Fluctuation-dissipation relations in driven dissipative systems

Yair Shokef,* Guy Bunin, and Dov Levine

Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel

(Received 16 November 2005; revised manuscript received 6 March 2006; published 25 April 2006)

Exact theoretical results for the violation of time-dependent fluctuation-dissipation relations in driven dissipative systems are presented. The ratio of the correlation to delayed response in the stochastic model introduced in [Phys. Rev. Lett. **93**, 240601 (2004)] is shown to depend on measurement time. The fluctuation temperature defined by this ratio differs both from the temperature of the environment performing the driving, and from other effective temperatures of the system, such as the average energy (or "granular temperature"). General explanations are given for the time independence of the fluctuation temperature for simple measurements or long measurement times.

The fluctuation-dissipation relation (FDR) [1] provides a fundamental connection between two small deviations from thermodynamic equilibrium—the autocorrelation function of an observable, and the response function of that observable to changes in its conjugate field. In equilibrium, there are two aspects to the FDR: (i) these functions have the same spatial and temporal (or frequency) dependence, and (ii) for every observable their ratio equals the system's temperature.

Away from equilibrium, there is no such theorem connecting correlation and response. This having been said, there is much theoretical [2–13] and experimental [14,15] interest as to what is the relation between these quantities. In particular, one may ask whether either aspect (i) or (ii) holds for small deviations from nonequilibrium steady states (NESS). Aspect (ii) has been tested extensively for low frequency measurements in various systems, and the effective temperature T_F, defined by the ratio of the correlation to response, clearly differs from the environment temperature [2,3,14]. On the other hand, numerical simulations indicate that T_F does coincide [16] in some cases with other effective temperatures defined for glasses [4], sheared dissipative foams [5], dense granular packings [6], and driven granular gases [7].

Regarding aspect (i), different frequency dependences of the correlation and response functions have been observed in simulations of glasses [2], sheared foam [8], and models of driven systems [9], as well as in bacterial bath experiments [17]. In all these cases, fast modes thermalize, while slower modes manifest a higher (usually frequency-independent) temperature reflecting driving or system history (e.g., temperature quench). Experiments on vibrated granular systems [15] have shown a weak dependence of T_F on measurement frequency, which may not obviously be attributed to thermalization of fast modes with the environment.

This work aims at understanding the relation between time- (or frequency-) dependent correlations and response in NESS. To our knowledge, these have not previously been calculated exactly for any driven dissipative system. Here we calculate the autocorrelation and response functions in the context of a simple stochastic model of a dissipative system. The model introduced in [13] is constructed about the essential features of any driven dissipative system: energy loss upon interaction between particles (or modes) and energy input through an external driving mechanism. As such, we have reason to hope that our results will be relevant to a broad class of driven dissipative systems [18].

Our main results are: (1) There are certain observables for which the correlation and response have identical temporal dependence, however, (2) for a general measurement, they have different temporal dependences. This notwithstanding, (3) for asymptotically long times, the time dependence of the correlation and response coincides. Spatial FDRs in our model are investigated in [19].

The model consists of N particles, each with two degrees of freedom (DOF), one "kinetic," e_i, and one "internal," x_i. The energy of particle i is $e_i - x_i F_i$, where F_i is an external field on particle i, which may depend on time. The $\{x_i\}$ may be thought of as positions or any other coordinates used to measure FDR. In our notation we emphasize the difference between *microscopic* DOF's $\{e_i\}$ and $\{x_i\}$ and *macroscopic* quantities $E \equiv \Sigma_{i=1}^N e_i$ and $X \equiv \Sigma_{i=1}^N x_i$. The system dissipates energy through interactions and is maintained in a steady state by being coupled to a thermal bath.

In every interaction two DOF's are randomly chosen and their energy is stochastically redistributed between them. In interactions between two "kinetic" DOF's ($e-e$) there is dissipation: only a fraction $0 \leq \alpha \leq 1$ (α is like a restitution coefficient) of the energy is conserved and the remaining is dissipated out of the system. Interactions between a "kinetic" and an "internal" DOF ($e-x$), and between a "kinetic" DOF and a DOF from the bath ($e-e_B$) conserve energy. When all interactions are elastic ($\alpha = 1$), detailed balance holds and the system reaches thermodynamic equilibrium with the bath. Rates of interactions are determined by the per-particle interaction rate Γ, and the dimensionless coupling strengths $0 \leq (f,h) \leq 1$ of the "kinetic" DOF's to the bath and to the "internal" DOF's, respectively.

The resulting stochastic equation of motion for particle i reads,

_____
*Formerly: Srebro

$$e_i(t+dt) = \begin{cases} \underline{\text{Value:}} & \underline{\text{Probability:}} \\ e_i(t) & 1 - \Gamma\, dt \\ z\alpha[e_i(t) + e_k(t)] & (1-f)\Gamma\, dt \\ z[e_i(t) + e_B] & f(1-h)\Gamma\, dt \\ z[e_i(t) - x_j(t)F_j(t)] & fh\Gamma\, dt, \end{cases} \quad \text{(1a)}$$

$$x_i(t+dt) = \begin{cases} \underline{\text{Value:}} & \underline{\text{Probability:}} \\ x_i(t) & 1 - fh\Gamma\, dt \\ z[x_i(t) - e_j(t)/F_i(t)] & fh\Gamma\, dt, \end{cases} \quad \text{(1b)}$$

where $j,k \in \{1,\dots,N\}$ are indices of particles with which particle i may interact, chosen randomly at every interaction (we do not allow $e_i - e_i$ interactions, therefore $k \neq i$); $z \in [0,1]$ is the fraction of repartitioned energy given to particle i in the interaction, chosen randomly from a uniform distribution at every interaction; e_B is the energy of a bath DOF chosen randomly at every interaction from the equilibrium distribution $p_B(e_B) = T_B^{-1} e^{-e_B/T_B}$ at the environment temperature T_B.

In the steady state we define *one-particle* and *N-particle* autocorrelation functions, $c(t) \equiv \langle x_i(t)x_i(0)\rangle - \langle x_i\rangle^2$ and $C(t) \equiv \langle X(t)X(0)\rangle - \langle X\rangle^2$, respectively, and corresponding delayed response functions $r(t) \equiv \partial\langle x_i(t)\rangle/\partial F_i(0)$ and $R(t) \equiv \partial\langle X(t)\rangle/\partial F(0)$. The responses $r(t)$ and $R(t)$ are to a sudden change at time $t=0$ of the field on the single particle i or of the uniform field on all particles, respectively. $\langle\ \rangle$ denotes the steady state ensemble average over possible states at the beginning of the measurement ($t=0$). In equilibrium ($\alpha=1$ or $f=1$), $c(t) = r(t)T_B$ for any x_i, and $C(t) = R(t)T_B$ for X [20]; we will be concerned with the nonequilibrium cases.

For one-particle FDR we consider a system evolving from a given state at $t=0$ until a later time t. We average Eq. (1b) over the stochasticity in the dynamics for this initial state, and denote this averaging by an overline,

$$\frac{2}{\Gamma}\frac{d\overline{x_i(t)}}{dt} = -fh\left(\overline{x_i(t)} + \frac{\overline{E(t)}}{NF_i(t)}\right). \quad \text{(2)}$$

By the central limit theorem, for large systems ($N \gg 1$) the relative fluctuation $(x_i - \langle x_i\rangle)/\langle x_i\rangle$ of x_i is much larger than the relative fluctuation $(E - \langle E\rangle)/\langle E\rangle$ of E, thus the steady state average $\langle E\rangle$ may be substituted for the instantaneous value $\overline{E(t)}$ in Eq. (2). For a constant field, noting that $\langle e\rangle \equiv \langle E\rangle/N = -\langle x_i\rangle F_i$ [21], this yields,

$$\overline{x_i(t)} = \langle x_i\rangle + [x_i(0) - \langle x_i\rangle]e^{-\gamma t}, \quad \text{(3)}$$

with $\gamma \equiv fh\Gamma/2$. The correlation is obtained by multiplying Eq. (3) by $x_i(0)$ and averaging over the steady state distribution for all possible initial ($t=0$) states of the system, which yields $c(t) = (\langle x_i^2\rangle - \langle x_i\rangle^2)e^{-\gamma t}$ [22].

The one-particle response is obtained by maintaining the system in a steady state with some external field, then, at time $t=0$, changing the field F_i acting on particle i from F_0 to F. We then follow $\overline{x_i(t)}$ and average over the system's steady state distribution at time $t=0$ to yield $\langle x_i(t)\rangle$. The steady state solution of Eq. (2) yields $-\langle x_i(0)\rangle F_0 = -\langle x_i\rangle F = \langle e\rangle$, thus

$\langle x_i(t)\rangle = -\langle e\rangle[F^{-1} + (F_0^{-1} - F^{-1})e^{-\gamma t}]$. After differentiating with respect to F_0 and taking the limit $F_0 \to F$, we get $r(t) = \langle e\rangle F^{-2}e^{-\gamma t}$.

We thus see that the one-particle correlation and response have the same temporal dependence and aspect (i) of the FDR exactly holds, with the time-independent fluctuation temperature [22],

$$T_F^1 \equiv \frac{c(t)}{r(t)} = T_G\frac{\langle x_i^2\rangle - \langle x\rangle^2}{\langle x\rangle^2}. \quad \text{(4)}$$

Aspect (ii), on the other hand, is violated [13], since T_F^1 generally differs both from the environment temperature T_B and from the granular temperature, defined as the average energy per DOF: $T_G \equiv \langle e\rangle$. Only in the equilibrium limits ($\alpha \to 1, f \to 1$) is the distribution of x_i the exponential Boltzmann distribution and $T_F^1 = T_G = T_B$.

The FDR is only proven for equilibrium, so one might expect the correlation and response to depend differently on time in a NESS. However, the correlation represents the system's return to steady state after deviating from it due to spontaneous fluctuations, and the response represents its return to steady state after being moved away from it by some external force. Both are governed by the same physical processes and thus generally posses the same time scales. In simple cases (as the one-particle measurement solved above) there is only one time scale and the correlation and response are not rich enough to have different time dependences. In order to observe violations of aspect (i) in the FDR we now consider a measurement on a macroscopic quantity. This possesses two time scales—just enough to exhibit a different temporal dependence for the correlation and response.

We assume uniform F and sum Eq. (2) over particles,

$$\frac{2}{\Gamma}\frac{d\overline{X(t)}}{dt} = -fh\left(\overline{X(t)} + \frac{\overline{E(t)}}{F(t)}\right). \quad \text{(5)}$$

Now, fluctuations in E are not negligible compared to fluctuations in X, so Eq. (5) must be solved in conjunction with the equation similarly derived from Eq. (1a),

$$\frac{2}{\Gamma}\frac{d\overline{E(t)}}{dt} = -fh\overline{X(t)}F(t) - A_1\overline{E(t)} + f(1-h)NT_B, \quad \text{(6)}$$

where $A_n \equiv n + (1-f)(1-2\alpha^n)$. The average evolution of the system from a state with $X(0)$ and $E(0)$ is hence given by the simultaneous solution of Eqs. (5) and (6):

$$\overline{X(t)} = \langle X\rangle + \sum_{\ell=1}^{2}\left[a_\ell[X(0) - \langle X\rangle] + b_\ell\frac{E(0) - \langle E\rangle}{F}\right]e^{-\gamma_\ell t}, \quad \text{(7)}$$

where we denote $k \equiv (A_1^2 - 2A_1 fh + 5f^2h^2)^{1/2}$, $a_1 \equiv (1 - A_1 + fh)/2k$, $a_2 \equiv (1 + A_1 - fh)/2k$, $b_1 \equiv fh/k$, $b_2 \equiv -fh/k$, $\gamma_1 \equiv (A_1 + fh + k)\Gamma/4$, and $\gamma_2 \equiv (A_1 + fh - k)\Gamma/4$, and the steady state values are given by $-\langle X\rangle F = \langle E\rangle = f(1-h)NT_B/(A_1 - fh)$.

To compute the correlation we multiply $\overline{X(t)}$ by $X(0)$ and average over the steady state distribution of the initial states [that is, over $X(0)$ and $E(0)$] [22]:

$$C(t) = \sum_{\ell=1}^{2} \left[a_\ell (\langle X^2 \rangle - \langle X \rangle^2) + b_\ell \frac{\langle XE \rangle - \langle X \rangle \langle E \rangle}{F} \right] e^{-\gamma_\ell t}. \quad (8)$$

For the response we change the field from F_0 to F at $t = 0$, and after averaging Eq. (7) over the initial states taken from the steady state corresponding to the field F_0, differentiating with respect to F_0 and taking the limit $F_0 \rightarrow F$ we have

$$R(t) = \frac{\langle E \rangle}{F^2} \sum_{\ell=1}^{2} a_\ell e^{-\gamma_\ell t}. \quad (9)$$

In this N-particle measurement $C(t)$ and $R(t)$ share the rates γ_1 and γ_2, however, with different prefactors, leading to violation of aspect (i) in the FDR [23]. If one insists on defining a fluctuation temperature $T_F^N(t) \equiv C(t)/R(t)$, it will depend on the measurement time, as can be seen in Fig. 1, which also demonstrates that the one-particle and N-particle measurements yield fluctuation temperatures differing both one from the other as well as from the granular temperature T_G and bath temperature T_B.

As the correlation and response generally share the same time scales, for measurement times longer than the maximal time scale of the system ($1/\gamma_2$ for the N-particle measurement in our model) this time scale dominates both correlation and response and aspect (i) of the FDR asymptotically holds for long times (see Fig. 1). It is interesting to speculate that this behavior may hold for general dissipative systems.

Although the FDR is generally not valid in NESS, it is hard to observe its violations in simulations or experiments on dissipative systems. One reason for this is that dissipative systems often have nearly Boltzmann distributions with some effective temperature, and deviations between different effective temperatures of the system (like T_F and T_G) are too small to be observed [10,13].

Another reason is that some measurements on some systems do not exhibit at all the FDR violations presented above. To demonstrate this we now show that in the inelastic Maxwell model [24] the frequency-dependent Kubo formula [25] holds exactly with an effective temperature coinciding with T_G. That is, the velocity autocorrelation $D(\omega) \equiv \int_0^\infty \langle v(0) v(t) \rangle e^{-i\omega t} dt$ relates to the mobility $\mu(\omega)$ by $D(\omega) = \mu(\omega) T_K$ with $T_K = T_G$ independent of frequency [26]. We shall first show that for our model T_K is frequency-independent and equals T_F^1, which generally differs from T_G, and subsequently show that for the Maxwell model $T_K = T_G$.

To make contact with the standard Kubo relations we will imagine the DOF's $\{x_i\}$ in our model as positions; thus the autocorrelation of the single-particle "velocity" $v_i \equiv dx_i/dt$ is obtained by twice differentiating $c(t)$ calculated above. The relation $c(t) = r(t) T_F^1$ with T_F^1 independent of t, as obtained for the one-particle FDR solved above, may be twice differentiated with respect to time and transformed to frequency domain. The Kubo formula then immediately follows with $T_K = T_F^1$. We have also verified this general result by explicitly calculating $D(\omega)$ and $\mu(\omega)$ in our model and obtained $D(\omega) = \mu(\omega) T_F^1 = (\langle x_i^2 \rangle - \langle x_i \rangle^2)/(\gamma + i\omega)$.

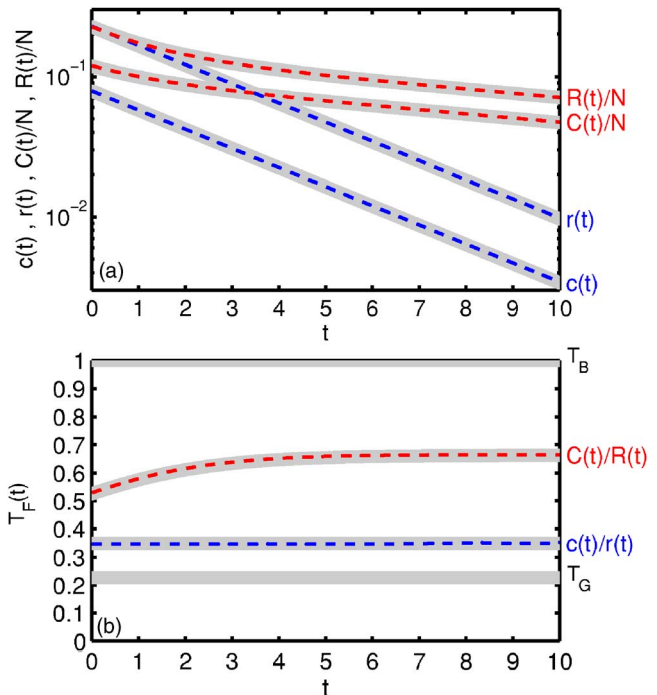Our method may now be applied to the one-dimensional

FIG. 1. (Color online) (a) Correlation and response functions and (b) resulting fluctuation temperatures vs. measurement time for the model with $\alpha = 0.6$, $f = 0.7$, and $h = 0.9$. Graphs are scaled to $T_B = 1$, $F = 1$, and $\Gamma = 1$. The resulting decay rates are $\gamma = 0.32$, $\gamma_1 = 0.72$, and $\gamma_2 = 0.07$. Numerical simulations with $N = 10^3$ (dashed lines) agree with theoretical results for $N \gg 1$ (thick gray lines). The bath temperature T_B and the granular temperature T_G are given for reference.

(1D) inelastic Maxwell model (the 2D and 3D versions follow trivially): During a finite time $dt > 0$, short compared to the collision rate Γ and to the coupling rate λ to the thermostat, the effect of the fluctuating force in the Langevin dynamics each particle undergoes is proportional to $\sqrt{dt}$ (see, e.g., [27]). Thus the velocity of particle i evolves as,

$$v_i(t+dt) = \begin{cases} \underline{\text{Value:}} & \quad \underline{\text{Probability:}} \\ (1 - \lambda dt) v_i(t) + \psi_i(t)\sqrt{dt} & \quad 1 - \Gamma \, dt \\ \dfrac{1-\alpha}{2} v_i(t) + \dfrac{1+\alpha}{2} v_j(t) & \quad \Gamma \, dt. \end{cases}$$

(10)

For stochastic thermostating, $\psi_i(t)$ is an uncorrelated random force with $\langle \psi^2 \rangle = 2\lambda T_B$, while for Gaussian thermostating, $\psi_i(t) = 0$ and $\lambda = (\alpha^2 - 1)\Gamma/4$. Averaging Eq. (10) yields $\langle v(0) v(t) \rangle = \langle v^2 \rangle e^{-\kappa t}$ with $\kappa \equiv \lambda + (1+\alpha)\Gamma/2$, thus $D(\omega) = \langle v^2 \rangle/(\kappa + i\omega)$. In the mobility measurement, a periodic acceleration $\xi(t) = \xi_0 e^{i\omega t}$ is added to a single particle, and from Eq. (10) we have

$$\frac{\overline{dv_i(t)}}{dt} = -\kappa \overline{v_i(t)} + \xi_0 e^{i\omega t}. \quad (11)$$

This has the steady solution $\langle v_i(t) \rangle = \xi_0 e^{i\omega t}/(\kappa + i\omega)$, thus $\mu(\omega) = 1/(\kappa + i\omega)$, and the Kubo formula holds with $T_K = T_G$.

The difference in T_K between our model and the Maxwell model derives from the fact that T_F and T_K measure fluctuations, expressed in second moments of DOF's. In the Maxwell model T_G is defined from $\langle v^2 \rangle$, which is the second moment of a DOF, while in our model $T_G \equiv \langle e \rangle$ is defined from the first moment of a DOF. Only in equilibrium do all moments of the exponential Boltzmann distribution yield a single energy scale and these measurements in both models give the actual temperature.

To our knowledge, we have presented the first exact results for time-dependent violations of FDR in driven dissipative systems by demonstrating that correlation and response functions have different temporal behaviors. This violation, like differences between different effective temperatures of a system, may be small, requiring sensitivity in simulations and experiments. We identified situations relevant to general driven dissipative systems where the correlation and response have the same time dependence: In sufficiently simple measurements the correlation and response share a single time scale and so have the same dependence on time, thus time-dependent FDR violations may be observed only in systems or measurements exhibiting multiple time scales. Finally, since the correlation and response typically share the same time scales, their temporal behavior for measurements with long waiting times asymptotically coincides.

---

[1] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).

[2] L. F. Cugliandolo, J. Kurchan, and L. Peliti, Phys. Rev. E **55**, 3898 (1997).

[3] P. C. Hohenberg and B. I. Shraiman, Physica D **37**, 109 (1989); K. Hayashi and S. I. Sasa, Phys. Rev. E **69**, 066119 (2004); T. Harada and S. I. Sasa, Phys. Rev. Lett. **95**, 130602 (2005).

[4] A. Barrat, J. Kurchan, V. Loreto, and M. Sellitto, Phys. Rev. Lett. **85**, 5034 (2000); Phys. Rev. E **63**, 051301 (2001); L. Berthier and J. L. Barrat, Phys. Rev. Lett. **89**, 095702 (2002).

[5] I. K. Ono, C. S. Ohern, D. J. Durian, S. A. Langer, A. J. Liu, and S. R. Nagel, Phys. Rev. Lett. **89**, 095703 (2002).

[6] H. A. Makse and J. Kurchan, Nature (London) **415**, 614 (2002); A. Barrat, V. Colizza, and V. Loreto, Phys. Rev. E **66**, 011310 (2002).

[7] A. Puglisi, A. Baldassarri, and V. Loreto, Phys. Rev. E **66**, 061305 (2002); A. Barrat, V. Loreto, and A. Puglisi, Physica A **334**, 513 (2004).

[8] C. S. O'Hern, A. J. Liu, and S. R. Nagel, Phys. Rev. Lett. **93**, 165702 (2004).

[9] T. Harada, e-print cond-mat/0310547; K. Hayashi and S. I. Sasa, Phys. Rev. E **71**, 046143 (2005).

[10] V. Garzó and J. Dufty, J. Stat. Phys. **105**, 723 (2001); V. Garzó, Physica A **343**, 105 (2004).

[11] S. Fielding and P. Sollich, Phys. Rev. Lett. **88**, 050603 (2002); A. Pérez-Madrid, D. Reguera, and J. M. Rubí, Physica A **329**, 357 (2003).

[12] A. Baldassarri A. Barrat, G. D'Anna, V. Loreto, P. Mayor, and A. Puglisi, J. Phys.: Condens. Matter **17**, S2405 (2005).

[13] Y. Srebro and D. Levine, Phys. Rev. Lett. **93**, 240601 (2004).

[14] X. L. Wu and A. Libchaber, Phys. Rev. Lett. **84**, 3017 (2000); T. Harada and K. Yoshikawa, Phys. Rev. E **69**, 031113 (2004).

[15] G. D'Anna *et al.*, Nature (London) **424**, 909 (2003).

[16] Dependence of T_F on observable was found in glasses [11], while kinetic theory approximations for granular gases [10] as well as exact solutions of a model system [13] showed T_F differs from the granular temperature.

[17] T. C. Lubensky (private communication).

[18] For example, our model qualitatively reproduces the energy distribution of driven granular gases [13].

[19] D. Levanony and D. Levine, e-print cond-mat/0601672.

[20] The more common FDR is obtained by time differentiating this [see e.g., D. Chandler, Introuction to Modern Statistical Mechanics (Oxford University Press, New York, 1987)].

[21] $\langle x \rangle$, $c(t)$, $r(t)$, $C(t)$, and $R(t)$ all diverge as $F \rightarrow 0$, however T_F and T_G are finite and independent of F [13], as is the case when x is height and F a gravitational field.

[22] The procedure used in [13] to calculate all moments of e_i has been used to obtain the steady state averaged second moments, which for $N \gg 1$ reduce to

$$\langle x_i^2 \rangle - \langle x \rangle^2 = f(1-h)T_B^2/F^2\{f^2(h^2+2h-1) + 2[f(1-3h)+A_1]$$
$$\times A_1 + f(1-h)(3-A_2)]/[(2A_2-fh)(A_1-fh)^2]\},$$

$$\langle XE \rangle - \langle X \rangle \langle E \rangle = -NT_B^2/Ff^2h(1-h)\{4A_1^2[3A_1+3+f(2-9h)$$
$$-A_2] + 2fA_1[15-27h-f(5-12h-3h^2)$$
$$-(5-9h)A_2] - f[18(A_2-1)(1-h)$$
$$+f(12-21h-3h^2) - f(A_2+f)(2-3h$$
$$-3h^2)]\}/[6(A_1-fh)^3(A_1+fh)(2A_2-fh)],$$

$$\langle X^2 \rangle - \langle X \rangle^2 = N[\langle x_i^2 \rangle - \langle x_i \rangle^2] - [\langle XE \rangle - \langle X \rangle \langle E \rangle]/F,$$

with $A_n \equiv n + (1-f)(1-2\alpha^n)$.

[23] This FDR violation is different in nature and origin from that observed in glasses [2] and driven systems [8,9,17], where fast modes thermalize while slower modes do not.

[24] E. Ben-Naim and P. L. Krapivsky, Phys. Rev. E **61**, R5 (2000).

[25] R. Kubo, M. Toda and N. Hashitsume, *Statistical Physics II* (Springer, Berlin, 1991).

[26] The Einstein relations, derived here from the Kubo formula with $\omega = 0$, were previously shown to hold for the 1D Maxwell model with a Gaussian thermostat [12].

[27] D. R. M. Williams and F. C. MacKintosh, Phys. Rev. E **54**, R9 (1996).