Wigner surmises and the two-dimensional homogeneous Poisson point process

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We derive a set of identities that relate the higher-order interpoint spacing statistics of the two-dimensional homogeneous Poisson point process to the Wigner surmises for the higher-order spacing distributions of eigenvalues from the three classical random matrix ensembles. We also report a remarkable identity that equates the second-nearest-neighbor spacing statistics of the points of the Poisson process and the nearest-neighbor spacing statistics of complex eigenvalues from Ginibre's ensemble of 2×2 complex non-Hermitian random matrices.

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Random matrix theory (RMT) was originally devised to model the complicated (and presumably unknowable) Hamiltonian of a heavy nucleus [1]. Despite its original motivation, RMT has developed into a subject of its own and has been applied in many different areas of mathematics and physics [2–4]. The most elementary result of classical RMT is the Wigner distribution

$$P_W(S) = \frac{\pi}{2} S \exp\left(-\frac{\pi}{4}S^2\right), \tag{1}$$

which is the nearest-neighbor spacing distribution (NNSD) of eigenvalues from the Gaussian ensemble of real symmetric 2×2 random matrices. The NNSD of eigenvalues $P(S)$ gives the probability $P(S)dS$ of finding two consecutive energy levels situated at a distance S apart with the lower at $E=\xi$ and the upper in the interval $\xi+S\leqslant E<\xi+S+dS$. Curiously, the Wigner distribution also describes the nearest-neighbor spacing statistics of one of the simplest and most important models of stochastic geometry: the homogeneous Poisson point process in $\mathbb{R}^2$ (henceforth denoted by $\mathbf{P}_2$). $\mathbf{P}_2$ is the limit of a simpler stochastic model: the binomial point process in $\mathbb{R}^2$ [5]. The binomial model consists of N independent uniformly distributed random points in a compact subset W of $\mathbb{R}^2$. For simplicity, suppose that W is a disk of radius R and center at the origin. If we take the limits $N\to\infty$ and $R\to\infty$ in such a way that $N/\pi R^2\equiv\rho$ remains constant, then the limiting stochastic point process is $\mathbf{P}_2$ (with intensity ρ). It can be shown that the NNSD for $\mathbf{P}_2$ is given by [6]

$$D(s) = 2\rho\pi s \exp(-\rho\pi s^2). \tag{2}$$

[The distribution $D(s)$ gives the probability $D(s)ds$ of finding the nearest neighbor to a given point of $\mathbf{P}_2$ at a distance between s and $s+ds$.] It is easy to verify that the distribution is normalized [i.e., $\int_0^\infty D(s)ds=1$] and that the mean nearest-neighbor distance

$$\bar{s} = \int_0^\infty sD(s)ds = \frac{1}{2\sqrt{\rho}}. \tag{3}$$

If we introduce the rescaled distance $S=s/\bar{s}$, then the distribution (2) becomes

$$D(S) = D(s=\bar{s}S)\times\left(\frac{ds}{dS}\right) = P_W(S). \tag{4}$$

This correspondence is not a new fact. It has already been shown in Ref. [7] that the NNSD of N random points uniformly distributed on a disk of radius R is (as $N\to\infty$) given by the Wigner distribution [21]. There is at once the intriguing question of whether $\mathbf{P}_2$ also has the same higher-order spacing statistics as eigenvalues from the Gaussian orthogonal ensemble (GOE). If not, is there any correspondence between the spacing statistics of $\mathbf{P}_2$ and the spacing statistics of eigenvalues from the other random matrix ensembles? It is these questions that we wish to answer in this Brief Report. As we will see, the answers are quite interesting and clearly show that there is a deep connection between $\mathbf{P}_2$ and classical RMT.

We first define the mathematical objects of interest in this paper. These are the following: (i) the kth-nearest-neighbor spacing distribution (kth-NNSD) $P(S;k,\beta)$ of eigenvalues [β is a parameter that labels the classical random matrix ensembles GXE ($X=O,U,S$) and will be specified below], which gives the probability $P(S;k,\beta)dS$ of finding two energy levels *separated by* $k-1$ *other energy levels* to be a distance S apart with the lower at $E=\xi$ and the upper in the interval $\xi+S\leqslant E<\xi+S+dS$ (with $k-1$ other energy levels in the interval $\xi<E<\xi+S$), and (ii) the kth-NNSD for $\mathbf{P}_d$ (the homogeneous Poisson point process in $\mathbb{R}^d$) $D(S;k,d)$, which gives the probability $D(S;k,d)dS$ of finding the kth nearest neighbor to a given point of $\mathbf{P}_d$ at a distance between S and $S+dS$. We have seen above that $D(S;1,2)=P_W(S;1,1)$, where $P_W(S;1,1)\equiv P_W(S)$ is the so-called Wigner surmise for the NNSD of eigenvalues from the GOE. The question posed above can now be put more succinctly as follows: is there any correspondence between $D(S;k,2)$ and the Wigner surmises for $P(S;k,\beta)$ [denoted by $P_W(S;k,\beta)$]?

The kth-NNSD for $\mathbf{P}_2$ is given by [6]

$$D(s;k,2) = \frac{2(\rho\pi)^k}{\Gamma(k)} s^{2k-1}\exp(-\rho\pi s^2). \tag{5}$$

It is once again easy to verify that the above distribution is normalized, and that the mean kth-nearest-neighbor distance is

$$\bar{s}=\int_0^\infty sD(s;k,2)ds=\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k)\sqrt{\rho\pi}}. \tag{6}$$

As before, if we transform to the random variable $S=s/\bar{s}$, the distribution (5) becomes

$$D(S;k,2)=\frac{2\alpha^k}{\Gamma(k)}S^{2k-1}\exp(-\alpha S^2), \tag{7a}$$

where

$$\alpha=\left[\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k)}\right]^2. \tag{7b}$$

The kth-NNSD for $\mathbf{P}_d$ is given in Ref. [7]; note however that there is a nontrivial error in the formula given there.

The Wigner surmises for the NNSD of eigenvalues from the GXE ($X=O,U,S$) are given by (see Ref. [3])

$$P_W(S;1,\beta)=\mathcal{A}(\beta)S^\beta\exp[-\mathcal{B}(\beta)S^2], \tag{8a}$$

where

$$\mathcal{A}(\beta)=2\frac{\left[\Gamma\left(\frac{\beta}{2}+1\right)\right]^{\beta+1}}{\left[\Gamma\left(\frac{\beta+1}{2}\right)\right]^{\beta+2}},\quad \mathcal{B}(\beta)=\frac{\left[\Gamma\left(\frac{\beta}{2}+1\right)\right]^{2}}{\left[\Gamma\left(\frac{\beta+1}{2}\right)\right]^{2}}, \tag{8b}$$

and the level-repulsion parameter $\beta=1,2,4$ labels the symmetry classes (i.e., the ensembles), which are Gaussian orthogonal (GOE), unitary (GUE), and symplectic (GSE), respectively. Upon comparison of Eqs. (7) and (8), we can immediately deduce that

$$D(S;(\beta+1)/2,2)=P_W(S;1,\beta),\quad \beta=1,3,5,\ldots. \tag{9}$$

This identity, although formally interesting, appears to be quite useless since the Wigner surmises for the NNSDs [Eq. (8)] are usually defined only for three integer values ($\beta=1,2,4$), which correspond to the three classical random matrix ensembles. It is, however, a commonly overlooked fact that $P_W(S;1,3)$ is identical to the NNSD of complex eigenvalues from Ginibre's ensemble of 2×2 general complex non-Hermitian random matrices [8]:

$$P_W(S;1,3)=P_G(S)=\frac{3^4\pi^2}{2^7}S^3\exp\left(-\frac{3^2\pi}{2^4}S^2\right). \tag{10}$$

Note that this ensemble yields cubic level repulsion [i.e., $P_G(S)\sim S^3$ for small values of S]. Cubic level repulsion in quantum spectra was found to be a universal property of dissipative quantum systems with a chaotic classical limit [9]. Ginibre-like statistics have also been observed for the nearest-neighbor spectral statistics of the lattice Dirac operator with nonzero chemical potential in lattice QCD [10]. In addition to these results, we have now the following remarkable equality:

$$D(S;2,2)=P_G(S). \tag{11}$$

We are not aware of any quantum system whose spectrum has a NNSD that is described by Eq. (8) with $\beta\geqslant 5$. There are, however, Gaussian ensembles of random matrices that would correspond to any $\beta>0$ (and, in particular, $\beta\geqslant 5$), the so-called β-Hermite ensembles [12], but the spacing statistics of eigenvalues from these ensembles are, to our knowledge, not known.

It might appear that the equality (11) already answers the original question we asked at the outset since the *second*-nearest-neighbor spacing statistics for $\mathbf{P}_2$ are clearly not GOE statistics. The *second*-nearest-neighbor spacing statistics for eigenvalues from the GOE should be the same as the *nearest*-neighbor spacing statistics for eigenvalues from the GSE [13], whereas the *second*-nearest-neighbor spacing statistics for the points of $\mathbf{P}_2$ are the same as the *nearest*-neighbor spacing statistics for eigenvalues from the Ginibre ensemble (11). However, these facts alone do not settle the matter and there is also the interesting possibility that GUE and GSE statistics might still appear in the higher-order spacing statistics of $\mathbf{P}_2$ despite the fact that the Wigner surmises with even-integer values of β (corresponding to *nearest*-neighbor GUE and GSE statistics) are curiously absent in relation (9) [22].

To settle these questions we consider the higher-order spacing distributions for eigenvalues from the three classical ensembles GXE ($X=O,U,S$). Interestingly, it was not long ago that the Wigner surmises for these higher-order spacing distributions were proposed [11]. The Wigner surmises for the kth-NNSDs $\mathcal{P}_W(s;k,\beta)$ were obtained in Ref. [11] subject to the conditions that $\int_0^\infty\mathcal{P}_W(s;k,\beta)ds=1$ and $\int_0^\infty s\mathcal{P}_W(s;k,\beta)ds=k$. However, if we transform to the scaled spacing $S=s/\bar{s}=s/k$ (as before), then the distribution $\mathcal{P}_W(s;k,\beta)$ becomes the distribution

$$P_W(S;k,\beta)=\mathcal{P}_W(s=\bar{s}S;k,\beta)(\mathrm{d}s/\mathrm{d}S)=\mathcal{P}_W(s=kS;k,\beta)\times k.$$

Using the formulas of Ref. [11] subject to the constraint of unit mean spacing (and expressing these in our notation), the Wigner surmises are then given by

$$P_W(S;k,\beta)=\mathcal{A}(k,\beta)S^{\upsilon(k,\beta)}\exp[-\mathcal{B}(k,\beta)S^2], \tag{12a}$$

where the level-repulsion exponent

$$\upsilon(k,\beta)=(k-1)+\frac{k(k+1)}{2}\beta, \tag{12b}$$

and the constants

$$\mathcal{A}(k,\beta)=2\frac{\left[\Gamma\left(\frac{\upsilon(k,\beta)}{2}+1\right)\right]^{\upsilon(k,\beta)+1}}{\left[\Gamma\left(\frac{\upsilon(k,\beta)+1}{2}\right)\right]^{\upsilon(k,\beta)+2}}, \tag{12c}$$

and

$$\mathcal{B}(k,\beta)=\frac{\left[\Gamma\left(\frac{\upsilon(k,\beta)}{2}+1\right)\right]^{2}}{\left[\Gamma\left(\frac{\upsilon(k,\beta)+1}{2}\right)\right]^{2}}. \tag{12d}$$

It is important to emphasize that the Wigner surmises (12) are analytical approximations to the exact kth-NNSD

$[P(S;k,\beta)]$ of eigenvalues from Gaussian ensembles of arbitrarily large random matrices [11].

Upon comparison of Eqs. (7) and (12), we may deduce several interesting relations between the higher-order spacing statistics of $\mathbf{P}_2$ and the higher-order spacing statistics of eigenvalues from the classical random matrix ensembles. The first set of identities relate $\mathbf{P}_2$ statistics and GOE statistics:

$$D(S;n(4n+3),2) = P_W(S;4n,1), \qquad (13a)$$

$$D(S;(4n+1)(n+1),2) = P_W(S;(4n+1),1), \qquad (13b)$$

where $n \in \mathbb{N}$. These identities explicitly show that GOE statistics are not exclusive to the nearest-neighbor spacing statistics of $\mathbf{P}_2$. Even more interesting is the fact that GUE and GSE statistics describe certain long-range spacing statistics of $\mathbf{P}_2$. The identities that relate $\mathbf{P}_2$ statistics to GUE and GSE statistics are

$$D(S;2n(n+1),2) = P_W(S;2n,2), \quad n \in \mathbb{N} \qquad (14)$$

and

$$D(S;n(4n+3),2) = P_W(S;2n,4), \quad n \in \mathbb{N} \qquad (15)$$

respectively.

Although it is only of subsidiary interest to the main discussion, we would like also to comment on the spacing statistics of $\mathbf{P}_1$ (the homogeneous Poisson point process on a line). It can be shown that

$$D(S;k,1) = \frac{k^k}{\Gamma(k)} S^{k-1} \exp(-kS). \qquad (16)$$

The NNSD for $\mathbf{P}_1$ is the well-known Poisson distribution [i.e. $D(S;1,1)=P_P(S)=\exp(-S)$], and it has been pointed out in Ref. [7] that $D(S;2,1)$ is the so-called semi-Poisson distribution (see Refs. [14,15]):

$$D(S;2,1) = P_{sP}(S) = 4S\exp(-2S).$$

There is also a more general correspondence between the distribution $D(S;k,1)$ and the "generalized semi-Poisson distribution"

$$P_{sP}(S;n,\beta) = \frac{(\beta+1)^{n(\beta+1)}}{\Gamma[n(\beta+1)]} S^{[n(\beta+1)-1]} \exp[-(\beta+1)S], \qquad (17)$$

which has been relevant in a number of different studies [23]. Comparison of Eqs. (16) and (17) reveals that, formally,

$$D(S;\beta+1,1) = P_{sP}(S;1,\beta). \qquad (18)$$

Note that in Refs. [16,17] there is no restriction on the value of the system parameter β in Eq. (17), but in Refs. [14,15] the parameter β takes only the values 1, 2, and 4. In the present context, Eq. (18) is a mathematical identity which is valid for all $\beta \in \mathbb{N}$. The correspondence (18) is equivalent to the result obtained in Ref. [19], where it was shown that the distribution (17) exactly coincides with the nth-NNSD of the so-called Poissonian "daisy model" of rank r. This model is obtained from retaining every $(r+1)$th level of a Poisson sequence. The authors of Ref. [19] have stated that this model has no "dynamical implications" and that "no link of such statistical spectra to quantized dynamical systems is known for $r>1$ [$\beta>1$]." This statement is incorrect since the distribution $P_{sP}(S;1,\beta)$ is actually the $(\beta+1)$th-NNSD of eigenvalues for a quantum system whose classical limit is integrable. We affirm this conclusion based on the results of Robnik and Veble [20], who showed that the long-range "gap function" statistics of typical integrable systems are indeed Poissonian (and therefore must have the same long-range spacing statistics as those of $\mathbf{P}_1$). The link between the spacing statistics that govern Poissonian daisy models and dynamical systems is thus the following: The NNSD for a daisy model of rank r describes the $(r+1)$th-NNSD of eigenvalues for a quantized integrable system.

In conclusion, there is an inscrutable correspondence between the distribution $D(S;k,2)$ which describes the spacing statistics of $\mathbf{P}_2$ and the Wigner surmises of RMT. The provocative result for the nearest-neighbor spacing statistics of $\mathbf{P}_2$ [Eq. (4)] is (formally) only a special case of a more general result, which is that the kth-NNSDs for $\mathbf{P}_2$ are exactly given by the Wigner surmises having odd-integer values of the level-repulsion parameter β [Eq. (9)]. The correspondence between the second-nearest-neighbor spacing statistics of $\mathbf{P}_2$ and Ginibre statistics (11), and the more general correspondences between the higher-order spacing statistics of $\mathbf{P}_2$ and the Wigner surmises [(13)–(15)] are the new results we wish to report. We should not fail to mention explicitly the astute change of variables that were instrumental in obtaining these results. The contrivance of rescaling a random variable by the mean of the random variable is prevalent in RMT. Indeed, the distance distributions for $\mathbf{P}_2$ coincide with the spacing distributions of RMT (i.e., the Wigner surmises) only when the distance between a given point and its kth nearest neighbor is rescaled with respect to the mean kth-nearest-neighbor distance.

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[21] The binomial point process on W is sometimes denoted by $[N/W]$. Strictly speaking, the situation considered in Ref. [7] is not $\mathbf{P}_2$, but rather the $N\rightarrow\infty$ limit of $[N/W]$. However, as long as this distinction is clear, the label of "Poissonian random process in the plane" as given there is contextually understood. The restriction of $\mathbf{P}_2$ to W (with $\rho=N/|W|$) is well approximated by $[N/W]$, provided $N\gg 1$, and so, it is common to refer to $[N/W]$ as a "Poisson point process on W" when N is large. We have shown above that the Wigner distribution is the NNSD for $\mathbf{P}_2$.
[22] The distribution $\mathrm{D}(S;k,2)$ does formally recover these cases for certain *noninteger* values of k. Specifically, $\mathrm{D}(S;\frac{3}{2},2)=P_W(S;1,2)$ and $\mathrm{D}(S;\frac{5}{2},2)=P_W(S;1,4)$. Of course, these two relations are meaningless in the sense that there is no actual point in the set that can be identified as the $\frac{3}{2}$th- or $\frac{5}{2}$th-nearest neighbor.
[23] The distribution $P_{sP}(S;n,\beta)$ [Eq. (17)] was introduced in Ref. [14] as the nth-NNSD for the short-range plasma model (SRPM); there is also unpublished work by A. Pandey [18], who investigated a model equivalent to the SRPM and derived Eq. (17) in the framework of random-banded matrices; and finally, the distribution $P_{sP}(S;n,\beta)$ was also shown to be relevant to a class of exactly solvable models with nearest- and second-nearest-neighbor interactions [16,17].