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Analytical investigation of the combined effect of fluid inertia and unsteadiness on low-Re particle centrifugation

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We analyze the explicit contribution of fluid inertia and fluid unsteadiness to the force acting on a solid sphere moving in a vertical solid-body rotation flow, in the limit of small Reynolds and Taylor numbers. This problem can be thought of as a test case where the flow induced by the particle is both unsteady (in the laboratory frame) and convected by the unperturbed flow. Many authors assume that the contributions of these two effects can be approximately superposed, and postulate that the particle motion equation is composed of the classical Boussinesq-Basset-Oseen equation (obtained by neglecting the fluid inertia) plus an additive lift force. In the present paper the simplicity of the unperturbed flow enables one to calculate analytically the explicit contribution of each term appearing in the perturbed flow equation (by using matched asymptotic expansions). Our results show how the convective terms and the unsteady term do contribute to the particle drag and lift coefficients in a very complex and nonadditive manner.

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The slow motion of isolated spherical solid particles in nonuniform flows is often investigated by using the equations of Maxey and Riley [1], which are valid in the limit of vanishing fluid inertia. These equations generalize the well-known Basset-Boussinesq-Oseen (BBO) equations, and give fairly good results, for example, when the particle moves freely in a vertical solid-body rotation flow (see Druzhinin and Ostrovsky [2] and Candelier *et al.* [3,4]). They are also extensively used in turbulent particle-laden flow simulations (see, for example, Elghobashi and Truesdell [5]). These equations read

$$m_p\ddot{\mathbf{X}}_p = m_p\mathbf{g} + \mathbf{F}^0 + \mathbf{F}^1,$$

where m_p and $\mathbf{X}_p(t)$ denote the particle mass and position, respectively, and $\mathbf{F}^0$ is the integral of the stress tensor corresponding to the unperturbed flow (it contains the well-known Archimedes force plus Tchen's force). The force $\mathbf{F}^1$ is the integral of the stress tensor of the flow induced by the inclusion, and reads [1]

$$\mathbf{F}^1 = -6\pi\mu a\mathbf{V}_s - \frac{1}{2}m_f\frac{d\mathbf{V}_s}{dt} - 6\rho a^2\sqrt{\nu\pi}\int_{-\infty}^{t}\frac{d\mathbf{V}_s}{d\tau}\frac{d\tau}{\sqrt{t-\tau}},$$

where $\mathbf{V}_s$ is the slip velocity, m_f is the mass of fluid within the particle volume, a is the particle radius, and ρ, μ, ν denote the fluid density, dynamical viscosity, and kinematical viscosity, respectively. The first term appearing in this force is Stokes' drag. The two other terms (added mass and Basset's history force) are due to the unsteadiness of the induced flow. These motion equations are valid when the unsteadiness of the induced flow dominates inertia effects.

Nevertheless, even in the limit where the particle Reynolds number is small, the motion equation of the induced flow does not always reduce to the unsteady Stokes equation, so that the above equation is not expected to be valid. Because these inertial terms are often responsible for lift effects (Saffman [6]), many authors propose to add a lift force into Maxey and Riley's equation:

$$\mathbf{F}^1 = -6\pi\mu a\mathbf{V}_s - \frac{1}{2}m_f\frac{d\mathbf{V}_s}{dt} - 6\rho a^2\sqrt{\nu\pi}\int_{-\infty}^{t}\frac{d\mathbf{V}_s}{d\tau}\frac{d\tau}{\sqrt{t-\tau}} + 6\pi\mu a\left(\frac{a^2\Omega}{\nu}\right)^{1/2}C_L\hat{\mathbf{\Omega}}\times\mathbf{V}_s, \qquad (1)$$

where Ω is the local shear rate of the unperturbed flow, $\hat{\mathbf{\Omega}}$ is the unit vector along the vorticity of the unperturbed flow, and $C_L=O(1)$ is the lift coefficient. The purpose of the present note is to examine the validity of Eq. (1) in the case of a solid particle moving slowly in a vertical solid-body rotation flow (see Fig. 1). First, we rewrite Eq. (1) in this case: see Eq. (3) below. Secondly, we solve asymptotically the fluid motion equation, and integrate the stress tensor around the inclusion, to derive a rigorous hydrodynamic force which will be compared to (3). Note that this force has

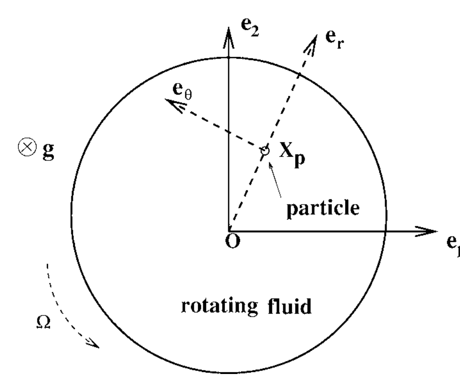

FIG. 1. Sketch of the rotating fluid and of the inclusion.

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already been obtained by Herron *et al.* [7] in the rotating frame attached to the inclusion, where the flow is steady. In contrast with these authors, and in order to follow the explicit contribution of fluid unsteadiness and fluid inertia to the resulting force, our calculations are performed in the nonrotating frame attached to the inclusion.

The rotation rate of the unperturbed flow is Ω, and the particle velocity is taken to be of the form

$$\dot{\mathbf{X}}_p(t) = U\mathbf{e}_r + R_p(t)\Omega_p\mathbf{e}_\theta + V_z\mathbf{e}_z, \tag{2}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_z$ denote the unit radial, azimuthal, and axial vectors, respectively, at the particle center, R_p is the distance to the rotation axis $O\mathbf{e}_z$, and $U=\dot{R}_p$.

The motion is such that both the particle Reynolds number $\mathrm{Re}=Ua/\nu$ and the particle Taylor number $\mathrm{Ta}=a^2\Omega/\nu$ are smaller than unity. In addition, we assume that $\mathrm{Ta}^{1/2}\gg\mathrm{Re}$. Under these conditions, experimental analyses [3,4] show that both V_z and U are constant. (The radial velocity U grows exponentially with a negligible growth rate of the order of $\mathrm{Ta}\Omega\ll\Omega$.) Also the angular velocity Ω_p of the particle is very close to Ω (no azimuthal slip). The horizontal motion of the inclusion has been shown to be independent of the vertical motion [4], so that one can take $V_z=0$ for the sake of simplicity. (For a rigorous derivation of V_z see Childress [8] and Herron *et al.* [7].) In particular the slip velocity reads $\mathbf{V}_s=U\mathbf{e}_r=U(\cos\Omega t\,\mathbf{e}_1+\sin\Omega t\mathbf{e}_2)$, and is a harmonic function of time. This enables one to calculate the various terms of Eq. (1), and we are led to

$$\mathbf{F}^1 = -6\pi\mu a\left[\mathbf{I}+\sqrt{\mathrm{Ta}}\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}+C_L\\ \frac{\sqrt{2}}{2}-C_L & \frac{\sqrt{2}}{2}\end{pmatrix}+O(\mathrm{Ta})\right]\cdot\mathbf{V}_s. \tag{3}$$

In order to check the validity of this result, we now solve asymptotically the fluid motion equation, to derive a rigorous expression for $\mathbf{F}^1$.

In a reference frame moving with the inclusion, with axes parallel to the laboratory frame, the particle-induced flow equations read

$$\nabla\cdot\mathbf{w}_f^1=0, \tag{4}$$

$$\frac{\partial\mathbf{w}_f^1}{\partial t}+\mathbf{w}_f^0\cdot\nabla\mathbf{w}_f^1+\mathbf{w}_f^1\cdot\nabla\mathbf{w}_f^0+\mathbf{w}_f^1\cdot\nabla\mathbf{w}_f^1=-\frac{1}{\rho}\nabla P^1+\nu\triangle\mathbf{w}_f^1, \tag{5}$$

where $\mathbf{w}_f^1(\mathbf{x},t)$ denotes the velocity field of the flow induced by the inclusion and P^1 is the modified pressure. The velocity field $\mathbf{w}_f^0$ in the absence of the particle reads, in the moving reference frame,

$$\mathbf{w}_f^0(\mathbf{x},t)=\mathbf{A}\cdot\mathbf{x}+\mathbf{A}\cdot\mathbf{X}_p-\dot{\mathbf{X}}_p, \tag{6}$$

where the matrix $\mathbf{A}$ corresponds to the solid-body rotation flow:

$$\mathbf{A}=\begin{pmatrix}0 & -\Omega & 0\\ \Omega & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

It is understood that the total velocity field is $\mathbf{w}_f^0+\mathbf{w}_f^1$, and that the gradient of the unperturbed flow is $\mathbf{A}$. The boundary conditions involve the particle velocity in the laboratory reference frame, which are given by Eq. (2).

In order to estimate the order of magnitude, at distance r from the particle center, of the various terms appearing in (5), one usually uses the steady Stokes solution, which clearly shows that $|\mathbf{w}_f^1|\sim Ua/r$ and $|\nabla\mathbf{w}_f^1|\sim Ua/r^2$. Also, the time derivative of $\mathbf{w}_f^1$ scales like ΩU, as suggested by the harmonic boundary condition below. Taking account of $\mathrm{Ta}^{1/2}\gg\mathrm{Re}$, the momentum equation reduces to

$$\frac{\partial\mathbf{w}_f^1}{\partial t}+\{(\mathbf{A}\cdot\mathbf{x})\cdot\nabla\}\mathbf{w}_f^1+\mathbf{A}\cdot\mathbf{w}_f^1=-\frac{1}{\rho}\nabla P^1+\nu\triangle\mathbf{w}_f^1. \tag{7}$$

The three terms on the left-hand side of (7) scale like ΩU, and none of them can be neglected so far. Under these assumptions the boundary condition at the particle surface is uniform and harmonic,

$$\mathbf{w}_f^1=\mathbf{V}_s=U(\cos\Omega t\mathbf{e}_1+\sin\Omega t\mathbf{e}_2)\quad\text{when } |\mathbf{x}|=a. \tag{8}$$

Also, the induced flow is set to vanish at infinity.

When the convective terms in Eq. (7) are set to zero, we recover the well-known Basset-Boussinesq problem along the axes $\mathbf{e}_1$ and $\mathbf{e}_2$. The corresponding hydrodynamic force reads

$$\mathbf{F}^1=-6\pi\mu a\left[\mathbf{I}+\sqrt{\mathrm{Ta}}\begin{pmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\ \sqrt{2}/2 & \sqrt{2}/2\end{pmatrix}+O(\mathrm{Ta})\right]\cdot\mathbf{V}_s, \tag{9}$$

where $\mathbf{I}$ is the unit matrix. When convective terms are taken into account, the resulting hydrodynamic force can be obtained (using matched asymptotic expansions) in the rotating reference frame where the induced flow is steady and the three left-hand side terms of Eq. (7) collapse into a Coriolis acceleration. The result is (Herron *et al.* [7])

$$\mathbf{F}^1=-6\pi\mu a\left[\mathbf{I}+\sqrt{\mathrm{Ta}}\begin{pmatrix}\frac{5}{7} & \frac{-3}{5}\\ \frac{3}{5} & \frac{5}{7}\end{pmatrix}+O(\mathrm{Ta})\right]\cdot\mathbf{V}_s \tag{10}$$

Clearly, since this last result has been obtained without neglecting the convective terms, it is expected to be more accurate than (9). However, because $\sqrt{2}/2\approx0.707$ and $5/7\approx0.714$, the drag corrections are so close that one could believe that inertia terms have almost no net effect on the drag, and only influence the lift force. Therefore, this result seems to confirm *a priori* that an additional lift force could be added to the classical Maxey and Riley's motion equation, as shown in Eq. (1), in order to correctly predict the particle trajectory in this case.

The goal of this note is to check whether this observation is correct, by solving (7) asymptotically, and by "following" the contribution of the convective terms. To achieve this goal we introduce two nondimensional multiplicative coefficients ("markers") α and β as follows:

$$\alpha\frac{\partial \mathbf{w}_f^1}{\partial t} + \beta[(\mathbf{A}\cdot\mathbf{x})\cdot\nabla\mathbf{w}_f^1 + \mathbf{A}\cdot\mathbf{w}_f^1] = -\frac{1}{\rho}\nabla P^1 + \nu\triangle\mathbf{w}_f^1, \tag{11}$$

and solve this last equation by using the classical matched asymptotic expansion approach (keeping in mind that, strictly speaking, both α and β are equal to 1, even though the following calculation is valid for any α and β held fixed as $\mathrm{Ta}\to 0$).

In the vicinity of the particle (inner problem) the induced flow is equal to the steady Stokes solution plus a corrective term which has to match the solution of the outer problem (Proudman and Pearson [9]):

$$\alpha\frac{\partial \mathbf{w}_f^1}{\partial t} + \beta[(\mathbf{A}\cdot\mathbf{x})\cdot\nabla\mathbf{w}_f^1 + \mathbf{A}\cdot\mathbf{w}_f^1] = -\nabla P^1 + \triangle\mathbf{w}_f^1 + 6\pi(\cos t\mathbf{e}_1 + \sin t\mathbf{e}_2)\delta(\mathbf{x}) \tag{12}$$

which corresponds to Eq. (7) written in a nondimensional form [by using the Ekman distance $a/\mathrm{Ta}^{1/2}$ for lengths, U for velocities, Ω for the time derivative, and $\rho(\Omega\nu)^{1/2}U$ for the pressure]. Also, the boundary condition at the particle surface is removed, and replaced by the classical Dirac source term on the right-hand side of the momentum balance (12). Following the classical approach, we solve this last equation by using the Fourier transform defined as

$$\widetilde{\mathbf{w}}(\mathbf{k},t) = \frac{1}{8\pi^3}\int_{R^3}\mathbf{w}_f^1(\mathbf{x},t)\exp(-i\mathbf{k}\cdot\mathbf{x})d^3\mathbf{x}.$$

We are led to

$$\alpha\frac{\partial\widetilde{\mathbf{w}}}{\partial t} + \beta\left[\left(k_1\frac{\partial}{\partial k_2} - k_2\frac{\partial}{\partial k_1}\right)\widetilde{\mathbf{w}} + \mathbf{A}\cdot\widetilde{\mathbf{w}}\right] = -i\mathbf{k}\widetilde{P} - k^2\widetilde{\mathbf{w}} + \frac{3}{4\pi^2}(\cos t\mathbf{e}_1 + \sin t\mathbf{e}_2). \tag{13}$$

It is convenient to introduce the cylindrical coordinates (k,η,k_3) in the Fourier space (see, for example, Gotoh [10]), $k_1=-\hat{k}\sin\eta$ and $k_2=\hat{k}\cos\eta$, so that (13) now reads

$$\alpha\frac{\partial\widetilde{\mathbf{w}}}{\partial t} + \beta\left(\frac{\partial\widetilde{\mathbf{w}}}{\partial\eta} + \mathbf{A}\cdot\widetilde{\mathbf{w}}\right) = -i\mathbf{k}\widetilde{P} - k^2\widetilde{\mathbf{w}} + \frac{3}{4\pi^2}(\cos t\mathbf{e}_1 + \sin t\mathbf{e}_2). \tag{14}$$

Because the solution is periodic we also write the time dependence as

$$\widetilde{\mathbf{w}} = \widetilde{\mathbf{w}}_+\exp(it) + \widetilde{\mathbf{w}}_-\exp(-it)$$

and

$$\widetilde{P} = \widetilde{P}_+\exp(it) + \widetilde{P}_-\exp(-it),$$

and after some algebra we get

$$\alpha i\widetilde{\mathbf{w}}_+ + \beta\left(\frac{\partial\widetilde{\mathbf{w}}_+}{\partial\eta} + \mathbf{A}\cdot\widetilde{\mathbf{w}}_+\right) = -i\mathbf{k}\widetilde{P}_+ - k^2\widetilde{\mathbf{w}}_+ + \frac{3}{8\pi^2}\mathbf{e}_1 + \frac{3}{8i\pi^2}\mathbf{e}_2. \tag{15}$$

Taking advantage of the periodicity in the spectral azimuth η we also write

$$\widetilde{\mathbf{w}}_+ = \widetilde{\mathbf{w}}_0 + \widetilde{\mathbf{w}}_{1+}\exp(i\eta) + \widetilde{\mathbf{w}}_{1-}\exp(-i\eta) + \widetilde{\mathbf{w}}_{2+}\exp(2i\eta) + \widetilde{\mathbf{w}}_{2-}\exp(-2i\eta), \tag{16}$$

and

$$\widetilde{P}_+ = \widetilde{P}_0 + \widetilde{P}_{1+}\exp(i\eta) + \widetilde{P}_{1-}\exp(-i\eta) + \widetilde{P}_{2+}\exp(2i\eta) + \widetilde{P}_{2-}\exp(-2i\eta). \tag{17}$$

By injecting these expressions into (15), using the continuity equation, and projecting on $\exp(ni\eta)$, we obtain an algebraic system with 20 unknowns, which is solved analytically. In this matched asymptotic expansion approach the nondimensional force correction is given by

$$6\pi\sqrt{\mathrm{Ta}}[(2\,\mathrm{Re}(I)\cos t - 2\,\mathrm{Im}(I)\sin t)\mathbf{e}_1 + (2\,\mathrm{Re}(J)\cos t - 2\,\mathrm{Im}(J)\sin t)\mathbf{e}_2], \tag{18}$$

where Re() and Im() denote the real and imaginary parts, respectively. I and J correspond to inverse Fourier transforms of the velocity in the limit where $|\mathbf{x}|\to 0$ (inner limit of the outer flow)

$$I = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}(\widetilde{u}_0 - \widetilde{u}_0^s)\hat{k}d\eta\,d\hat{k}\,dk_3,$$

$$J = \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi}(\widetilde{v}_0 - \widetilde{v}_0^s)\hat{k}d\eta\,d\hat{k}\,dk_3,$$

and the superscript "s" stands for "Stokeslet", and corresponds to the solution of our problem with $\alpha=\beta=0$ (Stokes problem). The variables $\widetilde{u}_0$ are the components of the Fourier transform of the velocity, namely $\widetilde{\mathbf{w}}_0=\widetilde{u}_0\mathbf{e}_1+\widetilde{v}_0\mathbf{e}_2+\widetilde{w}_0\mathbf{e}_3$. In order to achieve the analytical calculation of I and J we set $k_3=k'\sin\phi$ and $\hat{k}=k'\cos\phi$ with $\phi\in[-\pi/2;\pi/2]$ and $k'\in[0;\infty]$, and we obtain (coming back to dimensional variables)

$$\mathbf{F}^1 = -6\pi\mu a\left[\mathbf{I} + \sqrt{\mathrm{Ta}}\begin{pmatrix} M_{11}(\alpha,\beta) & -M_{12}(\alpha,\beta) \\ M_{12}(\alpha,\beta) & M_{11}(\alpha,\beta)\end{pmatrix} + O(\mathrm{Ta})\right]\cdot\mathbf{V}_s \tag{19}$$

with

$$M_{11}(\alpha,\beta) = \frac{(\alpha+\beta)^{3/2}(\alpha^2 - 12\alpha\beta + 57\beta^2) + (3\beta-\alpha)^3\sqrt{|3\beta-\alpha|}}{\frac{280}{\sqrt{2}}\beta^3} \tag{20}$$

and

TABLE I. Specific values of the mobility tensor coefficients.

	Herron *et al.* $\alpha=1$ $\beta=1$	Maxey and Riley $\alpha=1$ $\beta=0$	Gotoh $\alpha=0$ $\beta=1$
$M_{11}(\alpha,\beta)$:	$\frac{5}{7}$	$\sqrt{2}/2$	$\frac{3\sqrt{2}}{280}(19+9\sqrt{3})$
$M_{12}(\alpha,\beta)$:	$\frac{3}{5}$	$\sqrt{2}/2$	$\frac{3\sqrt{2}}{280}(19-9\sqrt{3})$

$$M_{12}(\alpha,\beta) = \frac{(\alpha+\beta)^{3/2}(\alpha^2-12\alpha\beta+57\beta^2)-|3\beta-\alpha|^3\sqrt{|3\beta-\alpha|}}{\frac{280}{\sqrt{2}}\beta^3}. \tag{21}$$

Equations (19)–(21) are the main result of this note. The mobility tensor of Herron *et al.* [7] is recovered when $\alpha=\beta=1$, as expected (see also Table I). Nevertheless, the differences between (19)–(21) and (3) are striking. Result (20) clearly shows that the convective terms do contribute to the particle drag coefficient (M_{11}), but in a very complex and nonadditive manner, in contrast with (3). The unsteadiness of the flow (manifested by the coefficient α) is also of major importance, and is strongly coupled to the convective terms.

In the unsteady creeping-flow limit ($\alpha=1$ and $\beta=0$) we recover the mobility tensor obtained by solving Maxey and Riley's equations [that is, Eq. (9)]. Gotoh's mobility tensor is recovered in the steady case ($\alpha=0$ and $\beta=1$). The reason why Gotoh's mobility tensor differs from the results of Herron *et al.* had already been investigated in the past by Miyazaki [11]. Our analysis differs in that we follow separately the contribution of unsteadiness and inertia, and with a different formalism.

Finally, one can check, by applying the very same formalism as the one used in this note, that if the particle is kept fixed in the rotating flow (and if condition $\mathrm{Ta}^{1/2} \gg \mathrm{Re}$ is still fulfilled), the force experienced by the particle corresponds to Gotoh's mobility tensor (see Table I). In particular, the lift force strongly differs from the one appearing in Eq. (3), since they have different absolute values and opposite signs. Clearly, this test case, like the centrifugated particle case, invalidate the systematic use of an additive Saffman lift force when the flow under the study is not a pure shear flow.

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