \rho r^{d+1}\xi^2 dr = -\int_0^R p\gamma r^{d+1}\left(\frac{d\xi}{dr}\right)^2 dr + \int_0^R \xi^2 r^d \frac{d}{dr}([d\gamma+2-2d]p)dr, \qquad (127)$$

where we have used an integration by parts in the second integral. The condition of dynamical stability ($\lambda \leqslant 0$) is therefore

$$\mathcal{F}[\xi] = \int_0^R \left\{ p\gamma r^{d+1}\left(\frac{d\xi}{dr}\right)^2 - \xi^2 r^d \frac{d}{dr}([d\gamma+2-2d]p)\right\} dr \geqslant 0. \qquad (128)$$

For a polytropic equation of state for which γ is uniform, the second term in brackets can be rewritten

$$-\xi^2 r^d (d\gamma+2-2d)\frac{dp}{dr}. \qquad (129)$$

Since $dp/dr<0$, the system is stable provided that $d\gamma+2-2d \geqslant 0$—i.e.,

$$\gamma \geqslant \gamma_{4/3} \equiv \frac{2(d-1)}{d}, \quad \frac{1}{n} \geqslant \frac{1}{n_3} \equiv \frac{d-2}{d}, \qquad (130)$$

where we have defined the polytropic index n by $\gamma=1+1/n$. This returns the stability condition obtained in [4] by minimizing the generalized free energy (7) of a polytropic system. By using the Poincaré theory of linear series of equilibria, it is shown furthermore in [4] that the system is unstable for $\gamma \leqslant \gamma_{4/3}$. In terms of the index n, a polytrope is stable with respect to the GSP system in $d=1$ for $n \geqslant 0$ and for $n \leqslant -1$, in $d=2$ for $n \geqslant 0$, in $d=3$ for $0 \leqslant n \leqslant 3$, and in $d>2$ for $0 \leqslant n \leqslant n_3$.

It is also possible to obtain an approximate analytical expression of the eigenvalue λ, valid for an arbitrary equation of state $p(\rho)$, by using the method developed by Ledoux and Walraven [24] in their investigation of stellar pulsations. We shall extend their results to a space of dimension d and adapt them to our problem. Considering first the polytropic case, Eq. (127) can be written

$$\frac{\lambda}{\chi}\int_0^R \rho r^{d+1}\xi^2 dr = -\int_0^R \left\{ p\gamma r^{d+1}\left(\frac{d\xi}{dr}\right)^2 - \xi^2 r^d (d\gamma+2-2d)\frac{dp}{dr}\right\} dr. \qquad (131)$$

Combining the condition of hydrostatic balance with the Gauss theorem, we obtain

$$\frac{dp}{dr} = -\rho\frac{d\Phi}{dr} = -\rho\frac{GM(r)}{r^{d-1}} = -\frac{GM(r)\, dM/dr}{S_d r^{2d-2}}. \qquad (132)$$

Substituting this relation into Eq. (131), we get

$$\frac{\lambda}{\chi}\int_0^R \xi^2 r^2 dM(r) = -\gamma\int_0^R \left(r\frac{d\xi}{dr}\right)^2 p dV - (d\gamma+2-2d)\int_0^R \xi^2 \frac{GM(r)}{r^{d-2}} dM(r). \qquad (133)$$

For $d\neq 2$, the potential energy can be written (see the Appendix)

$$W = -\frac{1}{d-2}\int_0^R \frac{GM(r)}{r^{d-2}} dM(r). \qquad (134)$$

The case $d=2$ will be treated independently. We introduce the moment of inertia,

$$I = \int_0^R r^2 dM(r), \qquad (135)$$

and the internal energy

$$U = \int \rho \int_0^\rho \frac{p(\rho')}{\rho'^2} d\rho' d\mathbf{r}. \qquad (136)$$

For a polytropic equation of state $p=K\rho^\gamma$, we have

$$U = \frac{1}{\gamma-1}\int p dV = \frac{2K}{d(\gamma-1)}, \qquad (137)$$

where, in Eq. (137), K is the kinetic energy. In term of these quantities, Eq. (133) can be rewritten

$$\frac{\lambda}{\chi} = \frac{-\gamma(\gamma-1)\int_0^R (r\xi')^2 dU + (d\gamma+2-2d)(d-2)\int_0^R \xi^2 dW}{\int_0^R \xi^2 dI}. \qquad (138)$$

From the theory of Sturm-Liouville problems, it is known that the above expression forms the basis of a variational principle. The function $\xi(r)$ which maximizes the RHS is the fundamental eigenfunction, and the maximum value of this expression gives the fundamental eigenvalue λ_0/χ. Furthermore, any trial function underestimates the value of λ_0, so this variational principle may prove the existence of instability but can only give approximate information concerning stability. As shown by Ledoux and Walraven [24], we can get a good approximation of the fundamental eigenvalue by taking $\xi(r)$ to be a constant. For the trial function ξ=const, we get[2]

$$\frac{\lambda}{\chi} = (d\gamma + 2 - 2d)(d-2)\frac{W}{I} \quad (d \neq 2). \tag{139}$$

In particular,

$$\frac{\lambda}{\chi} = (3\gamma - 4)\frac{W}{I} \quad (d = 3) \tag{140}$$

and

$$\frac{\lambda}{\chi} = -\gamma\frac{W}{I} \quad (d = 1). \tag{141}$$

For $d=2$, Eq. (133) can be rewritten

$$\frac{\lambda}{\chi}\int_0^R \xi^2 dI = -\gamma(\gamma-1)\int_0^R \left(r\frac{d\xi}{dr}\right)^2 dU - (\gamma-1)\int_0^R \xi^2 G dM(r)^2. \tag{142}$$

Considering again the trial function ξ=const, we obtain

$$\frac{\lambda}{\chi} = -(\gamma-1)\frac{GM^2}{I} \quad (d = 2). \tag{143}$$

We now come back to the general case where $\gamma(r)$ is position dependent. Equation (127) can be written

$$\frac{\lambda}{\chi}\int_0^R \rho r^{d+1}\xi^2 dr = -\int_0^R \left\{p\gamma r^{d+1}\left(\frac{d\xi}{dr}\right)^2 - \xi^2 r^d(d\gamma + 2 - 2d)\frac{dp}{dr} - d\xi^2 r^d p\frac{d\gamma}{dr}\right\}dr. \tag{144}$$

Integrating the last term by parts and simplifying the resulting expression, we get, after some rearrangements,

$$\frac{\lambda}{\chi}\int_0^R \rho r^{d+1}\xi^2 dr = -\int_0^R \left\{p\gamma\left(r\frac{d\xi}{dr} + d\xi\right)^2 - 2(1-d)\xi^2 r\frac{dp}{dr}\right\}r^{d-1}\, dr \tag{145}$$

or

$$\frac{\lambda}{\chi}\int_0^R \xi^2 dI = -\int_0^R p\gamma\left(r\frac{d\xi}{dr} + d\xi\right)^2 dV - 2(1-d)\int_0^R \xi^2\frac{GM(r)}{r^{d-2}}dM(r). \tag{146}$$

For $d \neq 2$, this can be rewritten

$$\frac{\lambda}{\chi}\int_0^R \xi^2 dI = -\int_0^R p\gamma\left(r\frac{d\xi}{dr} + d\xi\right)^2 dV + 2(1-d)(d-2)\int_0^R \xi^2 dW. \tag{147}$$

Taking now ξ=const, we obtain

$$\frac{\lambda}{\chi} = -\frac{\int_0^R p\gamma d^2 dV - 2(1-d)(d-2)W}{I}. \tag{148}$$

Defining

$$\bar{\gamma} = \frac{\int_0^R p(r)\gamma(r)dV}{\int_0^R p(r)dV}, \tag{149}$$

we find that

$$\frac{\lambda}{\chi} = -\frac{\bar{\gamma}d^2\int_0^R p dV - 2(1-d)(d-2)W}{I}. \tag{150}$$

Using the virial relation (22) with $\dot{I}=0$ and $P=0$, this can be finally written

$$\frac{\lambda}{\chi} = (d\bar{\gamma} + 2 - 2d)(d-2)\frac{W}{I} \quad (d \neq 2). \tag{151}$$

In $d=2$, we directly obtain

$$\frac{\lambda}{\chi} = -(\bar{\gamma}-1)\frac{GM^2}{I} \quad (d = 2). \tag{152}$$

Therefore, the results (139)–(143) can be generalized to an arbitrary equation of state provided that γ is interpreted as a properly averaged value of $\gamma(r)$, according to Eq. (149). We also emphasize that, for self-gravitating Brownian particles described by the GSP system, these results give the damping rate of the perturbation (in the stable case) while for barotropic stars described by the Euler equations they give the pulsation period. The connection between these two models (parabolic and hyperbolic) will be further discussed in paper II.

VII. CONCLUSION

In this paper, we have completed previous investigations concerning the properties of the Smoluchowski-Poisson and

[2]Note that ξ=const—i.e., $\delta r \propto r$—is the *exact* solution of the Sturm-Liouville equation (124) at the point of marginal stability λ =0 for a polytropic equation of state [38]; see also [39].

generalized Smoluchowski-Poisson systems describing self-gravitating Brownian particles. We have considered an arbitrary barotropic equation of state $p(\rho)$ in d dimensions. In summary, we have obtained the following results:

(i) We have derived the proper form of the virial theorem describing self-gravitating Brownian particles. In $d=2$, we have obtained a generalization of the Einstein relation including self-gravitating effects. We have obtained the exact expression of the critical temperature, $k_B T_c = G\Sigma_{\alpha\neq\beta} m_\alpha m_\beta / 4N$, entering in this relation (see also paper II).

(ii) We have obtained explicit results describing the evaporation and diffusion of isothermal distributions of particles in $d=2$ for $T>T_c$ and in $d>2$ when the domain is infinite, while previous works considered box-confined systems. We have also shown how the absence of boundary in $d=2$ at $T=T_c$ modifies the collapse properties of the system (the central density increases logarithmically instead of exponentially).

(iii) We have studied the dynamical stability of self-confined clusters and obtained explicit results for the damping rate and the growth rate of a perturbation in d dimensions by adapting to the present situation the results of Eddington and Ledoux and Walraven in astrophysics.

In the present paper, we have restricted our analysis to the case of overdamped models ($\xi \to +\infty$). The case of inertial models will be specifically discussed in paper II. On the other hand, many of the results obtain here in the case of self-gravitating Brownian particles can be applied to the problem of chemotaxis for bacterial populations by a proper reinterpretation of the parameters [7]. This application will be discussed specifically in another paper.

APPENDIX: THE POTENTIAL ENERGY TENSOR IN d DIMENSIONS

In this appendix, we generalize the potential energy tensor theory [22] to d dimensions, with particular attention devoted to the critical dimension $d=2$. The potential energy tensor is

$$W_{ij} = -\int \rho x_i \frac{\partial \Phi}{\partial x_j} d\mathbf{r}. \tag{A1}$$

Contracting the indices, we get the virial

$$W_{ii} = -\int \rho \mathbf{r} \cdot \nabla \Phi d\mathbf{r} \equiv -\mathcal{V}_d. \tag{A2}$$

Substituting the expression of the gravitational force in d dimensions,

$$\mathbf{F} = -\nabla \Phi = -G \int \rho(\mathbf{r}') \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^d} d\mathbf{r}', \tag{A3}$$

we find that

$$W_{ij} = -G \int \rho(\mathbf{r})\rho(\mathbf{r}') x_i \frac{x_j - x_j'}{|\mathbf{r}-\mathbf{r}'|^d} d\mathbf{r} d\mathbf{r}'. \tag{A4}$$

Exchanging the prime and unprimed variables, we obtain

$$W_{ij} = G \int \rho(\mathbf{r})\rho(\mathbf{r}') x_i' \frac{x_j - x_j'}{|\mathbf{r}-\mathbf{r}'|^d} d\mathbf{r} d\mathbf{r}'. \tag{A5}$$

Taking the half-sum of the resulting expressions, we get

$$W_{ij} = -\frac{G}{2} \int \rho(\mathbf{r})\rho(\mathbf{r}') \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{r}-\mathbf{r}'|^d} d\mathbf{r} d\mathbf{r}'. \tag{A6}$$

Under this form, the potential energy tensor is manifestly symmetric: $W_{ij}=W_{ji}$. Contracting the indices, we obtain

$$W_{ii} = -\frac{G}{2} \int \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^{d-2}} d\mathbf{r} d\mathbf{r}'. \tag{A7}$$

In particular, for $d=2$, we immediately find that

$$W_{ii} = -\frac{GM^2}{2}. \tag{A8}$$

On the other hand, for $d\neq 2$, the gravitational potential is

$$\Phi(\mathbf{r}) = -\frac{G}{d-2} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^{d-2}} d\mathbf{r}'. \tag{A9}$$

Recalling that the potential energy is

$$W = \frac{1}{2} \int \rho \Phi d\mathbf{r}, \tag{A10}$$

we get

$$W_{ii} = (d-2) W. \tag{A11}$$

On the other hand, for a spherically symmetric system, we have, according to the Gauss theorem,

$$\nabla \Phi = \frac{GM(r)}{r^{d-1}} \mathbf{e}_r. \tag{A12}$$

Therefore

$$W_{ii} = -S_d G \int \rho(r) M(r) r dr = -\int \frac{GM(r)}{r^{d-2}} dM(r). \tag{A13}$$

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