Long-time behavior of spreading solutions of Schrödinger and diffusion equations

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We investigate the asymptotic time behavior of the solutions of a large class of linear differential equations that generalize the free-particle Schrödinger and diffusion equations, containing the standard ones as particular cases. We find general scalings that depend only on characteristic features of both the arbitrary initial condition and the Green function associated with the evolution equation. Basically, the amplitude of a long-time solution can be expressed in terms of low order moments of the initial condition (if finite) and low order spatial derivatives of the Green function. These derivatives can also be of the fractional type, which naturally arise when moments are divergent. We apply our results to a large class of differential equations that includes the fractional Schrödinger and Lévy diffusion equations. In particular, we show that, except for threshold cases, the amplitude of a packet may follow the asymptotic law $t^{-\alpha}$, with arbitrary positive α.

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I. INTRODUCTION

Spreading of wave packets is omnipresent in physics. It has been connected, in many cases, with diffusion due to the likeness between Schrödinger and diffusion equations. However, a variety of dispersion behaviors can occur depending on each concrete situation. Free particle states, for instance, typically present a crescent spreading with time. In contrast, nonspreading cases are also possible [1,2]. Disorder also tends to suppress propagation and to favor localization [3]. Another interesting example is the phenomenon of wave packet revivals [4]. Furthermore, the multifractal dimensions of the energy spectrum and eigenfunction space have been shown to determine the asymptotic scaling of the width of a dispersing wave packet [5]. In any case, in order to characterize the temporal spreading of a wave packet, a common practice is to investigate the time evolution of the spatial variance. This variance can be estimated, for instance, by means of semiquantal or semiclassical approaches [4,6–8]. There are situations, however, where the variance does not exist (it is divergent). In such cases, it is convenient to consider the amplitude of a packet at a suitable position in order to measure its spreading, although other quantities are also useful [9,10].

When we restrict the above scenario to free particles, spreading of a quantum wave packet is a usual textbook subject only when its spatial integral is finite and non-null, being Gaussian states paradigmatic cases. Meanwhile, diverse other physically sound initial conditions have been investigated by several authors [11–16], showing that the results are highly dependent on the class of initial conditions. In the one-dimensional case, when the initial wave function is Gaussian, its amplitude decays asymptotically with time t as $t^{-1/2}$. However, this scenario changes when an arbitrary initial wave function is considered. Specifically, when the integral of the initial wave function over the whole space is not finite, a decay as $t^{-\alpha}$ with $\alpha<1/2$ is common [12–14]. Furthermore, logarithmic laws also arise in the borderline cases [11,14]. In addition, the asymptotic behavior $t^{-\alpha}$ with $\alpha=(2m+1)/2$, where m is a positive integer, has been found, too [15,16]. Since the structure of the usual free-particle Schrödinger equation is intimately connected with the standard diffusion equation in the absence of external forces, similar conclusions can be drawn for the diffusive spreading.

In this work, we investigate the asymptotic time solutions of a broad spectrum of free-particle Schrödinger and other linear diffusion-like equations, i.e, situations where initially "localized" packets spread indefinitely with time. The equations we deal with include those ones of space fractional order, containing as particular cases both the usual free-particle Schrödinger and diffusion equations. Fractional diffusion equations are suitable to describe transport in media with topological complexity where the normal diffusion scaling (in which the mean square displacement increases linearly with time) is violated and jumps occur in superdiffusive Lévy flights [17]. Within this context, the temporal law of spatial spreading is quite relevant in connection, for instance, with the efficiency of search algorithms and natural processes [18]. Concerning the quantum counterpart, let us remark that the time evolution of initially free wave packets is crucial in scattering phenomena. In particular, wave packet spreading alone gives an important contribution to ionization processes [19]. The fractional Schrödinger equation arises by extending Brownian path integrals to integration over the paths of Lévy motion [20].

Besides treating a relevant wide class of evolution equations, we also consider broad classes of initial conditions, both "short" and "long-tailed," as well as with different shapes around its center. For this broad class of physically relevant situations, we provide in the present work an ap-

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proach that allows to obtain the temporal laws of spatial spreading.

The paper is organized as follows. In Sec. II, we present the basic diffusion-like evolution equations we deal with, and to which we apply the general asymptotic analysis developed in Sec. III. This analysis provides a unified framework to predict the long-time shapes of both short and long tailed initial conditions. The performance of our approach is illustrated in Sec. III. Results developed in Sec. III for one dimension are extended to the N-dimensional case in Sec. IV. Finally, Sec. V contains concluding remarks.

II. EVOLUTION EQUATIONS

We basically investigate the class of differential equations

$$\frac{\partial \rho}{\partial t} = \mathcal{L}\rho, \tag{1}$$

where the operator $\mathcal{L}$ is such that it promotes a spatial spreading. In the most famous cases, one has $\mathcal{L}=\gamma\nabla^2$, that represents both the free-particle Schrödinger equation (for $\gamma=i\hbar/[2m]$, where m is the mass of the particle and $\hbar$ the Planck constant) and the usual diffusion equation (for $\gamma=D>0$, where D is the diffusion coefficient). In the one-dimensional case, the diffusive spreading can be generalized to

$$\mathcal{L}_\mu = \gamma \frac{\partial^\mu}{\partial |x|^\mu}, \tag{2}$$

for $0<\mu\leqslant 2$, where the fractional derivative is of Riesz type and can be defined through the Fourier transform $\mathcal{F}$ as follows: [21]

$$\mathcal{F}\left\{\frac{d^\mu f(x)}{d|x|^\mu}\right\} = -|k|^\mu \tilde{f}(k), \tag{3}$$

where $\tilde{f}(k)\equiv\mathcal{F}\{f(x)\}=\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-ikx}\mathrm{d}x$. When $0<\mu<2$, Eqs. (1) and (2) yield the one-dimensional free-particle fractional Schrödinger [20] and Lévy diffusion [21,22] equations, respectively. In those cases, γ is a generalized constant (either complex or real, respectively) with the correct physical dimensions. For $\mu=2$, of course, we have the usual Schrödinger and diffusion equations with the standard definitions of γ.

Let us write the solution of the diffusion-like equation in terms of a Green function $G(x,t)$, that is

$$\rho(x,t) = \int_{-\infty}^{\infty} G(x-x',t)\rho(x',0)\mathrm{d}x'. \tag{4}$$

An explicit form for the Green function can be obtained by Fourier-transforming Eq. (1), solving for $\tilde{\rho}(k,t)$ and anti-transforming. Then, for operator $\mathcal{L}_\mu$ defined in Eq. (2), one gets the Green function

$$G_\mu(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\gamma t|k|^\mu + ikx}\mathrm{d}k = \frac{1}{[\gamma t]^{1/\mu}} L_\mu\left(\frac{x}{[\gamma t]^{1/\mu}}\right), \tag{5}$$

where the Lévy distribution L_μ [21,22] is defined as

$$L_\mu(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-|k|^\mu + ikx}\mathrm{d}k. \tag{6}$$

Let us remark that, although our applications are connected to the general class of spreading operators defined in Eq. (2), our approach could be straightforwardly extended to linear evolution equations with other definitions of the operator $\mathcal{L}$ and/or containing, for instance, fractional time derivatives.

III. ASYMPTOTIC ANALYSIS

We will address our discussion to the asymptotic behavior of $\rho(x,t)$. This behavior is expected to be determined not only by the particular form of the spreading law but also by the initial condition. Therefore, our analysis will take both of them into account. Since we are dealing with diffusion-like processes in the absence of external forces, Green functions have as general property a spread that increases with time. The initial condition, however, may have an arbitrary shape. In order to characterize it, let us consider its moments where the nth order one is

$$\langle x^n \rangle_0 = \int_{-\infty}^{\infty} x^n \rho(x,0)\mathrm{d}x = i^n \left.\frac{\mathrm{d}^n}{\mathrm{d}k^n}\tilde{\rho}(k,0)\right|_{k=0} \tag{7}$$

that may exist (be finite) or not. The zero-order derivative stands for $\tilde{\rho}(0,0)$. Note that $\langle x^n\rangle_0$ is not the usual mean value of x^n employed in quantum mechanics.

Let us call n_o the order of the first nonvanishing moment and n_* the order of the first divergent one. In some cases n_* is infinite, as for Gaussian $\rho(x,0)$, but it may be finite in other ones, as for fat tailed initial conditions.

A. Short-tailed initial conditions

First, we will consider short-tailed $\rho(x,0)$, in the sense that the first nonvanishing moment is finite (hence, $n_o<n_*$). After a sufficiently long time, the Green function $G(x-x',t)$ will have dispersed much more than $\rho(x,0)$. In other words, for a sufficiently long t, $\rho(x,0)$ will become very narrow when compared to $G(x-x',t)$. By suitably choosing the coordinate system, we can assume, without loss of generality, that this narrow region encloses the origin. Moreover, we will reasonably assume that the Green function, at $x'=0$, is nonsingular and can be differentiated n_G times.

Following these considerations, we can Taylor-expand $G(x-x',t)$ around $x'=0$. Since we are dealing with short-tailed $\rho(x,0)$, in the sense defined above, one can truncate the Taylor expansion at an order smaller than that corresponding to the first divergent moment, obtaining

$$G(x-x',t) = \sum_{n=0}^{n_*-1} \frac{(-x')^n}{n!}\frac{\partial^n}{\partial x^n}G(x,t) + R_{n_*}(x',t), \tag{8}$$

where $R_{n_*}(x',t)$ is the residual of the Taylor formula. For the Gauss and Lévy Green functions, n_G is infinite. However, if n_G were finite and $n_G<n_*$, then the sum should run up to at most n_G.

By means of expansion (8), Eq. (4) becomes

$$\rho(x,t) \simeq \sum_{n=n_o}^{n_*-1} \frac{(-1)^n}{n!}\langle x^n\rangle_0 \frac{\partial^n}{\partial x^n}G(x,t). \tag{9}$$

Typically, for a given x and sufficiently long t, the first few nonvanishing terms in this expansion will be the dominant ones. This property is supported, for instance, by the fact that many Green functions, as those illustrated by Eq. (5), are of the form $G(x,t)=F(x/\phi(t))/\phi(t)$, $\phi(t)$ being an increasing function of t and taking arbitrarily large values for a convenient choice of t. Thus, for large enough t, it is common that the spatial and temporal behaviors of $\rho(x,t)$ will be ruled by either the Green function or its first derivatives, the ones corresponding to the first nonvanishing terms in the series given by Eq. (9).

Besides its simplicity, Eq. (9) enables us to obtain the asymptotic behavior for $\rho(x,t)$ for a large class of Green functions and initial conditions. Notice also that Eq. (9) can be employed to investigate other situations besides the nonconfining case. In fact, when the initial condition is sufficiently narrow, this expansion can be also used to investigate the time evolution of confined systems such as harmonic oscillators. In order to gain more understanding about Eq. (9) and its implications, let us discuss a few concrete examples where the Green function is G_μ. In next and later examples, the following expansion will be useful

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|k|^{\sigma}e^{-\gamma t|k|^{\mu}+ikx}dk = \sum_{n\geqslant 0}\frac{b_n}{(\gamma t)^{(\sigma+1)/\mu}}\left(\frac{x}{(\gamma t)^{1/\mu}}\right)^{2n}, \tag{10}$$

with $b_n=\frac{(-1)^n}{\pi\mu(2n)!}\Gamma\left(\frac{2n+\sigma+1}{\mu}\right)$ and $\sigma>-1$.

1. In the case $\mu=2$, the Green function is $G_2=\exp(-x^2/[4\gamma t])/\sqrt{4\pi\gamma t}$. If the initial packet is such that $\langle x^0\rangle_0$ is finite and non-null (hence, $n_o=0$), then the asymptotic solution, according to Eq. (9), is $\rho(x,t)\simeq\langle x^0\rangle_0 G_2(x,t)$. Thus, the initial condition manifests itself in the asymptotic behavior of $\rho(x,t)$ via $\langle x^0\rangle_0$ only. Then, the known decay $t^{-1/2}$ is immediately obtained, since in this case $\rho(0,t)\simeq\langle x^0\rangle_0[4\pi\gamma t]^{-1/2}$.

2. Analogously, in the general instance $0<\mu\leqslant 2$, $\rho(x,t)\simeq\langle x^0\rangle_0 G_\mu(x,t)$, as soon as the initial condition is such that $\langle x^0\rangle_0$ is finite and non-null. This implies that $\rho(0,t)\sim t^{-1/\mu}$ or, still more generally, that $\rho(x,t)\sim t^{-1/\mu}$ when $|x|\ll|\gamma t|^{1/\mu}$.

3. Initial conditions with vanishing zero-order moment are of interest mainly in the context of quantum wave packets. If $\langle x^0\rangle_0=0$ while $\langle x^1\rangle_0$ is finite (hence $n_o=1$), as, for instance, for $\rho(x,0)\propto x/(b^2+x^2)^\nu$ with $\nu>3/2$, then $\rho(x,t)\simeq-\langle x^1\rangle_0\partial_x G_\mu(x,t)\sim t^{-3/\mu}$, for $0<|x|\ll|\gamma t|^{1/\mu}$. Moreover, notice from Eq. (4) that $\rho(0,t)=0$, because $G_\mu(x,t)$ and $\rho(x,0)$ are even and odd in x, respectively.

4. As illustration of the case $n_o=2$, let us consider $\rho(x,0)\propto(x^2-a)/(b^2+x^2)^\nu$, with $\nu>5/2$, for a suitable choice of $a(a=b^2/[2\nu-3])$, such that $\langle x^0\rangle_0=0$. The leading term of $\rho(x,t)$ is $\langle x^2\rangle_0\partial_x^2 G_\mu(x,t)/2$, thus, the asymptotic behavior of $\rho(x,t)$ results proportional to $t^{-3/\mu}$ as in the $n_o=1$ case.

5. In some cases there are two terms of the series (9) contributing to the dominant asymptotic behavior. For instance, when $\rho(x,0)$ is a linear combination of $\rho_1(x)=x/(b^2+x^2)^\nu$ and $\rho_2(x)=[x^2-b^2/(2\nu-3)]/(b^2+x^2)^\nu$, with $\nu>5/2$. In fact, from the two previous examples, the asymptotic behavior of $\rho(x,t)$ is proportional to $t^{-3/\mu}$.

The possible asymptotic behaviors can be summarized as follows. In the long-time limit and at $|x|\ll|\gamma t|^{1/\mu}$,

$$\rho(x,t)\sim\begin{cases} t^{-(n_o+1)/\mu}, & \text{for even } n_o \\ t^{-(n_o+2)/\mu}, & \text{for odd } n_o, \end{cases} \tag{11}$$

recalling that n_o is the order of the first nonvanishing (finite) moment. Therefore, the decay law is characterized by a power-law exponent determined not only by the spreading process controlled by parameter μ but also by the shape of the initial packet, through n_o.

This very general scenario includes the well-known $t^{-1/2}$ power-law decay ($\mu=2$ and $n_o=0$). But decays slower or faster than the usual one may also occur. The smaller μ (more superdiffusive paths), the more rapid the long-time decay, for given n_o, as expected. Whereas, concerning the influence of the initial condition, the larger n_o, the more rapid the decay. Then the spreading rate can be related to the degree of localization of the packet. On one hand, the decay is faster for more localized packets in the sense that larger n_o corresponds to shorter tails. On the other, the decay is faster for packets less localized around the origin, as they must behave oscillatorily in order that the moments lower that the n_oth order one vanish. An implication of the decay dependence on n_o is that even at long times, the spread keeps a sort of memory of the initial condition, through a global feature of its x dependence. This persistence is connected with the fact that n_o is a conserved quantity along the free packets evolution.

The asymptotic behavior given by Eq. (11) also holds exactly at the origin, if n_o is even. For arbitrary x and odd n_o, two terms may lead to the contribution of order $t^{-(n_o+2)/\mu}$, i.e., $\rho(x,t)\simeq-\langle x^{n_o}\rangle_0\partial_x^{n_o}G_\mu(x,t)/n_o!+\langle x^{n_o+1}\rangle_0\partial_x^{n_o+1}G_\mu(x,t)/(n_o+1)!$, for finite $\langle x^{n_o+1}\rangle_0$. In example 3, the initial condition is odd, then only the first term contributes. In more general cases, as that one illustrated in example 5, both contributions may coexist. Examples illustrating the asymptotic behavior (11) are easily found by employing the initial condition $\rho(x,0)\propto d^{n_o}g(x)/dx^{n_o}$, where $g(x)$ is a function such that $\lim_{x\to\pm\infty}xg(x)=0$. For instance, $g(x)=1/(b^2+x^2)^\nu$, with $\nu>1/2$, was employed in the last three examples.

An alternative procedure to obtain Eq. (9) is based on the following representation of $\rho(x,t)$, obtained by Fourier transforming and anti-transforming Eq. (4):

$$\rho(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(k,t)\tilde{\rho}(k,0)e^{ikx}\mathrm{d}k. \tag{12}$$

As we have already discussed, if Eq. (1) is related to a nonconfining process [for instance, those given by Eq. (2)], the spread associated with the propagator $G(x-x',t)$ increases with t. In contrast, its Fourier transform $\tilde{G}(k,t)$ leads to increased localization with increasing t, consistently with the uncertainty principle for the quantum case. For instance, the Fourier transform of propagator $G_\mu(x,t)$ is $\widetilde{G}(k,t)=\exp(-\gamma t|k|^\mu)$. Then in the diffusive case ($\gamma>0$), it is clear that $\widetilde{G}(k,t)$ shrinks with time. In the quantum case, for instance when $\mu=2$, the Green function does not decay for long distances, but instead, presents rapid oscillations due the imaginary exponent. However, localization of the packet also arises due to cancellation provided by these oscillations, in such a way that the asymptotic behavior of the solutions does not depend on γ being real or imaginary. From a mathematically formal perspective, this identical behavior comes from facts such as suitable choice of the contour of integration in the complex plane and introduction of a convergence factor. This occurs, for instance, when dealing with Eq. (10). Within this framework, the transform $\widetilde{G}(k,t)$, when employed in equations such as Eq. (12), can be seen as effectively more localized as time goes by since it leads to less localized packets. Thus, the dominant region of the integral in Eq. (12) becomes more and more restricted with time. Taking this consideration into account, our analysis will focus on the neighborhood of $k=0$. In doing so we will retain only the main contribution to $\widetilde{\rho}(k,0)e^{ikx}$, for small $|k|$. When x is sufficiently large, e^{ikx} is highly oscillating even if $|k|$ becomes small. Therefore, we must keep the full form of e^{ikx} in order to preserve the spatial asymptotic behavior of $\rho(x,t)$. On the other hand, the initial condition $\widetilde{\rho}(k,0)$ is typically smooth for small k. More precisely, we invoke an approximation for $\widetilde{\rho}(k,0)$ useful for small $|k|$. In particular, if $\rho(x,0)$ is such that $\langle x^0\rangle_0$ is finite and non-null, we have $\widetilde{\rho}(0,0)=\langle x^0\rangle_0$. In this case, the first term in Eq. (9) is immediately obtained if we consider $\widetilde{\rho}(k,0)\simeq\widetilde{\rho}(0,0)$. The generalization is straightforward when the first nonvanishing (finite) moment is higher than the zero-order one. In fact, in the Fourier scenario, one can use the expansion

$$\widetilde{\rho}(k,0) \simeq \sum_{n=n_o}^{n_*-1} \frac{k^n}{n!} \left.\frac{d^n}{dk^n}\widetilde{\rho}(k,0)\right|_{k=0}, \tag{13}$$

where n_o corresponds to the first non-null derivative of $\widetilde{\rho}(k,0)$ at the origin and the expansion runs up to the term of order n_*-1, consistently with Eq. (7). Of course, by substitution of Eq. (13) into Eq. (12) and employing Eq. (7), one recovers the expansion in Eq. (9).

Although this alternative representation does not provide new information on the short-tailed case, it allows to establish a link with the treatment used in the next section.

B. Long-tailed initial conditions

Let us call long-tailed cases those in which the first nonvanishing moment is divergent (ill defined). This class includes those initial conditions for which $\langle x^0\rangle_0$ is divergent. In particular, these cases may be of interest in the context of quantum wave packets where the requirement is of square integrability.

Some care is necessary when employing the terms short-tailed and long-tailed in the present sense since some short-tailed initial conditions can present a slower decay than some long-tailed ones. For instance, according to the definitions above, $\rho_1(x,0)=x(b^2+x^2)^{-1.4}$ is long-tailed and $\rho_2(x,0)=(b^2+x^2)^{-0.6}$ is short-tailed, however $|\rho_1(x,0)|\sim|x|^{-1.8}$ and $|\rho_2(x,0)|\sim|x|^{-1.2}$ for large $|x|$.

If the first nonvanishing moment is divergent, then an expansion as in Eq. (9) does not exist. In such case, we must consider either Eqs. (4) or (12) directly. In what follows, we will base our analysis on the Fourier transform of $\rho(x,0)$.

As we have seen in the short-tailed case, following Eq. (13), the first term for $\widetilde{\rho}(k,0)$ is proportional to k^{n_o}, where n_o is a positive integer. Motivated by this property and in order to treat long-tailed initial conditions, we first address our discussion to the asymptotic situation such that $\widetilde{\rho}(k,0)\simeq A|k|^\sigma$ where A is a constant and σ a real parameter. In particular, when $-1<\sigma<n_*$, one gets even initial conditions such that $\rho(x,0)\sim|x|^{-(\sigma+1)}$, for large $|x|$. By replacing this approximation for $\widetilde{\rho}(k,0)$ in Eq. (12), we obtain

$$\rho(x,t) \simeq \frac{A}{2\pi}\int_{-\infty}^{\infty}|k|^\sigma \widetilde{G}(k,t)e^{ikx}dk = -A\frac{\partial^\sigma}{\partial|x|^\sigma}G(x,t), \tag{14}$$

where we have employed Eq. (3) to identify the fractional derivative of order σ.

To treat odd initial conditions, one should consider $\widetilde{\rho}(k,0)\simeq iA'|k|^{\sigma-1}k$, instead of $\widetilde{\rho}(k,0)\simeq A|k|^\sigma$. In this situation, we have $\rho(x,0)\sim x|x|^{-(\sigma+2)}$, for large $|x|$, and

$$\rho(x,t) \simeq -A'\frac{\partial}{\partial x}\frac{\partial^{\sigma-1}}{\partial|x|^{\sigma-1}}G(x,t), \tag{15}$$

for long time. As a consequence, even when the original equation does not contain fractional derivatives, fractional derivatives may naturally arise when we explore the possibility of long-tailed initial conditions.

In Sec. III A, we have observed the following scenario for the short-tailed case: the dominant term in the expansion given by Eq. (9) is that corresponding to the derivative of the Green function of smallest order, associated with the first nonvanishing term of the expansion. Notice that this scenario remains unchanged when fractional derivatives are involved, as in Eqs. (14) and (15). Then, in general, both in short and long-tailed cases, the asymptotic behavior is ruled by derivatives of $G(x,t)$, be they of integer or fractional order.

Let us consider some examples where the first nonvanishing moment is divergent and the Green function is G_μ.

1. If $\rho(x,0)\propto 1/(b^2+x^2)^\nu$, with $0<\nu<1/2$, then $\rho(x,0)$ is such that its 0th moment is divergent ($n_*=0$). In this case, following Eq. (14), one gets $\rho(x,0)\propto|x|^{-(\sigma+1)}$ and $\widetilde{\rho}(k,0)\propto|k|^\sigma$ with $\sigma=2\nu-1(-1<\sigma<0)$ for large $|x|$ (small $|k|$). Therefore, $\rho(x,t)\propto\partial^\sigma_{|x|}G_\mu(x,t)\sim t^{-(\sigma+1)/\mu}=t^{-2\nu/\mu}$ when $|x|\ll|\gamma t|^{1/\mu}$.

2. When $\rho(x,0)\propto x/(b^2+x^2)^\nu$, with $1/2<\nu<3/2$, then $\rho(x,0)\propto x|x|^{-(\sigma+2)}$ and $\widetilde{\rho}(k,0)\propto|k|^{\sigma-1}k$, with $\sigma=2\nu-2(-1<\sigma<1)$ for large $|x|$ (small $|k|$). Note that $\langle x^1\rangle_0$ is divergent. Moreover, $\langle x^0\rangle_0=0$, when $\nu>1(n_*=1)$ and divergent otherwise ($n_*=0$). Then, $\rho(x,t)\sim t^{-(\sigma-1)/\mu}=t^{-2\nu/\mu}$ when $0<|x|\ll|\gamma t|^{1/\mu}$.

3. For $\rho(x,0)\propto[x^2-b^2/(2\nu-3)]/(b^2+x^2)^\nu$, with $1<\nu<5/2$, we have $\tilde{\rho}(k)\propto|k|^\sigma$ with $\sigma=2\nu-3(-1<\sigma<2)$. Then, we have $\rho(x,t)\sim t^{-(\sigma+1)/\mu}=t^{-(2\nu-2)/\mu}$ when $|x|\ll|\gamma t|^{1/\mu}$.

4. From the two previous examples, if $\rho(x,0)$ is a linear combination of $\rho_1(x)=x/(b^2+x^2)^\nu$ and $\rho_2(x)=[x^2-b^2/(2\nu-3)]/(b^2+x^2)^\nu$, with $1<\nu<5/2$, then the asymptotic behavior of $\rho(x,t)$ is dominated by $\rho_2(x)$. Thus, $\rho(x,t)\sim t^{-2(\nu-1)/\mu}$.

In these examples, lower bounds for ν were employed to characterize a minimum degree of localization for $\rho(x,0)$, in the sense that $\lim_{x\to\pm\infty}\rho(x,0)=0$. The above behaviors can be summarized as follows:

$$\rho(x,t)\sim\begin{cases} t^{-(\sigma+1)/\mu}, & \text{for even } \rho(x,0)\\ t^{-(\sigma+2)/\mu}, & \text{for odd } \rho(x,0),\end{cases} \tag{16}$$

where σ defines the asymptotic decay $\rho(x,0)\sim|x|^{-(\sigma+1)}$ for large $|x|$. As for short-tailed packets discussed in connection with Eq. (11), faster decays occur for smaller μ. Concerning the influence of the shape of the initial condition, faster decays are observed for larger σ, that is, for higher localization in the sense of less heavy tails.

For completeness, let us remark that logarithmic corrections of the form $\tilde{\rho}(k,0)=A|k|^\sigma\ln|k|$ or $\tilde{\rho}(k,0)=iA'|k|^{\sigma-1}k\ln|k|$ may occur as marginal situations in the previous examples. In the following ones, these borderline cases are illustrated.

1. If $\rho(x,0)\propto1/(b^2+x^2)^{1/2}$, its 0th moment is divergent and $\tilde{\rho}(k,0)\propto-2(\xi+\ln|kb/2|)$, for small $|k|$, where ξ is the Euler constant. Then $\rho(x,t)\sim t^{-1/\mu}\ln t$ for $|x|\ll|\gamma t|^{1/\mu}$ and $b\ll|\gamma t|^{1/\mu}$.

2. For $\rho(x,0)\propto d[1/(b^2+x^2)^{1/2}]/dx\propto x/(b^2+x^2)^{3/2}$, $\tilde{\rho}(k,0)\propto2i(\xi+\ln|kb/2|)k$ for small $|k|$. Then $\rho(x,t)\sim t^{-3/\mu}\ln t$, for $0<|x|\ll|\gamma t|^{1/\mu}$ and $b\ll|\gamma t|^{1/\mu}$.

3. When $\rho(x,0)\propto d^2[1/(b^2+x^2)^{1/2}]/dx^2$, i.e., $\rho(x,0)\propto[x^2-b^2/(2\nu-3)]/(b^2+x^2)^{5/2}$, we obtain $\tilde{\rho}(k,0)\propto2(\xi+\ln|kb/2|)k^2$ for small $|k|$. Thus, $\rho(x,t)\sim t^{-3/\mu}\ln t$ for $|x|\ll|\gamma t|^{1/\mu}$ and $b\ll|\gamma t|^{1/\mu}$.

Aside from logarithmic behaviors, the following scenario holds *both for short and long-tailed* initial conditions: If one considers a spreading as defined by Eq. (2), one obtains, for long t and small $|x|$, asymptotic power-laws

$$\rho(x,t)\sim t^{-\alpha}, \tag{17}$$

where α can take any positive value, depending on μ and on the initial shape. Naturally, this conclusion can be extended to other diffusion-like equations (hence, to other propagators).

A general consequence concerns the loss of localization. For bell-shaped packets, even in the cases of divergent second moment, a typical width can be viewed as the inverse of the height [9]. Then a temporal power-law decay at the origin signals a width growth with a positive power of time. In particular, in the quantum problem, the faster the spreading, the more rapid the loss of localization and sooner trajectories become less classical [6].

IV. *N*-DIMENSIONAL CASE

Up to now we focused on one-dimensional processes but our approach may be directly adapted to N-dimensional problems. As before, let us write the solutions of diffusion-like equations in terms of the associated Green function $G(\mathbf{x},t)$, i.e.,

$$\rho(\mathbf{x},t)=\int G(\mathbf{x}-\mathbf{x}',t)\rho(\mathbf{x}',0)\mathrm{d}^N\mathbf{x}'. \tag{18}$$

In what follows, only selected special cases will be focused under the light of the one-dimensional analysis.

When the relevant moments of $\rho(\mathbf{x},0)$ are finite, the generalization of Eq. (9) becomes

$$\rho(\mathbf{x},t)\simeq\sum_{n=n_o}^{n_*-1}\frac{(-1)^n}{n!}\sum_{\{i_1,\ldots,i_n\}}\langle x'_{i_1}\ldots x'_{i_n}\rangle_0\frac{\partial^nG(\mathbf{x},t)}{\partial x_{i_1}\ldots\partial x_{i_n}}. \tag{19}$$

In contrast, when the first non-null moment is not finite, extensions of Eqs. (14) and (15) are necessary. For instance, for $\rho(\mathbf{x},0)=\Pi_{j=1}^N\rho_j(x_j)$, one has

$$\rho(\mathbf{x},t)\simeq-A\prod_{j=1}^N\mathcal{D}_jG(\mathbf{x},t), \tag{20}$$

where

$$\mathcal{D}_j=\begin{cases}\dfrac{\partial^{\sigma_j}}{\partial|x_j|^{\sigma_j}}, & \text{for } \rho_j(x_j) \text{ even}\\[2ex] \dfrac{\partial}{\partial x_j}\dfrac{\partial^{\sigma_j-1}}{\partial|x_j|^{\sigma_j-1}}, & \text{for } \rho_j(x_j) \text{ odd}.\end{cases} \tag{21}$$

For spatially symmetric or antisymmetric initial conditions, other possibilities are

$$\rho(\mathbf{x},t)\simeq-A\frac{\partial^\sigma}{\partial|\mathbf{x}|^\sigma}G(\mathbf{x},t) \tag{22}$$

and

$$\rho(\mathbf{x},t)\simeq-A'\frac{\partial}{\partial x_j}\frac{\partial^{\sigma-1}}{\partial|\mathbf{x}|^{\sigma-1}}G(\mathbf{x},t) \tag{23}$$

for arbitrary $1\leq j\leq N$. Naturally, other combinations and possibilities can occur as well.

For the usual Schrödinger or diffusion equations in N dimensions, the Green function is factorizable as $G_2(\mathbf{x},t)=\Pi_{j=1}^NG_2(x_j,t)$ and has spherical symmetry since $G_2(x_j,t)$ is Gaussian. In fractional cases, if we employ a generalized Laplacian such that $\mathcal{F}\{\nabla_\mu^2f(\mathbf{x})\}=-|\mathbf{k}|^\mu\mathcal{F}\{f(\mathbf{x})\}$, spherical symmetry is preserved and the Green function is

$$G_\mu(\mathbf{x},t)=\frac{1}{(2\pi)^N}\int e^{-\gamma t|\mathbf{k}|^\mu+i\mathbf{k}\cdot\mathbf{x}}d^N\mathbf{k}. \tag{24}$$

If we consider instead a generalized Laplacian as sum of N fractional derivatives, spherical symmetry is lost and we have $G_{\bar{\mu}}(\mathbf{x},t)=\Pi_{j=1}^NG_{\mu_j}(x_j,t)$.

Let us illustrate possible asymptotic behaviors in N dimensions. When the initial condition and the Green function are given by $\rho(\mathbf{x},0)=\Pi_{j=1}^N\rho_j(x_j)$ and $G(\mathbf{x},t)=\Pi_{j=1}^NG_{\mu_j}(x_j,t)$,

TABLE I. Asymptotic temporal behavior in the N-dimensional case, when the evolution is ruled by the Green function $G_\mu(\mathbf{x},t)$ given by Eq. (24). Two cases are considered, related, respectively, to (i) $\langle x_j^0\rangle_0 \neq 0$, for some j, and (ii) $\langle x_j^0\rangle_0=0$, $\forall j$, but $\langle x_j^1\rangle_0 \neq 0$ for some j. In the first column the initial condition is described: the tail shape determines the finiteness or not of the first non-null moment. In the second column we find the dominant term of its Fourier transform $\tilde{\rho}(\mathbf{k},0)$. In the last column the asymptotic behavior of $\rho(\mathbf{x},t)$ for small $|\mathbf{x}|$ is presented.

$\langle x_j^0\rangle_0 \neq 0$, for some j $\rho(\mathbf{x},0) \sim |\mathbf{x}|^{-(\sigma+N)}$	$\tilde{\rho}(\mathbf{k},0)$	$\rho(0,t)$
Short tail ($\sigma>0$)	$|\mathbf{k}|^0$	$t^{-N/\mu}$
Threshold case ($\sigma=0$)	$a+b\ln|\mathbf{k}|$	$t^{-N/\mu}\ln t$
Long tail ($\sigma<0$)	$|\mathbf{k}|^\sigma$	$t^{-(\sigma+N)/\mu}$
$\langle x_j^0\rangle_0=0$, $\forall j$, and $\langle x_j^1\rangle_0 \neq 0$, for some j $\rho(\mathbf{x},0) \sim x_j|\mathbf{x}|^{-(\sigma+N+1)}$	$\tilde{\rho}(\mathbf{k},0)$	$\rho(\mathbf{0},t)$
Short tail ($\sigma>1$)	k_j	$t^{-(N+2)/\mu}$
Threshold case ($\sigma=1$)	$k_j(a+b\ln|\mathbf{k}|)$	$t^{-(N+2)/\mu}\ln t$
Long tail ($\sigma<1$)	$k_j|\mathbf{k}|^{\sigma-1}$	$t^{-(\sigma+N+1)/\mu}$

respectively, the asymptotic behavior of $\rho(\mathbf{x},t)$ is decomposed in a product of time-dependent factors as those obtained in the previous section. In particular, the case $\mu=2$, corresponding to the usual Schrödinger and diffusion equations, for $N=1$ and 3, has been partially discussed in Refs. [11–15]. For spatially symmetric or antisymmetric initial conditions, different possibilities, related to the properties of the 0th and 1st moments, are summarized in Table I. Naturally, the possibilities related to upper order moments follow the same scheme presented in this table. In fact, if we employ the initial condition $\rho(\mathbf{x},0) \propto \partial^{n_o} g(\mathbf{x})/\partial x_j^{n_o}$, with a suitable $g(\mathbf{x})$, we extend the results of Table I to the n_oth moment. For instance, for $g(\mathbf{x})=1/(b^2+|\mathbf{x}|^2)^\nu$ with $\sigma=2\nu-N$, we can obtain, for long time and small $|\mathbf{x}|$, $\rho(\mathbf{x},t)\sim t^{-\alpha}$ where α can take arbitrary values depending on the choice of ν.

As in the one-dimensional case, aside from logarithmic corrections arising in marginal cases, power laws with arbitrary exponent can arise. Then, previous general observations for one dimension also apply to the N-dimensional case. Now, besides depending on the form of the propagator (through μ) and on the shape of the initial packet (through either n_o or σ), the exponent also depends on the dimensionality N of the spreading process. The higher the dimensionality, the faster the decay.

V. FINAL REMARKS

Summarizing, the temporal and spatial behaviors of a packet obeying the evolution equations here considered (diffusion-like equations), and after a sufficiently long time has elapsed, can be expressed, in general, in terms of the moments of the initial distribution (if finite) and the spatial derivatives of the corresponding Green function. Let us remark that these derivatives can also be of the fractional type, which occurs when the moments are divergent. Furthermore, the dominant contribution to the asymptotic expression contains terms corresponding to the lowest order derivatives.

This picture remains essentially valid for the N-dimensional case but the combination of partial derivatives applied over the Green function leads to a richer spectrum than in the one-dimensional case. We remark that, aside from logarithmic behaviors in threshold cases, the asymptotic time decay $t^{-\alpha}$, where α is an arbitrary positive constant, naturally arises for the solutions of diffusion-like equations starting from appropriate initial conditions. Therefore, the temporal decay may be arbitrarily faster or slower than normal ($\alpha=N/2$) depending on the characteristics of the initial packet and the spreading process.

Our present approach could be straightforwardly extended to other linear evolution equations (1) with operators different from the one given in Eq. (2). Also evolution equations with fractional time derivatives could be analyzed by means of the present treatment.

As a perspective, besides being relevant for collision phenomena, our findings could be employed to investigate survival and nonescape probabilities in quantum systems [16], since free-particle states would give, after a sufficiently long time, an approximation to situations with a localized potential energy. Additionally, our methods and results can be useful for the design of efficient strategies in random search or optimization algorithms, where rapid occupation of the available space is desired.

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