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Probing criticality with linearly varying external fields: Renormalization group theory of nonequilibrium critical dynamics under driving

Fan Zhong
Department of Physics, Zhongshan University, Guangzhou 510275, People's Republic of China
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A renormalization group theory for the nonlinear and nonequilibrium responses to linearly varying external probes is formulated for the critical dynamics of a time-dependent Ginzburg-Landau equation with a scalar nonconserved order parameter. A series of nonequilibrium dynamic critical exponents and their scaling relations characterizing the time-dependent probes and the nonequilibrium hysteresis induced are derived analytically and systematically and agree well with numerical results. The three-dimensional dynamic critical exponent z is accordingly determined as $z=2.036(11)$. These show that the linearly varying external fields can be applied to probe criticality effectively.

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To study the properties of a system, a usual way is to apply to it a small external field and then measure its response. Though the field drives the system out of equilibrium, the well-known linear response theory expresses remarkably the linear response in terms of such equilibrium properties as correlation functions evaluated in the absence of that perturbation [1]. At a continuous phase transition point, the critical point T_c, however, long-range fluctuations give rise to a singular response, which signals its nonlinear nature. In fact, for a magnetic system, the relation between the magnetization M and the applied field H at T_c is well known, viz., $M \propto H^{1/\delta}$, where δ is a critical exponent [2]. In spite of the nonlinearity, measuring of the response still yields an equilibrium property, δ. Below T_c, the equation of states provides more information. Though equilibrium properties of the critical phenomena have been well established [3], nonequilibrium dynamical ones have not [4,5]. A problem of concern here is the notoriously critical slowing down that renders equilibrium hard to achieve. So, can one obtain both equilibrium and nonequilibrium properties by taking advantage of the nonequilibrium response inherent in a time-dependent driving that imposes an external time scale to circumvent the critical slowing down? Probing criticality this way may probably be profitable upon noticing the wide emergences of critical and critical-like scale-free phenomena.

In fact, we have used a linearly varying external magnetic field and temperature to study the dynamic scaling of hysteresis [6], and the results of the latter have been favorably tested experimentally [7]. For critical dynamics, we have brought in recently some new nonequilibrium critical exponents to characterize the response of a two-dimensional (2D) Ising model to a linearly swept temperature across its critical point by extending the Monte Carlo (MC) renormalization group (RG) methods [8,9]. Scaling laws relating them to the other usual critical exponents have also been found numerically [10]. A method that uses a linearly swept external field was suggested earlier to determine the dynamic critical exponent z [11], which was cited, among others, to compare with that obtained from other methods in 2D [4]. More recently, we have applied the RG theory to first-order phase transitions driven by a linearly swept field and found that they are governed by new fixed points [12]. As there is no exposition of the method used there, it is, therefore, also desirable to apply it to the well-established critical point to clarify the method used, to check the nonequilibrium effects stemming from the driving, as well as to set the other limit for the crossover from the first-order phase transitions to the continuous one.

The problem to be solved is essentially the usual time-dependent Ginzburg-Landau equation but containing a linearly time-dependent external probe [see Eqs. (1) and (2) below]. The linear response theory is apparently inapplicable here owing to the inherent nonlinearity at T_c. It is difficult, if not impossible, to solve it directly. Lack of an effective small parameter also invalidates direct perturbation expansions. Special approaches, therefore, have to be devised to tackle the problem. To this end, we use the field theoretic version of the RG theory [13]. We renormalize first the corresponding field theory at the critical point, where the temperature deviation $T-T_c$ or the external field vanish, and then make an expansion about the critical theory [13,14] by taking as insertions the deviations arising from the driving away from that point. In this way, the time-dependent external probes can be naturally accounted for. Moreover, the renormalization at the critical point enables us to make direct contact with the original situation to which no time-dependent field is applied, and thus to solve the problem analytically almost without any additional labor. In this paper, we show, for the linearly varying external probes that are experimentally used as, for example, in the standard thermal analysis, our RG theory yields analytically and systematically a series of nonequilibrium dynamic critical exponents and scaling relations that characterize the probes and the ensuing nonequilibrium hysteresis, and that agree quite well with numerical results and thus provide alternative methods for determining the usual critical exponents [10]. The three-dimensional (3D) dynamic critical exponent is accordingly determined.

Consider the model with the following Ginzburg-Landau functional in an external field H,

$$F[\varphi] = \int d\mathbf{r}\left\{\frac{1}{2}\tau\varphi^2 + \frac{1}{4!}g\varphi^4 + \frac{1}{2}[\nabla\varphi]^2 - H\varphi\right\}, \quad (1)$$

where g is a coupling constant and τ is proportional to the temperature T. Its dynamics is governed by the Langevin equation for the scalar nonconserved order parameter φ

$$\partial\varphi/\partial t = -\lambda\,\delta F[\varphi]/\delta\varphi + \xi, \quad (2)$$

i.e., Model A [15], with a Gaussian white noise ξ satisfying $\langle\xi(\mathbf{r},t)\rangle=0$ and $\langle\xi(\mathbf{r},t)\xi(\mathbf{r}',t')\rangle=2\lambda\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$, where λ is a kinetic coefficient. We shall consider two driven nonequilibrium situations in which one starts with a sufficiently ordered state and increases linearly either the temperature $T=Rt$ or the external field $H=Rt$ with a small rate constant R across the critical point.

To start, we recast the dynamics into a field-theoretic form with a dynamic functional [16],

$$I[\varphi,\tilde{\varphi}] = \int d\mathbf{r}dt\left\{\tilde{\varphi}\left[\dot{\varphi} + \lambda(\tau - \nabla^2)\varphi + \frac{1}{3!}\lambda g\varphi^3 - \lambda H\right] - \lambda\tilde{\varphi}^2\right\}, \quad (3)$$

by introducing an auxiliary response field $\tilde{\varphi}$ [17], where the Jacobian of the transform has been dropped to cancel the self-loop of the response field. The generating functional is then $W[h,\tilde{h}]=\ln\int D(\varphi,\tilde{\varphi})\exp[-I[\varphi,\tilde{\varphi}]+\int d\mathbf{r}dt(h\varphi+\tilde{h}\tilde{\varphi})]$. Expectation values can then be obtained by taking appropriate derivatives with respect to the external sources h and $\tilde{h}$ that conjugate, respectively, to φ and $\tilde{\varphi}$.

Accordingly to the renormalized field theory [13], one notes first that since the variation of T and H is spatially uniform and linearly time dependent with a small R, no new divergence except the extrinsic one at $t\to\infty$ or $\omega\to 0$ in frequency domain is generated. As a result, no new renormalization factor Z besides the usual φ^4-theory ones has to be introduced to cure the divergences. As pointed out above, to deal with the time-dependent external probes, we perform the renormalization at the critical point, and take as insertions the deviations arising from the driving away from that point. In this prescription, the renormalization becomes standard [16]. The theory can be rendered finite by introducing the following Z factors:

$$\varphi \to \varphi_0 = Z_\varphi^{1/2}\varphi, \tilde{\varphi} \to \tilde{\varphi}_0 = Z_{\tilde{\varphi}}^{1/2}\tilde{\varphi},$$

$$\lambda \to \lambda_0 = (Z_\varphi/Z_{\tilde{\varphi}})^{1/2}Z_\lambda\lambda, g \to g_0 = N_d\mu^\epsilon Z_\varphi^{-2}Z_u u,$$

$$\tau \to \tau_0 = Z_\varphi^{-1}Z_{\varphi^2}\tau + \tau_c, \quad \text{when varying } \tau,$$

$$H \to H_0 = Z_\varphi^{-1/2}H, \quad \text{when varying } H, \quad (4)$$

where $\epsilon=4-d$, $N_d=2/[(4\pi)^{d/2}\Gamma(d/2)]$ with Γ being the Euler Gamma function, d is the space dimensionality, μ an arbitrary momentum scale, and τ_c the fluctuation shift of the critical point, which can be neglected if dimension regulations [18] are employed. Anyway, we shall henceforth still use τ to denote $\tau-\tau_c$ that is proportional to $T-T_c$. Consequently, the critical point at $\tau=0$ and $H=0$ can be chosen to correspond to $t=0$ by proper time translations. In Eq. (4), the subscripts 0 indicate unrenormalized bare variables. The possible initial slip [16] has been neglected since the transition seems independent of the initial condition when we start with a sufficiently ordered state. By exploiting the fact that the bare quantities are independent of μ and expanding the averaged order parameter $m=\langle\varphi\rangle=G_{10}$ in a Taylor's series in τ or H at every definite time instant, the RG equations in the two driven cases are then

$$\left(\mu\partial_\mu + \varsigma\lambda\partial_\lambda + \beta(u)\partial_u + \gamma_{\varphi^2}\tau\partial_\tau + \frac{1}{2}\gamma\right)m = 0, \quad (5)$$

$$\left(\mu\partial_\mu + \varsigma\lambda\partial_\lambda + \beta(u)\partial_u + \frac{1}{2}\gamma H\partial_H + \frac{1}{2}\gamma\right)m = 0, \quad (6)$$

respectively, where ∂_i indicates partial derivative with respect to i, and the Wilson's functions are defined as derivatives at constant bare parameters

$$\varsigma = \mu\partial_\mu\lambda, \; \gamma_{\varphi^2} = \mu\partial_\mu\tau, \quad \gamma = \mu\partial_\mu Z_\varphi, \quad \beta(u) = \mu\partial_\mu u. \quad (7)$$

At the infrared stable fixed point $u=u^*$, $\beta(u^*)=0$, the usual scaling form can be found, respectively, from Eqs. (5) and (6) and dimension analyses to be

$$m(t,\lambda,\tau) = \rho^{\beta/\nu}f_T(\lambda t^2\rho^z, \tau\rho^{-1/\nu}), \quad (8)$$

$$m(t,\lambda,H) = \rho^{\beta/\nu}f_H(\lambda t^2\rho^z, H\rho^{-\beta\delta/\nu}) \quad (9)$$

with the critical exponents given by

$$\eta = \gamma^*, \quad \nu^{-1} = 2 - \gamma^*_{\varphi^2}, \quad \beta = \nu(d + \eta - 2)/2,$$

$$\delta = (d - \eta + 2)/(d + \eta - 2), \quad z = 2 + \varsigma^*, \quad (10)$$

where f's are scaling functions, and ρ is a running variable.

To complete the theory, we now follow the driving by substituting $\tau=Rt$ ($H=Rt$) in Eq. (8) [Eq. (9)]. This gives, for the two driven cases, respectively,

$$R' = R\rho^{-r_T}, \quad \text{with } r_T = z + 1/\nu, \quad (11)$$

$$R' = R\rho^{-r_H}, \quad \text{with } r_H = z + \beta\delta/\nu \quad (12)$$

for R at the scale ρ. Choosing freely R and τ (H) as independent variables and setting $\rho=R^{1/r}$, one obtains from Eqs. (8) and (11) [Eqs. (9) and (12)],

$$m(\tau,R) = R^{\beta/\nu r_T}f'_T(\tau R^{-1/\nu r_T}), \quad (13)$$

$$m(H,R) = R^{\beta/\nu r_H}f'_H(HR^{-\beta\delta/\nu r_H}) \quad (14)$$

for the two cases, where f's are also scaling functions.

The nonequilibrium driving imposes an external time scale to the relaxation of the system. As a result, equilibration is intervened and nonequilibrium hysteresis ensues. The latter can readily be characterized by finding the value of T and H at which $m=0$. Equations (13) and (14) then dictate that the hysteresis scales with R with hysteresis exponents

TABLE I. Nonequilibrium dynamic critical exponents.

d	r_T	r_H	n_T	n_H	$\beta/\nu r_T$	$\beta/\nu r_H$	n'_T	n'_H
2	3.1665(12)	4.0415(12)	0.3158(1)	0.4639(1)	0.03947(1)	0.03093(1)	0.3553(1)	0.4948(1)
3	3.623(10)	4.518(13)	0.4380(15)	0.5493(12)	0.1430(3)	0.1147(3)	0.5811(18)	0.6640(15)
4(MF)	4	5	1/2	3/5	1/4	1/5	3/4	4/5

$n_T=1/(1+\nu z)=1/\nu r_T$ and $n_H=\beta\delta/(\beta\delta+\nu z)=\beta\delta/\nu r_H$ for the temperature driving and field driving, respectively.

As we perform the renormalization at the critical point, all the Z factors are identical to the usual critical dynamics in which dynamics decouples further from statics [20] because of the dimension regulations and minimal subtraction scheme used. In particular, $Z_\lambda=1$ and thus the fluctuation-dissipation theorem (FDT) is satisfied automatically within the present theory, which relies on the validity of the expansion and is applicable to the cases in which the driving does not introduce new divergences that need to be remedied by new Z factors and in which the inverse time rate of the driving (R^{-1} in the present case) is sufficiently longer than the relaxation times to far-off-critical states so that the initial condition appears to have no effect as has been pointed out above. This is in contrast to the recent study of the aging phenomena where FDT is generically violated [19]. Therefore, all the static critical exponents and the dynamic critical exponent z determined from Eq. (10) are identical to those of the usual Model A. All static critical exponents of the φ^4 model have been determined by Borel summation of 6- to 7-loop ϵ expansion [21]. On the other hand, z has been calculated to three loops as $z=2+c\eta$ with $c=0.7261(1-0.1885\epsilon)$ [22], which gives $z=2.101$ and 2.022 for 2D and 3D, respectively. The former is only a little smaller than 2.1665(12) obtained from the variation and projection method [4], while the latter agrees well with $z=2.02(4)$ from MC simulations [23]. Using the exact 2D and four-dimensional (4D) [mean-field (MF)] critical exponents and $\beta=0.3265(3)$, $\delta=4.789(2)$, and $\nu=0.6301(4)$ for 3D [24], along with $z=2.1665(12)$, 2.036(11) obtained below, and 2 for 2D, 3D, and 4D, respectively, we collect the nonequilibrium exponents in Table I, where the errors are calculated from those cited.

Now we compare the theory with numerics to check the validity of the theory. First comes the temperature driving. Note that the last part of Eq. (11) is just another form of the scaling law, Eq. (5) in [10] found from dynamic MCRG. Also true is the scaling relation of the hysteresis exponent n_T, though there n_T was found from the peaks of correlation functions [10]. Moreover, upon replacing τ with $T-T_c$, the scaling form, Eq. (13), becomes identical to Eq. (11) in [10]. Furthermore, the numerical $r_T=3.17(14)$ and $n_T=0.325(23)$ for 2D in [10] conform well to those listed in Table I. Therefore, the present nonequilibrium theory for varying T across T_c agrees well with the dynamic MCRG analysis, and thus provides a theoretical basis for it.

We move now to the field driving case. In Table I, we

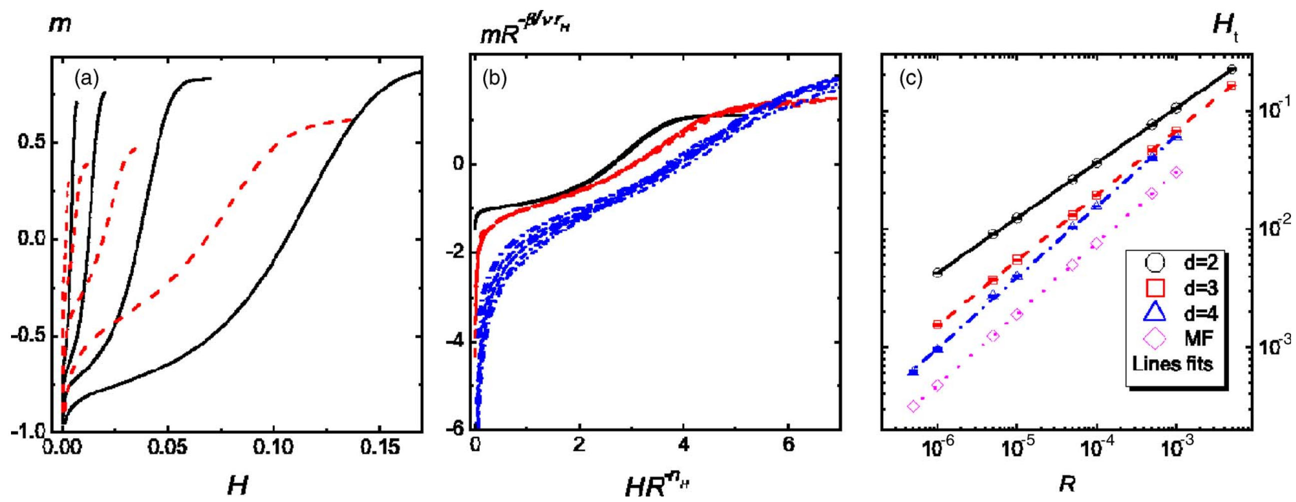

FIG. 1. (Color online) (a) Magnetization curves for the rate $R=10^{-6}$, 10^{-5}, 10^{-4}, and 10^{-3} (from left to right) for 2D (solid line) and 3D (dashed) at their respective T_c. (b) Rescaled magnetization curves, $mR^{-\beta/\nu r_H}$ vs HR^{-n_H}, for 2D, 3D, and 4D (dash dotted) (from upper left to lower right) and all those rates R displayed in (c). We used the n_H found from (c) but those $\beta/\nu r_H$ listed in Table I. Note that the scaling form, Eq. (14), is nevertheless quite good for 2D and 3D and also for 4D near $m=0$, confirming our theory, though it exhibits larger fluctuations for 4D possibly due to the logarithmic corrections at this upper critical dimension and its small size and the accompanying large fluctuations. (c) Rate dependence of the transition field H_t at $m=0$. Lines are fits to constant$\times R^{n_H}$. The error bars have been drawn but are generally smaller than the symbols' sizes. Each symbol is an average of 200 plus samples except for the mean-field (MF) ones that have no fluctuation. The lateral sizes of the lattices used are 256, 50, and 20, and the dimensionless $T_c=2/\ln(1+\sqrt{2})$, 1/.221 654 59, and 1/1.947 for 2D, 3D, and 4D [26], respectively. The line types are uniform for the three subfigures.

have also included two exponents n'_T and n'_H that represent the scaling of hysteresis loop areas A with respect to R. Since $A=\oint m dH$, one gets $n'_T=(1+\beta)/\nu r_T$ and $n'_H=\beta(1+\delta)/\nu r_H$ for the temperature and field driving, respectively. The MF exponents for $d\geqslant 4$ in Table I can also be attained by a simple scaling analysis to the MF dynamic equation [12] and can easily by verified by direct numerical solutions of the dynamic equation. Moreover, $n'_H=4/5$ has also been found for the scaling of hysteresis loop areas of a multidimensional laser at its threshold [25], the critical point. The hysteresis exponents n_H in Table I agree well with $n_H=0.4650(20)$, 0.5493(15), 0.599(12), and 0.5997(5) of our MC simulations of the field-driven critical Ising model in 2D, 3D, 4D, and MF solutions, respectively, as obtained from Fig. 1(c), confirming our theory. Moreover, the first two n_H thus obtained give back $z=2.158(17)$ and 2.036(11) for 2D and 3D, respectively. The former agrees well with previous results, and the latter improves the previous value 2.02(4) [23] and has been used to produce Table I.

Our n'_H and its associated scaling law disagree, however, with those found in [11], an extension of a previous RG analysis of the dynamic scaling of hysteresis [27] to the critical point. There are two apparent problems in that work that invalidate its results. First the claimed scale invariance of the k^2t factor in the only-one-loop correlation is only compatible with $z=2$, though the wave number k is to be integrated away. Second, its MF area exponent n'_H is not that at T_c but below it [6]! Thus its scaling relation cannot sustain even the MF exponents.

In summary, we have used time-dependent external probes to study critical dynamics. The probes drive the system out of equilibrium and lead to such nonlinear and nonequilibrium effects as hysteresis in the vicinity of the critical point. Nevertheless, we are able to develop a RG theory to tackle circuitously but analytically the probes and have derived systematically a series of nonequilibrium dynamic critical exponents and their scaling relations to characterize the time-dependent probes and the hysteresis. Their agreement with the numerical simulations, together with the exact MF results, makes in turn the theory effective and appeared accurate. In fact, we have improved the precision of the 3D dynamic critical exponent z via the hysteresis scaling to $z=2.036(11)$.

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