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Synchronization is optimal in nondiagonalizable networks

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We consider maximization of the synchronizability of oscillator networks by assigning weights and directions to the links of a given interaction topology. By extending the master stability formalism to all possible network structures, we show that, unless some oscillator is linked to all the others, maximally synchronizable networks are necessarily nondiagonalizable and can always be obtained by imposing unidirectional information flow with normalized input strengths. The results provide insights into hierarchical structures observed in complex networks in which synchronization is important.

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Under extensive study in recent years is how the collective dynamics of a complex network is influenced by the structural properties of the network [1], such as clustering coefficient [2], average network distance [3], connectivity distribution [4], assortativity [5], and weight distribution [6,7]. The effects of these properties on synchronization has particularly attracted the attention of researchers, partly because of the elegant analysis due to Pecora and Carroll [8], which allows us to isolate the contribution of the network structure in terms of the eigenvalues of the coupling matrix.

Synchronizability of complex networks of oscillators generally has been shown to improve as the average network distance decreases, with one notable exception: in random scale-free networks, which are characterized by a strong heterogeneity of the connectivity distribution [4], synchronization was shown to become more difficult as the heterogeneity increases [9], even though the average network distance decreases at the same time. Motivated by this counterintuitive effect, researchers have pursued ways to enhance the synchronizability of scale-free networks by introducing directionality and weight to each link in the network [6,10]. A natural question arising in this context is as follows: Given a network of oscillators with a fixed topology of interactions, which assignment of weights and directions maximizes its synchronizability? By maximization, we mean that the synchronized states are stable for the widest possible range of the parameter representing the overall coupling strength.

The study of such a question provides us with insights into the dynamics of real-world complex networks and may guide us in designing large artificial networks. Metabolic networks are prototypic examples where the weights and directions of feasible links (metabolic fluxes) are adjusted to optimize fitness, which is likely to account for the robustness of synchronized behavior against environmental changes [11]. Other examples range from the enhancement of neuronal synchronization for a given topology of synaptic connections in the brain, to the design of interaction schemes that optimize the performance of computational tasks based on the synchronization of processes in computer networks [12].

Here we show that the answer to the question of maximum synchronizability falls outside the framework of the Pecora-Carroll analysis, which is built on the assumption that the network dynamics can be linearly decomposed into eigenmodes, i.e., the coupling matrix of the network is diagonalizable. Indeed, we show that maximally synchronizable networks are always *nondiagonalizable* (except for the extreme configurations where a node is connected to all the others) and can be constructed for any given interaction topology by imposing that the network: (i) embeds an *oriented spanning tree*, (ii) has *no directed loops*, and (iii) has *normalized* input strength in each node. The fact that networks are not necessarily diagonalizable has been largely overlooked in the literature, apparently because most previous works have focused on networks of symmetrically coupled oscillators, which are guaranteed to be diagonalizable. However, the same does not hold true in general when the network is directed, as required in the realistic modeling of many complex systems. Here we develop a new theory that extends the Pecora-Carroll analysis to the case of nondiagonalizable networks. We show that in this case the synchronizability is still determined by the eigenvalues of the coupling matrix, but the speed at which the system converges toward the synchronized state may be significantly slower. This theory is an example of going beyond the traditional framework for studying complex systems based on either decomposition into eigenmodes or some sort of superposition principle.

Consider n identical oscillators whose individual dynamics without coupling is governed by $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$, $\mathbf{x}\in\mathbb{R}^m$. Now consider the network of these oscillators coupled via an output signal function $\mathbf{H}:\mathbb{R}^m\to\mathbb{R}^m$ along a network with a symmetric adjacency matrix $A=(A_{ij})$ defined by $A_{ij}=1$ if oscillators i and j $(\neq i)$ are connected and $A_{ij}=0$ otherwise. Let $W_{ij}\geqslant 0$ denote the strength of the coupling that oscillator i receives from j. Thus, A represents the topology of interactions and $W=(W_{ij})$ represents the assignment of weights and directions. The system of equations governing the dynamics of the oscillator network can then be written as $\dot{\mathbf{x}}_i=\mathbf{F}(\mathbf{x}_i)+\sigma\Sigma_{j=1}^n A_{ij}W_{ij}[\mathbf{H}(\mathbf{x}_j)-\mathbf{H}(\mathbf{x}_i)]$ or, equivalently,

$$\dot{\mathbf{x}}_i=\mathbf{F}(\mathbf{x}_i)-\sigma\sum_{j=1}^{n}L_{ij}\mathbf{H}(\mathbf{x}_j),\quad i=1,\ldots,n, \tag{1}$$

where σ is the parameter controlling the overall coupling strength and $L=(L_{ij})$ is the coupling matrix of the directed weighted network, defined by $L_{ij}=-A_{ij}W_{ij}$ if $i\neq j$ and $L_{ii}=-\Sigma_{j\neq i}L_{ij}$. Note that L is not necessarily symmetric because the network is not constrained to be undirected.

The maximization problem considered in this paper can be formulated as follows. For a given topology of interactions between oscillators (represented by A), we want to find the assignment of weights and directions (represented by W) that maximizes the synchronizability of the network. In order to address this question, we need a condition for the network to synchronize. For any solution $\mathbf{x}=\mathbf{s}(t)$ of the individual dynamics $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$, the completely synchronous state $\mathbf{x}_i=\mathbf{s}(t)$ $i=1,\ldots,n$ is automatically a solution of the entire system (1). The question then is to determine when this solution is stable against small perturbations. This synchronization condition can be derived by extending the linear stability analysis of Pecora and Carroll [8] to the case where L is not necessarily diagonalizable, as follows.

The starting point of our analysis is the observation that, for any $n\times n$ matrix L, there exists an invertible matrix P of generalized eigenvectors of L which transforms L into Jordan canonical form as $P^{-1}LP=J$, where

$$J=\begin{pmatrix} 0 & & & \\ & B_1 & & \\ & & \ddots & \\ & & & B_l \end{pmatrix}, \quad B_j=\begin{pmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix}, \qquad (2)$$

and λ is one of the (possibly complex) eigenvalues of L. The stability of the synchronous solution of Eq. (1) is determined by the variational equation $\dot{\xi}=D\mathbf{F}(\mathbf{s})\xi-\sigma D\mathbf{H}(\mathbf{s})\xi L^T$, where $\xi=(\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_n)$ and $\boldsymbol{\xi}_i$ is the perturbation to the ith oscillator. By applying the change of variable $\eta=\xi P^{-T}$, we get

$$\dot{\eta}=D\mathbf{F}(\mathbf{s})\eta-\sigma D\mathbf{H}(\mathbf{s})\eta J^T. \qquad (3)$$

Each block of the Jordan canonical form corresponds to a subset of equations in Eq. (3). For example, if block B_j is $k\times k$, then it takes the form

$$\dot{\boldsymbol{\eta}}_1=[D\mathbf{F}(\mathbf{s})-\alpha D\mathbf{H}(\mathbf{s})]\boldsymbol{\eta}_1, \qquad (4)$$

$$\dot{\boldsymbol{\eta}}_2=[D\mathbf{F}(\mathbf{s})-\alpha D\mathbf{H}(\mathbf{s})]\boldsymbol{\eta}_2-\sigma D\mathbf{H}(\mathbf{s})\boldsymbol{\eta}_1, \qquad (5)$$

$$\vdots$$

$$\dot{\boldsymbol{\eta}}_k=[D\mathbf{F}(\mathbf{s})-\alpha D\mathbf{H}(\mathbf{s})]\boldsymbol{\eta}_k-\sigma D\mathbf{H}(\mathbf{s})\boldsymbol{\eta}_{k-1}, \qquad (6)$$

where $\alpha=\sigma\lambda$ and $\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\ldots,\boldsymbol{\eta}_k$ are perturbation modes in the generalized eigenspace of eigenvalue λ.

For α regarded as a complex parameter, Eq. (4) is a master stability equation and its largest Lyapunov exponent $\Lambda(\alpha)$, called master stability function [8], determines the stability of Eq. (4): it is linearly stable iff $\Lambda(\sigma\lambda)<0$. The condition for Eq. (5) to be stable is apparently more involved but can be formulated as follows. The linear stability of Eq. (4) implies that $\eta_1\to 0$ exponentially as $t\to\infty$. Assuming that the norm of $D\mathbf{H}(\mathbf{s})$ is bounded, we have that the second term in Eq. (5) is exponentially small. Then, the same condition $\Lambda(\sigma\lambda)<0$, now applied to Eq. (5), guarantees the stabilizing effect of both the first and second terms, resulting in the exponential convergence $\boldsymbol{\eta}_2\to 0$ as $t\to\infty$. The same argument applied repeatedly shows that $\boldsymbol{\eta}_3,\ldots,\boldsymbol{\eta}_k$ must also converge to zero if $\Lambda(\sigma\lambda)<0$. This shows that $\Lambda(\sigma\lambda)<0$ is a necessary and sufficient condition for the linear stability of the equations corresponding to each full block B_j. This condition is valid not only in diagonalizable [8] but also in nondiagonalizable networks.

However, it is worthwhile noting a crucial difference between the diagonalizable and nondiagonalizable cases. If L is diagonalizable, then all Jordan blocks are 1×1, so there would be no equations like Eqs. (5) or (6), and each mode of perturbation is decoupled from the others. Thus, the exponential convergence occurs independently and simultaneously. On the other hand, if L is not diagonalizable, some modes of perturbation may suffer from a long transient. For instance, if we have a network of linearly coupled phase oscillators, $\dot{\theta}_i=\omega-\sigma\Sigma_j L_{ij}\theta_j$, $\theta_i\in S^1$, then we can explicitly solve Eqs. (4)–(6) for the solution $s(t)=\omega t$ to obtain the last perturbation mode $\eta_k=e^{-\alpha t}\Sigma_{i=0}^{k-1}c_i t^i$, where the constants c_i depend on the initial condition. Therefore, the larger the size k of the Jordan block, the longer the transient.

Turning our attention back to the maximization problem, we first note that the eigenvalues $\lambda_1,\ldots,\lambda_n$ of matrix L can be ordered such that $0=\lambda_1\leqslant\mathrm{Re}\,\lambda_2\leqslant\cdots\leqslant\mathrm{Re}\,\lambda_n$, where one eigenvalue is always zero because L has zero row sum and all the others are guaranteed to have non-negative real parts because of the Gerschgorin Circle Theorem. Thus, taking all the Jordan blocks into account, it follows from our stability analysis that the synchronous solution is stable if and only if

$$\Lambda(\sigma\lambda_i)<0 \text{ for } i=2,\ldots,n.$$

Here $\Lambda(\sigma\lambda_1)=\Lambda(0)\geqslant 0$ is the largest Lyapunov exponent of the individual oscillators and corresponds to the stability along the synchronization manifold. We next note that $\mathrm{Re}\,\lambda_2>0$ if and only if the network embeds an oriented spanning tree, i.e., there is a node from which all other nodes can be reached by following directed links. This condition follows from Ref. [13] and generalizes the notion of connectedness to directed networks. We assume this condition here to ensure that the network is compatible with synchronization.

In most of the previously studied cases, the master stability function $\Lambda(\alpha)$, determined by $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{s}$, has been found to be negative in a single convex bounded region of the complex plane [14]. This implies the existence of a single interval $(\sigma_{\min},\sigma_{\max})$ of the overall coupling strength σ for which synchronization is stable. Thus, the synchronizability of the network can be measured in terms of the relative interval $\sigma_{\max}/\sigma_{\min}$: the network becomes *more synchronizable* as $\sigma_{\max}/\sigma_{\min}$ becomes larger. In the special case of undirected networks, the eigenvalues of L are real, and this measure of synchronizability is proportional to the ratio λ_2/λ_n [15].

A critical observation is that in order for the ratio $\sigma_{\max}/\sigma_{\min}$ to achieve absolute maximum for any given $\Lambda(\alpha)$ with a convex stability region, all nonzero eigenvalues must be real and equal to each other. The condition that the eigenvalues must be real follows from the convexity of the stabil-

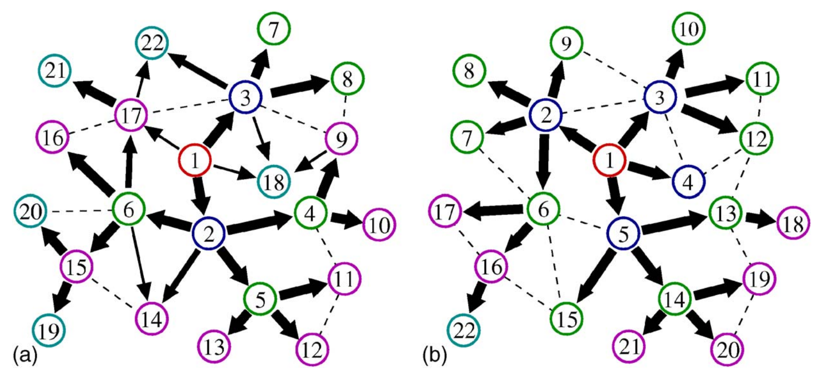

FIG. 1. (Color online) (a) Example of optimal assignment of weights and directions within a given interaction topology. The total input strength in each node is normalized to λ, where thick, medium, and thin arrows indicate weight λ, $2\lambda/3$, and $\lambda/3$, respectively, and dashed lines have zero weight. Nodes are numbered and colored to show the hierarchical structure, in which connections are only from a higher level to a lower level, with no feedback loops. (b) Example of oriented spanning tree within the same interaction topology as in (a), constructed by the breadth-first search.

ity region and the fact that complex eigenvalues appear in conjugate pairs, while the condition that they must be equal follows from the fact that, for real eigenvalues, the ratio $\sigma_{\max}/\sigma_{\min}$ is proportional to λ_2/λ_n. Thus, a network with

$$0=\lambda_1<\lambda_2=\cdots=\lambda_n \tag{7}$$

has the *widest possible range of coupling strength* in which synchronization is stable, independently of the individual node dynamics $\mathbf{F}$, output function $\mathbf{H}$, and synchronous state $\mathbf{s}$, as long as the stability region is convex [16].

Under the mild assumption that the interaction topology allows no oscillator to interact with all the others, *any maximally synchronizable network is necessarily nondiagonalizable*. This comes from the fact that if L is diagonalizable and satisfies the optimality condition (7) with nonzero eigenvalues equal to $\lambda>0$, then all the rows of the characteristic matrix $L-\lambda I$ must be equal. In terms of the network structure, this means that each node must either have uniform output to all the other nodes (at least one of them must do so) or have no output at all. These exceptional cases include globally connected networks and directed star configurations. However, it is uncommon in a large complex network that an oscillator can communicate with all the other oscillators. Therefore, our extension of the master stability analysis to nondiagonalizable networks was indeed necessary to properly address the optimization problem.

Having observed that optimal networks are rarely diagonalizable, we now show that, for *any connected topology of interactions*, there are assignments of directions and weights for which *the resulting network is nondiagonalizable and maximally synchronizable*. We first note that maximum synchronizability can always be achieved by imposing that the network (i) embeds an oriented spanning tree, (ii) has no directed loops, and (iii) has normalized input strengths in each node, i.e., the total input is the same for all nodes that have input. Condition (i) guarantees that Re $\lambda_2>0$, condition (ii) guarantees that the eigenvalues are real, and condition (iii) then implies the identity (7) among the nonzero eigenvalues. In such optimal networks, we can always rank the nodes so that each node has inputs only from nodes that are higher in the ranking [see Fig. 1(a) for an example]. In this hierarchical structure, information flows only from top to bottom in the ranking, without feedback. The optimality can be formally confirmed by noting that indexing nodes according to the ranking makes L a lower triangular matrix with $0,\lambda,\ldots,\lambda$ on the diagonal, which means that $\lambda_2=\cdots=\lambda_n=\lambda$, where $\lambda>0$ is the total input strength in $n-1$ of the nodes. An important class of such maximally synchronizable networks consists of the oriented spanning trees themselves, where condition (iii) leads to uniform weights for all links of the tree [see Fig. 1(b) for an example]. This example shows that any interaction topology admits at least $n-1$, but usually many more, optimal nondiagonalizable networks. Indeed, from the Matrix-Tree Theorem it follows that the number of all oriented spanning trees is $\Pi_{i=2}^{n}\mu_i$, where $\mu_2,\ldots,\mu_n$ are the nonzero Laplacian eigenvalues of the underlying *undirected* network defined by matrix A. For a globally connected network, for example, the number is n^{n-1}, which is huge even for relatively small networks. All these oriented spanning trees are nondiagonalizable, except for the star configuration. Oriented spanning trees can be explicitly constructed by the well-known procedure called the breadth-first search, which spans all nodes starting from an arbitrary root node.

Physically, the optimality conditions (i)–(iii) can be understood as follows. The top node in the ranking has no input and acts as a *master oscillator* that dominates the network dynamics. If the coupling strength σ is chosen so that $\Lambda(\sigma\lambda)<0$, then the oscillators that are immediately lower in the hierarchy and have input from the master will synchronize themselves with the master. Any oscillator having input only from these oscillators and the master must also synchronize, since normalization of the total input strength makes the equation effectively look as if it were having input from a single oscillator that is synchronized with the master. Repeating the same argument for the rest of the network, we see that under conditions (i)–(iii) all oscillators must eventually synchronize and they do so for the entire range of σ where $\Lambda(\sigma\lambda)<0$.

Interestingly, *undirected* tree networks have been found to be among the most difficult to synchronize [17], in striking contrast to our result that *directed* spanning trees lead to the most synchronizable configurations. This highlights the significance of directionality of the interactions in determining the synchronizability of networks [18]. On the other hand, the choice of the master oscillator in a maximally synchronizable network is completely arbitrary, despite the intuition that the nodes with largest connectivity would be the most natural choice. Moreover, the directions of the links in such a network are not necessarily related to the properties of the nodes they connect, even though there has been a suggestion that it would be related to the age of the nodes [19]. In contrast, under the stricter constraint that all feasible input connections have the same strength in each node, it was found [6] that maximum synchronizability is achieved when the individual input strength is inversely proportional to the connectivity of the node, which is consistent with our result that normalization is key to ensuring optimality.

The optimality conditions (i)–(iii) suggest that in designing a network for which synchronization is desired, it is generally advantageous to avoid feedback loops and to normalize input strength. Because these conditions typically lead to assigning nonzero weights only to a subset of all possible links, this interesting result can be interpreted as a synchronization version of the paradox of Braess for traffic flow [20], in which removing links leads counterintuitively to improved performance of the network. Furthermore, such assignment of weights not only maximize the synchronizability, but also *minimize* the coupling cost. The coupling cost can be defined as the sum of the input strengths of all nodes at the synchronization threshold [6]. If $\Lambda(\alpha)<0$ in (α_1,α_2), then the coupling cost for any network can only be as small as $\alpha_1(n-1)$, which can be achieved by networks with global uniform coupling. A surprising fact, however, is that this minimum can also be achieved by the maximally synchronizable networks as well. In other words, our optimality conditions allow a network constrained by an arbitrary topology to synchronize with the best possible efficiency. Interestingly, loopless networks are also obtained in the optimization of transportation networks [21].

Our characterization of the maximally synchronizable networks can be used to test the widely assumed hypothesis that synchronizability plays an important role in the evolution of many real-world complex networks. The loop structure of the metabolic network of *E. coli* suggests that having fewer loops may have been beneficial for the cell (the details will be published elsewhere), while recent experimental findings [22] suggest the significance of hierarchical structures in neuronal networks. Exploring more real data to systematically test this hypothesis is fundamental for a better understanding of complex networks.

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