Quantum-classical transition of the open quartic oscillator: The role of the environment

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We investigate the classical and quantum dynamics of the open quartic oscillator model. Typically quantum behavior such as collapses and revivals (also squeezing) are induced by the nonlinearity of the model. We show that purely diffusive environments, as expected, attenuate such phenomena. We obtain analytical results in both regimes classical and quantum and discuss the effect of a diffusive reservoir in the two cases. We show that "separation times" as usually defined in the literature are strongly observable (and initial condition) dependent, rendering a solid definition of a unique classical limit rather difficult. In particular, the separation time for the variance $\langle \Delta\hat{x} \rangle$ can be smaller than that for the expectation value of the position $\langle \hat{x} \rangle$ of the centroid of the wave packet. We find a hierarchy of time scales which depends on the observable and the reservoir.

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I. INTRODUCTION

The transition from quantum to classical dynamics constitutes an unsolved problem which has raised permanent interest since the foundation of quantum mechanics. Although much attention has been paid to quantum systems with classically chaotic counterparts [1–10], nonlinearities present in integrable systems can be a major source of discrepancies between quantum and classical dynamical behaviors. Currently, these discrepancies are believed to disappear if the quantum system is subjected to the action of an environment. A famous example where environmental effects were found to destroy typical quantum dynamical characters is the experiment performed in Paris [11], where a coherent superposition of states (Schrödinger cat state) of a cavity field was observed gradually to evolve into a statistical mixture due to the interaction between the electromagnetic mode and the cavity walls. Also, interference patterns in multislit experiments involving fullerene molecules have been shown to be strongly affected by air molecules or by thermal emission of radiation [12].

The characterization of the quantum-classical transition is also a matter of debate: discussions of the classical limit are often based on Ehrenfest's theorem, which states that, under certain conditions, the centroid of a quantum wave packet will follow a classical trajectory up to what is called a "quantum break time" [1,13–19]. As discussed in Ref. [20], the conditions for the applicability of Ehrenfest's theorem are neither necessary nor sufficient to define the classical regime. In addition to that, it has been found that it is impossible to characterize quantum dynamics through a unique time scale such as the Ehrenfest time [21]. As we will show in the present contribution, if one considers the variance of the wave packet, we observe that the deviation between classical and quantum predictions occurs before the corresponding deviation for the centroid ("Ehrenfest time"). Although strictly speaking, Ehrenfest's theorem is only valid for classical trajectories, it has been argued that a more "fair" comparison between quantum and classical physics is given by the difference between the probability distributions of a quantum state and a classical ensemble constructed in such a way as to have the same initial marginal (position and momentum) distributions.

Of course, an important question in this context is in which sense the system becomes "classical," given the fundamentally different aspects of quantum and classical kinematics. The most sensible definition of a classical limit should be related to experimental precision [10]. We show that the typical quantum effect (revival) does not strictly vanish when the coupling to the reservoir is included. It is only the amplitude of the revivals that can become orders of magnitude smaller, in a way that it can never be measured for lack of experimental precision. Does that mean that the classical limit has been achieved? No, it simply means that, at that level of precision, quantum effects become unaccessible to any known measurement process. The same argument can be also used when one defines a classical limit in the present model as the limit in which the intensity of the coherent states considered is large compared with the unity. In this case, even without a reservoir, the revival will occur at later and later times [22]. It does not mean, however, that the reservoir is not necessary to achieve the classical limit, since the Wigner function can be completely different from the corresponding classical distribution function, even for short times, as it has been shown in Ref. [21]. Since quantum signatures in a state can develop at short times, it is possible, in principle, to measure them.

The present contribution aims at shedding some light on the quantum-classical transition in the quartic oscillator subjected to the action of a purely diffusive environment. In this model, the parameters of interest are chosen in such way that the nonlinearity constant is smaller than the natural frequency of the oscillator and both are greater than the effec-

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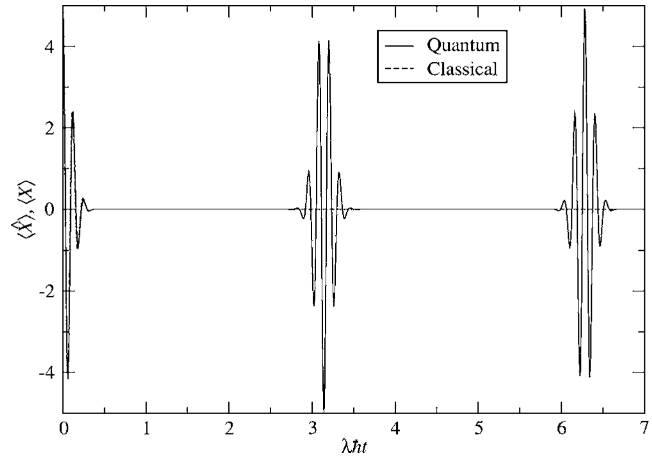

FIG. 1. Expectation value of position. Plot of the mean value of position for the quantum (solid line) and classical (dashed line) versions of the quartic oscillator without diffusion ($\kappa=0$). In the quantum mechanical case, the initial state is the coherent state $|\alpha_0\rangle\langle\alpha_0|$, where $\alpha_0=5$. The valid equivalent classical distribution is the Gaussian one centered at $\alpha=5$ in the complex phase space, and variance equal to unity. The quantities in the vertical axis are adimensional: $\langle\hat{X}\rangle=\mathrm{Re}\langle\hat{a}\rangle$ and $\langle X\rangle=\mathrm{Re}\langle\alpha\rangle$. Here, we make $\lambda\hbar/\omega=0.1$.

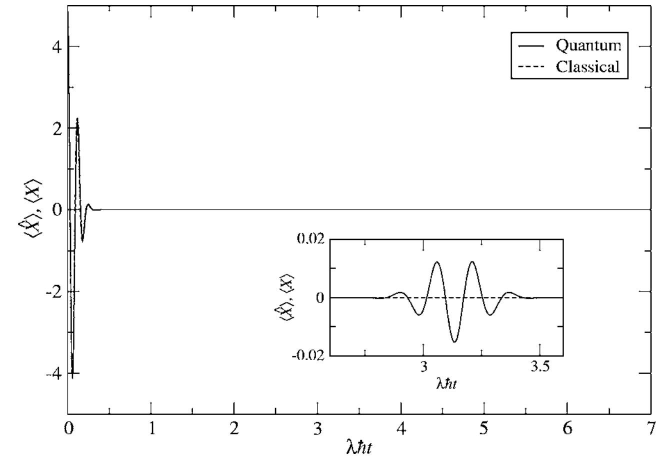

FIG. 2. Expectation value of position. Plot of the mean value of position for the quantum (solid line) and classical (dashed line) versions of the quartic oscillator with diffusion $[\kappa/(\lambda\hbar)=0.05]$. The initial conditions and the values of the other parameters are the same as Fig. 1. The inset graphic shows a "zoom" of the region of the first revival. Note that the revival is present, though damped.

tive diffusion constant. With this choice, the essentially quantum phenomenon (as reflected by the expectation value of the position operator) possesses a very different time scale clearly separated from the harmonic oscillator one. By virtue of the relative simplicity of the model, fully analytical results can be obtained allowing for a clear characterization of the quantum-classical transition and the sense in which quantum mechanics becomes classical. In the present contribution we consider four versions of the quartic model: quantum, quantum diffusive, classical, and classical diffusive. Analytical expressions are obtained for the states as a function of time as well as for the expectation value of the position operator and its variance. We show the existence of a quantum effect in a "pre-Ehrenfest" time scale connected to the variance time scale. Again the effect of the reservoir is to attenuate the difference.

II. THE MODEL

A. The quantum model with diffusion

The time evolution of the purely diffusive quantum quartic oscillator is given by the usual Lindblad master equation widely used in quantum optics [23] in the limit of zero dissipation constant k and infinite average number of thermal photons $\bar{n}$, such as the product $k\bar{n}=\kappa$ is considered constant. In this limit, the master equation is given by

$$\dot{\hat{\rho}}(t)=\frac{1}{i\hbar}[\hat{\rho},\hat{H}_0]+2\kappa[\hat{a}^\dagger\hat{\rho}(t)\hat{a}+\hat{a}\hat{\rho}(t)\hat{a}^\dagger-\hat{a}^\dagger\hat{a}\hat{\rho}(t)-\hat{\rho}(t)\hat{a}\hat{a}^\dagger], \tag{1}$$

where the density operator $\hat{\rho}(t)$ represents the state of the system, $\hat{a}$ and $\hat{a}^\dagger$ are creation and annihilation operators of the harmonic oscillator and κ is the effective diffusion constant. $\hat{H}_0$ is the Hamiltonian of the free quartic oscillator given by

$$\hat{H}_0=\hbar\,\omega\hat{a}^\dagger\hat{a}+\lambda\hbar^2(\hat{a}^\dagger)^2\hat{a}^2. \tag{2}$$

Here, ω is the natural frequency of the oscillator and $\lambda\hbar$ gives the strength of the nonlinearity. This model has been widely used by many authors [15,18,21–29] for several purposes.

The limit of applicability of this equation is the following $\omega\gg\lambda\hbar\gg\kappa$. To guarantee the validity of Eq. (1) in all numerical examples, we have chosen λ such that $\omega\langle\hat{a}^\dagger\hat{a}\rangle\gg\lambda\hbar\langle(\hat{a}^\dagger)^2\hat{a}^2\rangle$. If we start with a general initial density operator such as $\hat{\rho}(0)=\Sigma_{m,n}\rho_{mn}|m\rangle\langle n|$, where $|m\rangle$ is a Fock state, after a tedious calculation, we have

$$\begin{aligned}\hat{\rho}(t)=\sum_l\sum_j\sum_{m,n}\rho_{m+j,n+j}&\frac{\sqrt{(m+j)!\,(n+j)!\,(m+l)!\,(n+l)!}}{m!\,n!\,l!\,j!}\\&\times\gamma_{m-n}^{j+l}(t)\zeta_{m-n}^{m+n+1}(t)\exp[-i(m-n)(\omega-\lambda\hbar)t]\\&\times|m+l\rangle\langle n+l|,\end{aligned} \tag{3}$$

where

$$\gamma_n(t)=\frac{2\kappa\sinh(\Delta_n t)}{\Delta_n\cosh(\Delta_n t)+\Lambda_n\sinh(\Delta_n t)},$$

$$\zeta_n(t)=\frac{\Delta_n}{\Delta_n\cosh(\Delta_n t)+\Lambda_n\sinh(\Delta_n t)},$$

and $\Delta_n=\sqrt{\Lambda_n^2-4\kappa^2}$, $\Lambda_n=i\lambda\hbar n+2\kappa$.

In order to compare with the classical Liouville evolution, we evaluate the expectation value of the destruction operator

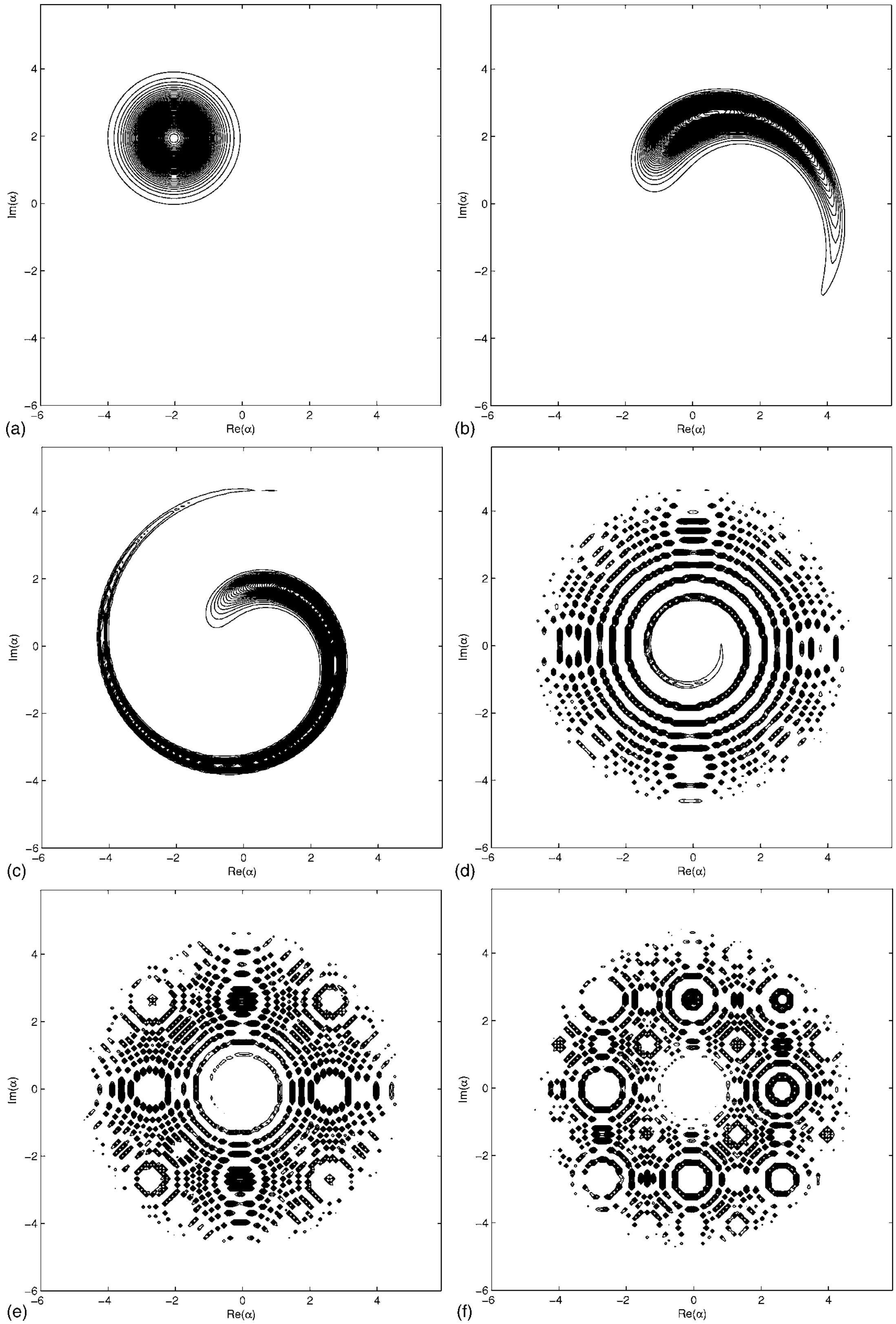

FIG. 3. Evolution of the classical probability distribution $\sigma(\alpha,\alpha^*,t)$ in the complex phase space for the quartic oscillator without diffusion ($\kappa=0$). (a) shows the contour of the initial Gaussian distribution centered in $\alpha=-2+2i$. The rest of the plots shows the evolved distribution at the following instants: (b) $t=\pi/(50\lambda\hbar)$, (c) $t=\pi/(20\lambda\hbar)$, (d) $t=\pi/(2\lambda\hbar)$, (e) $t=\pi/(\lambda\hbar)$, and (f) $t=2\pi/(\lambda\hbar)$.

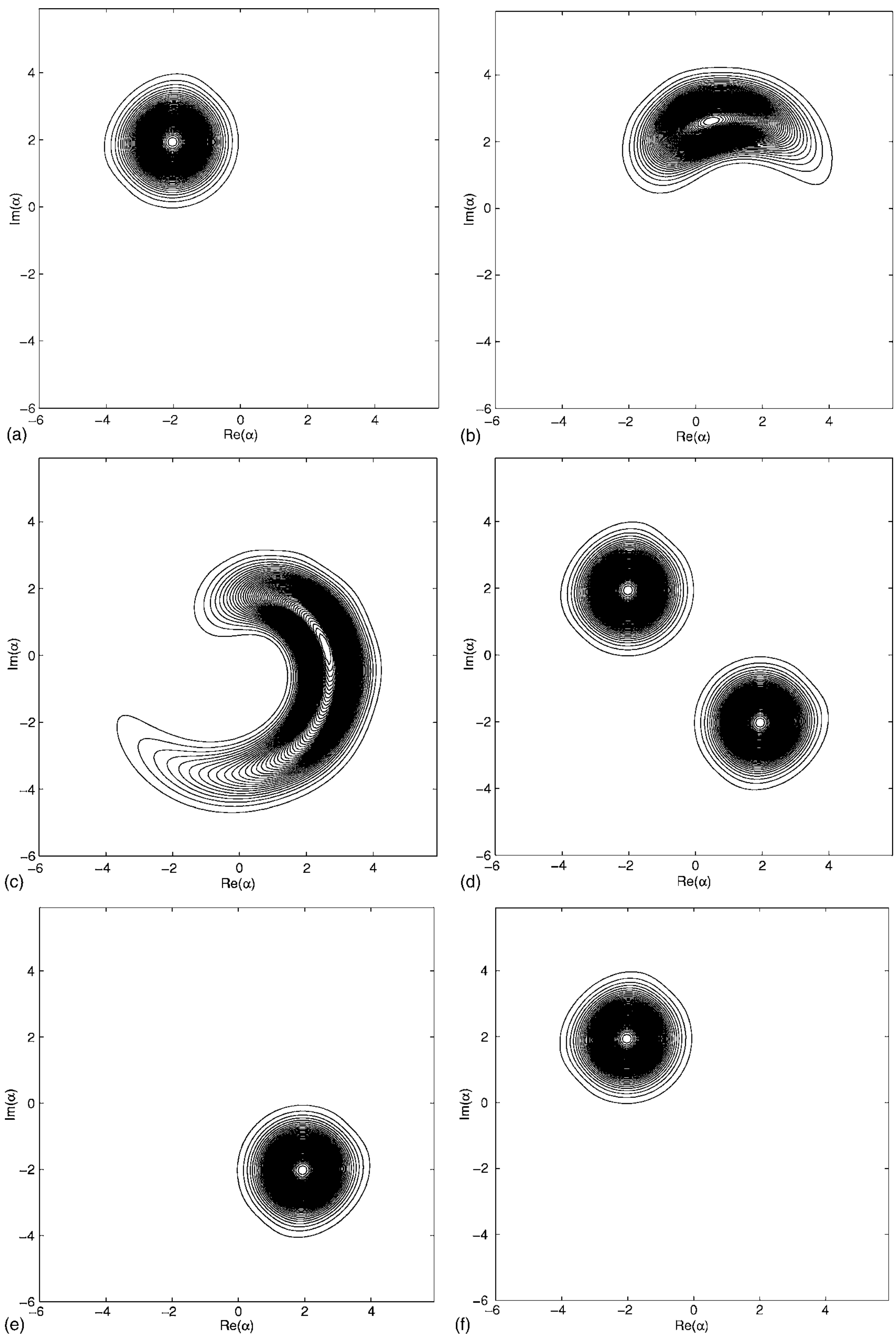

FIG. 4. Evolution of the Husimi function $Q(\alpha,\alpha^*,t)$ in the complex phase space for the quartic oscillator without diffusion ($\kappa=0$). (a) shows the contour of the Husimi function associated to the initial coherent state $|\alpha_0\rangle\langle\alpha_0|$, where $\alpha_0=-2+2i$. The rest of the plots shows the Husimi distribution of the evolved state at the following instants: (b) $t=\pi/(50\lambda\hbar)$, (c) $t=\pi/(20\lambda\hbar)$, (d) $t=\pi/(2\lambda\hbar)$, (e) $t=\pi/(\lambda\hbar)$, and (f) $t=2\pi/(\lambda\hbar)$. In (d), the state corresponds to a coherent superposition of the two bumps shown in the plot. In (e), there is an antirevival, i.e., the state is a coherent state with amplitude equal to $-\alpha_0$. In (f), there is a revival, i.e., the initial state is recovered.

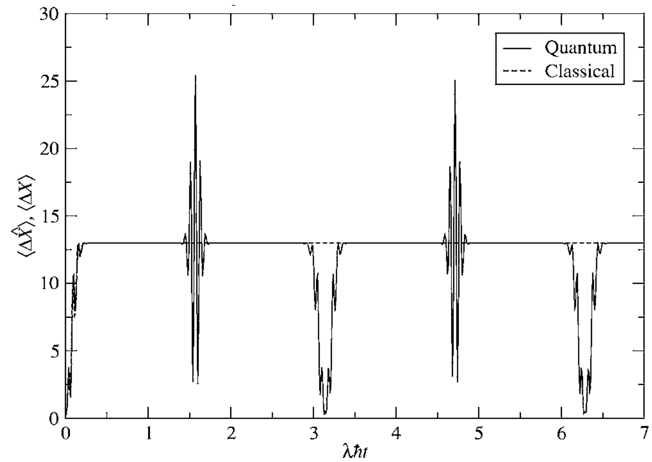

FIG. 5. Variance of position. Plot of the variances of the adimensional positions $\langle \hat{X}\rangle$ and $\langle X\rangle$ for the quantum (solid line) and classical (dashed line) versions of the quartic oscillator without diffusion ($\kappa=0$). The initial conditions and the values of the other parameters are the same as Fig. 1. Note the squeezing in the quantum mechanical case in the instants $\pi/(\lambda\hbar)$ and $2\pi/(\lambda\hbar)$.

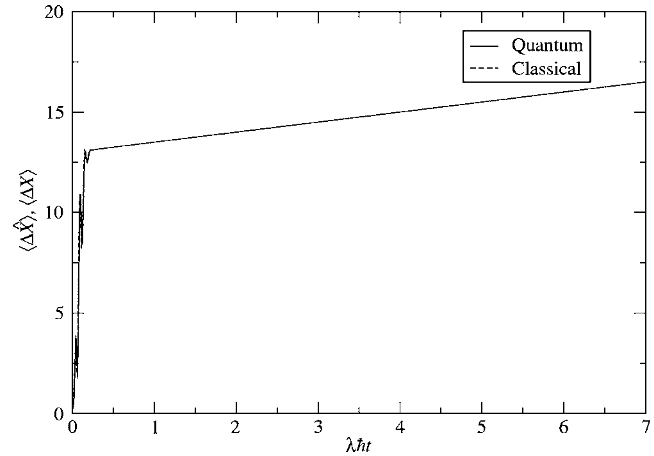

FIG. 6. Variance of position. Plot of the variances of the adimensional positions $\langle \hat{X}\rangle$ and $\langle X\rangle$ for the quantum (solid line) and classical (dashed line) versions of the quartic oscillator with diffusion $[\kappa/(\lambda\hbar)=0.05]$. The initial conditions and the values of the other parameters are the same as Fig. 1.

$$\langle \hat{a}(t)\rangle = \sum_l \sum_j \sum_n \rho_{n+j+1,n+j} \frac{(n+l+1)!\,\sqrt{(n+j+1)!\,(n+j)!}}{(n+1)!\,n!\,l!\,j!}$$

$$\times \gamma_1^{j+l}(t)\zeta_1^{2n+2}(t)\exp[-i(\omega-\lambda\hbar)t].$$

For an initial coherent state $\hat{\rho}(0)=|\alpha\rangle\langle\alpha|$, we have

$$\langle \hat{a}(t)\rangle = \alpha\left[\frac{\zeta_1(t)}{1-\gamma_1(t)}\right]^2 \exp\left\{-|\alpha|^2\left[1-\gamma_1(t)-\frac{\zeta_1^2(t)}{1-\gamma_1(t)}\right] - i(\omega-\lambda\hbar)t\right\}. \tag{4}$$

In order to calculate the variance we will need

$$\langle \hat{a}^2(t)\rangle = \alpha^2\left[\frac{\zeta_2(t)}{1-\gamma_2(t)}\right]^3 \exp\left\{-|\alpha|^2\left[1-\gamma_2(t)-\frac{\zeta_2^2(t)}{1-\gamma_2(t)}\right] - 2i(\omega-\lambda\hbar)t\right\}. \tag{5}$$

B. Classical diffusive model

The dynamics of the classical version of the diffusive quartic oscillator is governed by the master equation

$$\frac{d}{dt}\sigma = 2\kappa\frac{\partial^2\sigma}{\partial\alpha^*\,\partial\alpha} - \{H_0,\sigma\}, \tag{6}$$

where $\sigma=\sigma(\alpha,\alpha^*,t)$ represents the classical probability distribution function in the phase space, and $H_0=\hbar\omega|\alpha|^2+\lambda\hbar^2|\alpha|^4$ is the Hamiltonian of the free system. Here, $\{\cdot,\cdot\}$ stands for the Poisson brackets, i.e.,

$$\{f,g\} = \frac{1}{i\hbar}\left[\frac{\partial f}{\partial\alpha}\frac{\partial g}{\partial\alpha^*} - \frac{\partial g}{\partial\alpha}\frac{\partial f}{\partial\alpha^*}\right].$$

The adimensional complex variable α is related with the phase space coordinates q and p by the following expression:

$$\alpha = q\sqrt{\frac{M\omega}{2\hbar}} + ip\sqrt{\frac{1}{2M\omega\hbar}},$$

where M is the mass of the oscillator.

For an initial Gaussian state $\sigma(\alpha,\alpha^*,0)=(2\pi)^{-1}\exp(-|\alpha_0-\alpha|^2)$, one can easily verify that the solution of Eq. (6) is

$$\sigma(\alpha,\alpha^*,t) = \frac{1}{2\pi}\frac{1}{2\kappa t+1}\exp\left[-\frac{|\alpha(t)-\alpha|^2}{2\kappa t+1}\right], \tag{7}$$

where $\alpha(t)$ is the solution of the equation

$$\frac{d}{dt}\alpha = -i(\omega+2\lambda\hbar|\alpha|^2)\alpha,$$

with initial condition $\alpha(0)=\alpha_0$. Thus the expectation value of α is given by

$$\langle\alpha(t)\rangle = e^{-i\omega t}\sqrt{\frac{1}{2}}\left[\frac{q+ip}{A(t)^2}\right]$$

where we have used $M\omega=1$ and $A(t)=i\lambda\hbar t+(2\kappa t+1)^{-1}$. The expectation value of x^2 reads

$$\langle x^2(t)\rangle = \frac{1}{2}\left[q^2+p^2+\frac{1}{2\kappa t+1}+\mathrm{Re}(I_{\alpha^2})\right],$$

where

$$I_{\alpha^2} = \frac{q^2-p^2}{(2\kappa t+1)^3A(t)^2}\exp\left\{\frac{q^2+p^2}{(2\kappa t+1)^2A(t)} - \frac{(q^2+p^2)}{2\kappa t+1}\right\}.$$

At this point some comments are in order: as pointed out by Ballentine and collaborators [20,30,31], and mentioned above, the classical limit of a quantum state is an ensemble

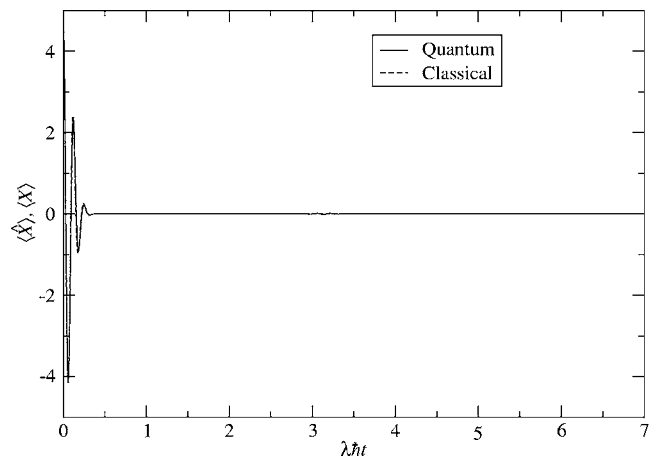

FIG. 7. Expectation value of position. Plot of the mean value of position for the quantum (solid line) and classical (dashed line) versions of the quartic oscillator with diffusion $[\kappa/(\lambda\hbar)=0.002]$. In the quantum mechanical case, the initial state is the coherent state $|\alpha_0\rangle\langle\alpha_0|$, where $\alpha_0=5$. The valid equivalent classical distribution is the gaussian one centered in $\alpha=5$ in the complex phase space, and variance equal to unity. The quantities in the vertical axis are adimensional: $\langle\hat{X}\rangle=\mathrm{Re}\langle\hat{a}\rangle$ and $\langle X\rangle=\mathrm{Re}\langle\alpha\rangle$. Here, we make $\lambda\hbar/\omega=0.1$.

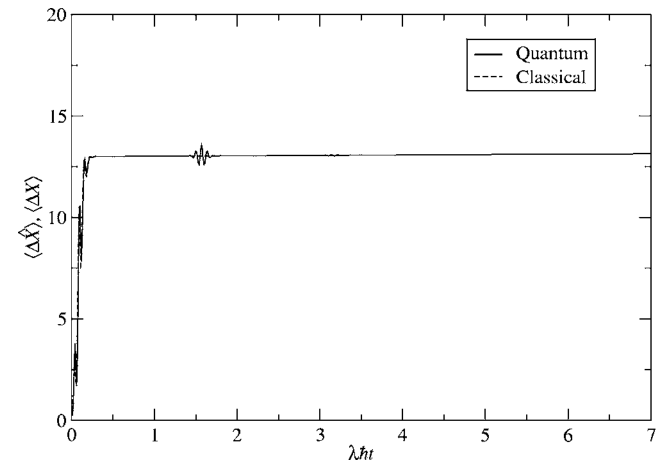

FIG. 8. Variance of position. Plot of the variances of the adimensional positions $\langle\hat{X}\rangle$ and $\langle X\rangle$ for the quantum (solid line) and classical (dashed line) versions of the quartic oscillator with diffusion $[\kappa/(\lambda\hbar)=0.002]$. The initial conditions and the values of the other parameters are the same of Fig. 7.

of classical orbits, not a single classical orbit. This is, in fact, the classical limit we should get for an open system. The increasing of the average number of excitations in the quantum regime is given by

$$\frac{\langle\hat{a}^\dagger\hat{a}\rangle(t)}{\langle\hat{a}^\dagger\hat{a}\rangle(0)} = 1 + \frac{2\kappa t}{\langle\hat{a}^\dagger\hat{a}\rangle(0)}. \tag{8}$$

Thus it is possible to choose $\kappa[\langle\hat{a}^\dagger\hat{a}\rangle(0)]^{-1}$ negligible in the classical domain. Otherwise, quantum effects prevail. This deviation characterizes the mesoscopic regime.

III. RESULTS

We start by showing the expectation value of the adimensional position operator $\hat{X}=\hat{a}+\hat{a}^\dagger$ and its classical counterpart $X=\alpha+\alpha^*$ in the limit where no diffusion is included. Both dynamics start with coherent states. As mentioned in the introduction, the constants of the model are chosen as $\omega \gg \lambda\hbar \gg \kappa$, which is also necessary if one derives the quantum model from first principles. Figure 1 shows the quantum result and its classical counterpart. Clearly we note the revival phenomenon in the quantum unitary version, absent from its classical counterpart.

When the environment is included (see Fig. 2) the revival seems to be absent, in the time scale we use. An instructive way to understand what happens is by looking at the purity loss of the quantum system, that can be conveniently measured by the linear entropy or idempotency defect [32], defined as

$$\delta(t) = 1 - \mathrm{tr}[\hat{\rho}^2(t)]. \tag{9}$$

The linear entropy is a direct indicative of the influence of the environment on the dynamics of the oscillator. In the limit where diffusion is absent, when $t=\pi/(2\lambda\hbar)$ both classical and quantum expectation values are close to zero, but the quantum one results from a nontrivial negative interference phenomenon, completely absent from the classical evolution. At this time, the wave function becomes a superposition of two coherent states (Schrödinger cat state). Figures 3 and 4 show the Husimi function and its classical analog for $\kappa=0$. It is clear that the quantum probability distribution is very different from the classical one even for short times.

For short times, the rate of increasing of the linear entropy is given by

$$\dot{\delta}(t) = 4\kappa + 8\kappa|\alpha|^2\left(1 - \exp\{-|\alpha|^2[1-\cos(2\lambda\hbar t)]\}\right). \tag{10}$$

At $t=\pi/(2\lambda\hbar)$ the quantum system is being most dramatically affected by the environment, since the diffusion attenuates quantum effects and the state evolves into statistical mixture. Moreover, the larger the $|\alpha|$, the smaller κ which is needed to destroy interference patterns generated by nonlinearity. This is intuitive, and the dependence of the linear entropy on a classical parameter has been found in several different situations [2,33–40].

Let us next take a closer look at the time where the revival is expected in the absence of the environment. The inset graphic in Fig. 2 shows the result. Note that a revival is clearly to be seen and there quantum mechanics is perfectly at work. Note also the very small amplitude in the figure (vertical axis). The role of the reservoir is to attenuate the quantum behavior, *not* to destroy it as has often been proposed in the literature. From this standpoint, the quantum-classical transition is a matter of measurement precision. It is achieved for all practical purposes, when quantum and classical results "coincide" for the observables and initial state in question. Figure 5 shows the quantum variance $\langle\Delta\hat{X}\rangle$ without diffusion and its classical analog. We clearly note a squeezing in the quantum case. For very short times both expectation values coincide. As we include the diffusion, the squeezing effect is also suppressed in the same sense as the revival in the expectation value (see Fig. 6). Comparing Figs. 1 and 5, we clearly see that the separation time for the variance is completely different from corresponding separation time for the expectation value of the position or the momentum.

As a result of the previous analysis, we conclude that there is a hierarchy of time scales that includes separation times for the expectation values and the decoherence characteristic time. In fact, for this model and considering an initial coherent state, the separation time for the mean value of position is $\tau_{\langle X\rangle}=\pi/(\lambda\hbar)$, and the separation time for the variance is $\tau_{\langle\Delta X\rangle}=\pi/(2\lambda\hbar)$, which is smaller than the first by a factor of 2. Finally, the shortest of these characteristic times is the decoherence time scale $\tau_{\rm dec}=(8\kappa|\alpha|^2)^{-1}$, $\kappa|\alpha|^2\gg 1$. The "classical limit" is attained when $\tau_{\rm dec}\ll\tau_{\Delta x}<\tau_{\langle x\rangle}$. Of course, it is possible to choose values of $|\alpha|$ and κ in a way that quantum effects cannot be detected by measuring the expectation value of position or momentum, but can be detected if a measurement of the variance is performed. In Figs. 7 and 8 we have an example of this situation. While the revival was suppressed, as shown in Fig. 7, Fig. 8 clearly exhibits squeezing in the same scale.

In summary, we have investigated time scales for the departure of the expectation values of several quantum observables and their classical counterpart. A diffusive environment was introduced and it shown to attenuate all typically quantum effects. Completely analytical expressions for both quantum and classical cases are explicitly given. From the present results we conclude that characteristic times for the deviation of quantum and classical predictions for the expectation value of observables is strongly observable and initially condition dependent. Also, as discussed previously, the notion of a classical limit relies strongly on comparison between expectation values of observables, which reflect only partially the real states (quantum or classical). As we have explicitly shown for the case of the revivals strictly speaking, quantum mechanics is always present, inspite of the fact that environmental effects tend to attenuate its typical behavior to a degree which is incompatible with experimental observation.

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