Domain motion in the voter model with noise

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We study the voter model with noise on one-dimensional chains using Monte Carlo simulations and finite-size scaling techniques. We observe that the system evolution toward consensus is deeply affected by the addition of noise, and that the time to reach complete ordering increases with the noise parameter q. In particular, the simulations show that the average domain size scales as $\xi \sim q^{-1/2}$ whereas the magnetization scales with the number of nodes as $m \sim N^{-1/2}$.

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I. INTRODUCTION

The voter model is a simple nonequilibrium model for spreading phenomena. It presents some interesting features for physicists such as being analytically solvable and is identical to the Glauber dynamics of the Ising model at zero temperature [1,2]. The voter model has also been used to study the phase separation or domain coarsening [3–5]. Coarsening patterns have a correlation length that grows with time as $L(t) \sim t^{1/z}$, where z is the dynamical exponent. This quantity should be considered as the analog of the domain size $\xi(t) \sim t^n$, with $n=1/z$ and $z=2$, when the dynamics is nonconservative.

In the voter model each voter is represented by a binary spin variable S_i, associated with the ith site of the lattice. It can have two possible values $S_i = \pm 1$ corresponding to the two opposite opinions such as, for instance, the vote in favor of one out of two candidates or the options *yes* and *no* in a referendum. During the update, a randomly selected spin assumes the state of one of its $2k$ neighbors, also chosen at random. Previous works on the voter model considered the spins either on regular d-dimensional lattices or on small-world networks interpolating between regular lattices and random graphs [6]. On regular lattices with $d=1$ and 2, the voter model converges to an ordered state with all spins in the same state. Since there are no links connecting nodes with opposite opinions, this is an absorbing state which can never be left once it is reached. In higher dimensions ($d > 2$) the system does not order in the thermodynamic limit [4,7]. The question of how the introduction of long-range interactions, such as the shortcuts present in small-world networks, affects the ordering process of the voter model was discussed by Castellano and co-workers [8,9]. For finite networks, they found that the system always converges to a state of complete order, whereas, in the limit of infinite-size networks, the system remains in a stationary state presenting domains with coexisting opinions. Very recently, a study of the voter model dynamics in complex networks provided an analysis of the role of the topology of the network on the ordering dynamics of the voter model [10,11].

In the present work we consider a modified version of the voter model, which we shall call the voter model with noise in analogy with the majority-vote model with noise [12–17]. Its dynamical evolution is the same as that of the standard voter model, but now the voter (spin) assumes the opposite opinion of the chosen neighbor spin with probability q and it assumes the same opinion with probability $1-q$. For a system with fixed size N, the probability of disagreement q, also called the noise parameter, is the only significant parameter. Our goal is to investigate how the introduction of noise affects the evolution of the voter model toward the absorbing state.

The ordering parameter commonly used in the study of the voter model is the density n_a of active bonds, that is, the fraction of bonds connecting sites with opposite opinions through which the dynamics takes place. One has $n_a=0$ for the absorbing state, whereas a finite value of n_a means that the system has domains of average size $\xi \sim 1/n_a$. Numerical simulations of the standard voter model on one-dimensional chains showed that the system evolves toward a configuration where the domains grow as $\xi \sim t^{1/2}$ [7]. In this work we perform Monte Carlo simulations to calculate the time evolution of n_a for chains with various number of nodes N and different values of the noise parameter q. Through a finite-size scaling analysis we determine the exponent governing the domain sizes dependence with the parameter q, and a scaling relation which allows a data collapse of the results for different lattice sizes and small values of q.

The remainder of the paper is organized in the following way: In Sec. II we describe the model used in the simulations. In Sec. III we present the numerical results along with a finite-size scaling analysis of the relevant quantities, and in Sec. IV we present our conclusions.

II. THE MODEL

We consider the voter model on a one-dimensional regular lattice composed of N nodes or sites with periodic bound-

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ary conditions. A spin variable $S_i = \pm 1$, denoting the opposite states *yes* and *no*, is associated with each site of the lattice. Moreover, in the most general case, each node interacts with $2k$ nearest neighbors, that is, each node is connected by a direct link with $2k$ other nodes. The case $k=1$ corresponds to a chain with nearest-neighbor interactions in which each node interacts only with one node on the left and one on the right.

To simulate the voter model with noise we start from a disordered configuration $\{S_i, i=1,\ldots,N\}$, with randomly distributed spins, and during the update a site chosen at random has its spin S_i made equal to one of its $2k$ nearest neighbors, also selected at random, with probability $1-q$. Therefore, the site or voter can disagree with probability q. The spin-flip rate $w(S_i) \equiv w(S_i \rightarrow -S_i)$ reads

$$w(S_i) = \frac{1}{2}[1 - (1-2q)S_i S_{i+\delta}], \qquad (1)$$

where the parameter q is the noise parameter defined in the interval [0, 1], and $S_{i+\delta}$ is one of the $2k$ spins connected to the spin S_i, that is, the set of nearest-neighboring sites of site i. This procedure is then repeated N times, such that each site in the chain is selected once on the average. After this, the Monte Carlo time is incremented by one unit.

Thus, we follow the temporal evolution of the fraction of active bonds, n_a, defined by the density of links connecting sites with opposite opinions. Another quantity monitored is the magnetization, m, which measures the system ordering state. For given values of N and q, averages over several samples are used to obtain the quoted values for n_a and m.

In what follows we present our results and discuss the dependence of the fraction of active bonds, n_a, and magnetization, m, on N and t, in the entire space of noise parameters q.

III. RESULTS AND DISCUSSION

First we present our results for the size and duration of domains, through the calculation of the fraction of active bonds n_a as function of time.

In Fig. 1 we plot the time dependence of n_a for $N=400$, $k=2$, and various values of the noise parameter q. The result for the voter model without noise ($q=0$) is also included for comparison. After a transient time, where there is an initial decrease, the domains remain with a constant size, on average, and the curves for n_a present a plateau. For fixed system size N, the duration of the plateau grows with the parameter q and the time needed for the system to reach the absorbing state increases as the noise increases. The approach to the absorbing state is exponential and it is due to a standard finite-size effect in the presence of the absorbing state. In the thermodynamic limit, when $N \rightarrow \infty$, the system will stay in everlasting activity, i.e., it remains indefinitely in the stationary state. This is better illustrated in Fig. 2 where we show the dependence of the width of the plateaus with system sizes N, for $N=400, 500, 600, 700$ and two values of q. We have also observed similar behaviors for the cases of $k=1$ and 3, and we will return to this point later.

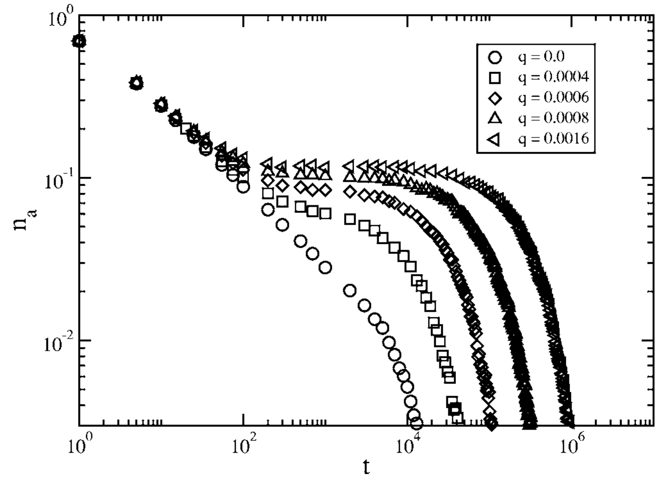

FIG. 1. Dependence of the fraction n_a of active bonds on time for $N=400$ and several values of the noise parameter q. We consider that each node is connected with $2k$ nearest neighbors. The data points were obtained from averages over 1000 samples and correspond to the case of $k=2$. A similar behavior was observed for $k=1$ and 3.

The results of Fig. 3, obtained from simulations on a much larger system with $N=2000$, show the dependence of the heights of the plateaus on the noise parameter q. Notice that even these small values of $q=0.0002$, 0.0004, and 0.0008 are enough to settle the system in a state with constant activity.

The dependence of the fraction of active bonds, n_a, on the time and noise parameter is better explored by considering the following approach. In regular networks the domain size $\xi \sim 1/n_a$ grows as $t^{1/2}$ and in small-world networks a behavior $\xi \sim 1/p$ has been observed [18], where p is the concentration of long-range connections (shortcuts) and ξ denotes the characteristic size or the average distance between two nodes. In the present case, we can infer how the characteristic size depends on the noise parameter q in a similar way.

In Fig. 4 we have plotted $\log_{10} \xi$ as a function of $\log_{10} q$, for a fixed value of N and $k=1$, 2, and 3. We clearly observe

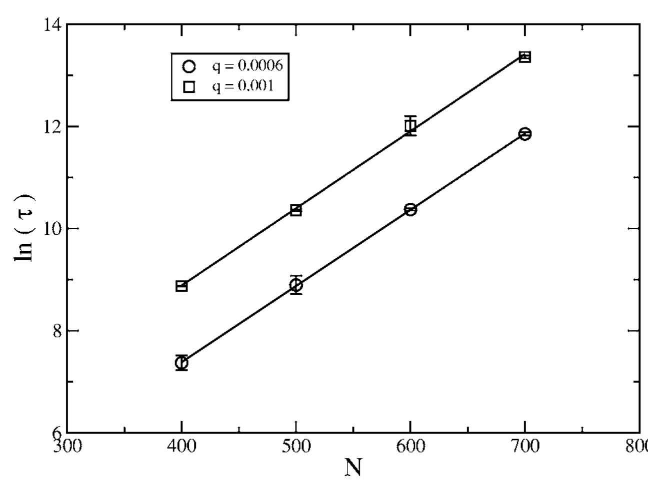

FIG. 2. The characteristic time τ as a function of the number of nodes N in the chain, for $k=2$ and two values of the parameter q. The solid lines are the best linear fits to the data points.

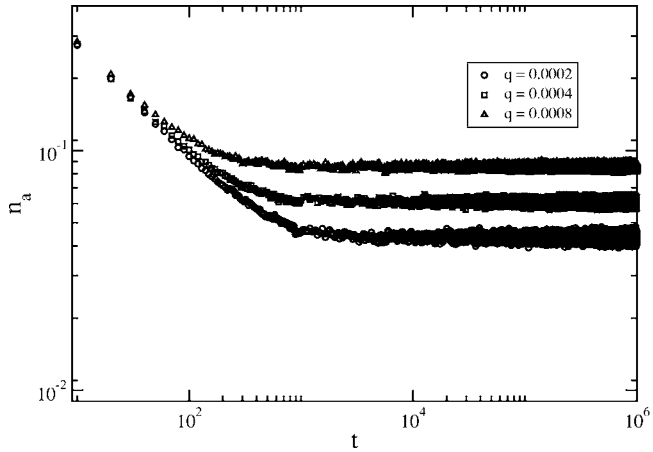

FIG. 3. The same as in Fig. 1, but for a much larger system with $N=2000$. Here the dependence of the heights of the plateaus on noise parameter q is clearly shown.

a power-law behavior $\xi \sim q^{-\gamma}$ and the slope of the straight lines fitted to the data points gives us $\gamma=1/2$. From our simulations it is possible to obtain the following expression for the mean domain size:

$$\log_{10}[\xi(t,q)] = \log_{10}(1/n_a) = -\log_{10}(k+1) + \log_{10} q^{-1/2}. \tag{2}$$

The straight lines in Fig. 4 were drawn using the above equation with $k=1$, 2, and 3.

For small values of q, the evolution of the system is almost the same as that of the standard voter model without noise ($q=0$), so we expect $n_a \sim t^{-1/2}$. Besides, in this regime, we have obtained the power-law behavior $n_a \sim q^{1/2}$ given by Eq. (2). Therefore, it is possible to write down the following scaling relation:

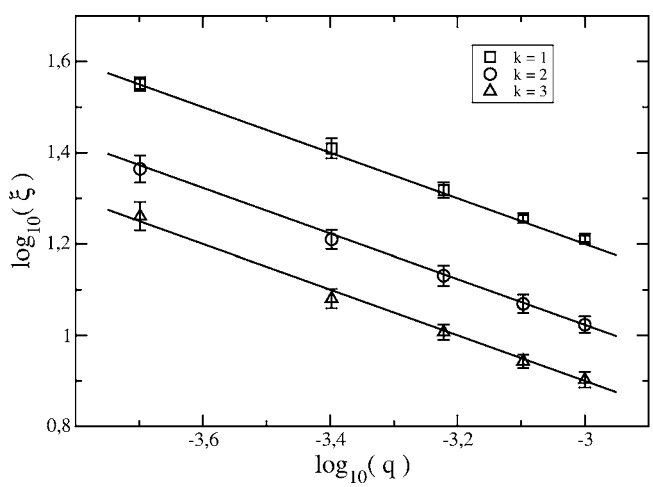

FIG. 4. Log-log plot of $1/n_a(q)$ for $N=2000$, with $k=1$, 2, and 3. For all networks considered we have verified the same power-law behavior $\xi \sim q^{-\gamma}$ with $\gamma=1/2$. The solid lines are drawn from Eq. (2). Data are averaged over 100 samples.

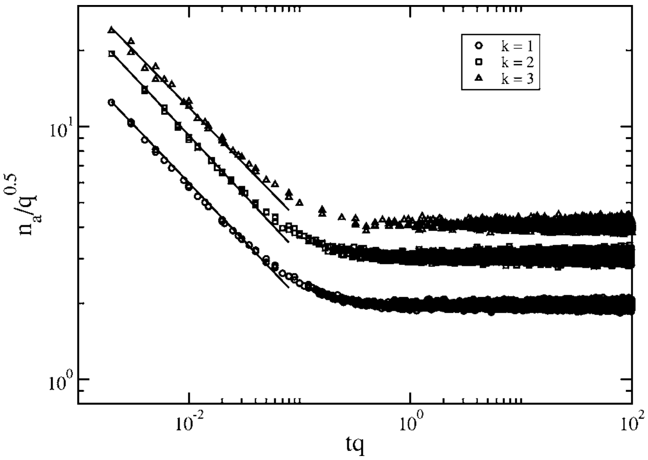

FIG. 5. Data collapse of the results for the density of active sites, for $N=2000$ and $k=1,2,3$, using the scaling form given by Eq. (3). The straight lines correspond to Eq. (4).

$$n_a(t,q) = q^{1/2} F_k(t/q^{-1}), \tag{3}$$

where $F_k(u)$ is a scaling function of the combined scaling variable $u=t/q^{-1}$ and is independent of N, q, and t. Using the above relation we have collapsed the data points for $n_a(t,q)$ into a single smooth curve $F_k(u)$. This is shown in Fig. 5, for $N=2000$ and $k=1,2,3$. Moreover, from our simulations we obtain for the scaling function the results

$$F_k(u) = \begin{cases} 0.72u^{-0.46}, & k=1, \\ 1.06u^{-0.47}, & k=2, \\ 1.50u^{-0.45}, & k=3, \end{cases} \tag{4}$$

for $u<1$, whereas for $u>1$ it approaches a k-dependent constant value

$$F_k(u) = (k+1). \tag{5}$$

The above relations for $F_k(u)$ are in good agreement with the results for the constant domain sizes given by Eq. (3).

During the simulation, in addition to the data for the fraction of active links, we have obtained the magnetization m, defined by

$$m = \left\langle \left\langle \frac{1}{N} \left| \sum_{i=1}^{N} S_i \right| \right\rangle_t \right\rangle_C , \tag{6}$$

where N is the total number of sites, $\langle \cdots \rangle_t$ denotes Monte Carlo averages over a given configuration, and $\langle \cdots \rangle_C$ means averages over different samples. In order to get a better understanding of the nature of the domains we have calculated the magnetization considering averages only over those configurations in the region of plateaus, that is, during the time for which the domains remain with constant size. In Fig. 6 we plot the magnetization as a function of q, for the cases of $k=1$, 2, and 3. We use $N=400$ and 1000 samples. In the region of small noise parameter q the magnetization has a fast decrease with increasing value of the noise parameter; then, for $q>0.1$, it shows a much slower variation.

The size dependence of the magnetization is presented in Fig. 7, where we plot $\log_{10}[m(q,N)]$ as a function of $\log_{10} N$,

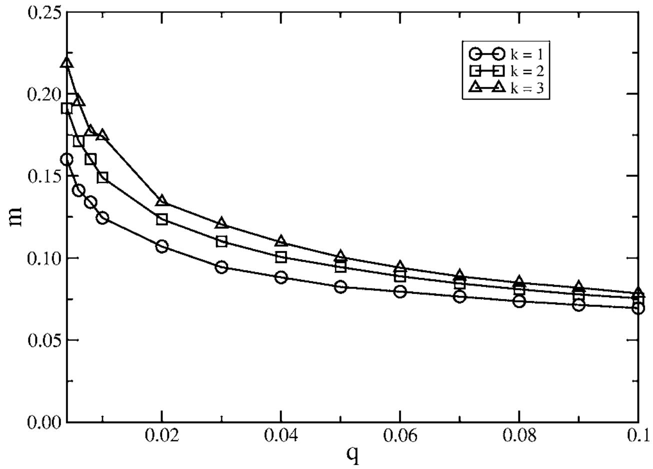

FIG. 6. Dependence of the magnetization m in the region of the plateaus on the noise parameter q for a linear chain of size $N=400$. The curves for $k=1$, 2, and 3 are obtained assuming that each node interacts with $2k$ nearest neighbors. Averages are performed over 1000 samples.

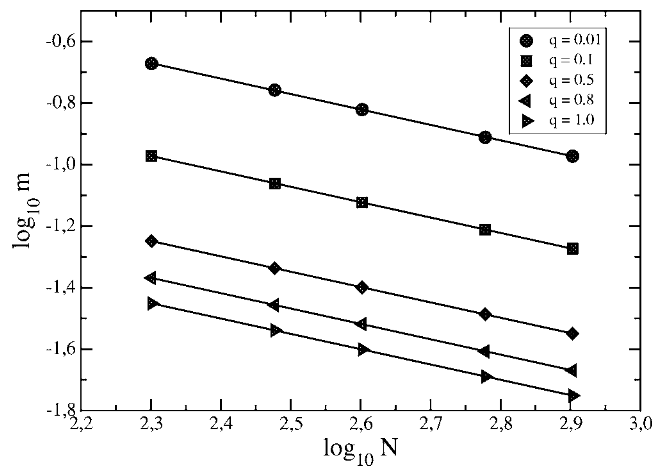

FIG. 7. Log-log plot of $m(q,N)$ for $k=2$ and several values of the noise parameter q. All straight lines have slope $-1/2$ and correspond to least-squares linear fits to the data points for $N=200$, 300, 400, 600, and 800. Data are based on 100 samples.

for several values of the noise parameter q and $k=2$. The best linear fit to the data points gives a slope equal to $-1/2$, in other words, the magnetization decays with a power law as $m(N)\sim N^{-1/2}$. We should note that simulations with other values of k yielded the same behavior.

IV. CONCLUSIONS

We have investigated how the introduction of noise affects the dynamics of the voter model. The present Monte Carlo simulations on chains of different sizes N and values of the noise parameter q have shown that the model system with noise does not converge to a state of complete order in the thermodynamic limit. In contrast with previous numerical simulations on the standard ($q=0$) model in one-dimensional regular lattices, our results imply that, as $N\to\infty$, the system goes to a disordered state in which the magnetization is zero. For finite lattices, before the system reaches a state of complete order it remains in a stationary state during a time that increases exponentially with the system size. In this regime, the domain size dependence on the parameter q is given by $\xi\sim q^{-1/2}$. Moreover, we have obtained a scaling relation which allows a data collapse of the results for different lattice sizes and small values of q.

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