

Second-order variation in elastic fields of a tensile planar crack with a curved front

M. Adda-Bedia and E. Katzav

Laboratoire de Physique Statistique de l'École Normale Supérieure, 24 rue Lhomond, F-75231 Paris Cedex 05, France

D. Vandembroucq

Surface du Verre et Interfaces, Unité Mixte de Recherche CNRS/Saint-Gobain, 39 Quai Lucien Lefranc, 93303 Aubervilliers Cedex, France

(Received 23 December 2005; published 30 March 2006)

We derive the second-order variation in the local static stress intensity factor of a tensile crack with a curved front. We then discuss the relevance of this result to the stability analysis of such fronts, and propose an equation of motion of planar crack fronts in heterogeneous media that contains two main ingredients—irreversibility of the propagation of the crack front and nonlinear effects.

DOI: [10.1103/PhysRevE.73.035106](https://doi.org/10.1103/PhysRevE.73.035106)

PACS number(s): 62.20.Mk, 64.60.Ht, 81.40.Np

The propagation of a crack front in a brittle material is the playground of a number of physical phenomena, which range from dynamic instabilities of fast moving cracks [1] to quasistatic instabilities of crack paths [2,3], or of crack fronts [4–8]. Although the actual theory of brittle fracture mechanics succeeded to explain a number of instabilities, the experimentally observed self-affine roughness of a crack front propagating through a heterogeneous medium remains the subject of theoretical debate [5–7]. This phenomenon is of fundamental importance, because it may be regarded as an archetype of self-affine patterns induced by advancing fronts. Wetting of a disordered substrate being another example of systems with a similar structure [9,10].

In the framework of linear elastic fracture mechanics, an important step was performed by Rice [11] following a work of Meade and Keer [12]. He gave a general formula for the first-order variation in elastic fields of a planar curved crack front and subsequent analysis was mainly based on this work [5,6,13–16]. However, aspects related to crack-front roughness and stability could not be derived within this first-order perturbation solution. A possible explanation, which has been suggested in the context of the wetting problem [10], is that higher order variations might be necessary for the study of the stability and roughening properties of these fronts.

This paper aims at the determination of the second-order variation in elastic fields of a tensile crack front. The present approach is different from [11] and can be generalized to higher orders. It uses a methodology introduced in [8] for the study of the peeling-induced crack-front instability in a confined elastic film. Since the present study is performed in the framework of linear elastic fracture mechanics, our perturbation analysis is expected to hold as long as the radius of the curvature of the crack front remains larger than the size of the process zone where the plastic effects become dominant.

This solution is intended to be used for understanding the roughening of interfaces whose front dynamics does not belong to the Kardar-Parisi-Zhang (KPZ) [17] universality class. For this purpose, we propose a generalized equation of motion of the planar crack fronts in heterogeneous media that includes both the irreversibility of crack-front propagation and the nonlinear effects.

The problem of a half-plane crack located in the plane

$y=0$ with a curved front (see Fig. 1) can be solved by using the linear equations of elasticity. It has been shown [12] that these equations are satisfied for a tensile loading that is symmetric to the crack plane if displacement components (u_y, u_x, u_z) are written as

$$Eu_y = -2(1 - \nu^2)\Phi + (1 + \nu)y \partial \Phi / \partial y, \quad (1)$$

$$Eu_x = (1 + \nu) \partial (F + y\Phi) / \partial x, \quad (2)$$

$$Eu_z = (1 + \nu)y \partial (F + y\Phi) / \partial z, \quad (3)$$

where $F(x, y, z)$ and $\Phi(x, y, z)$ are harmonic functions related by $\partial F / \partial y = (1 - 2\nu)\Phi$. E is the Young modulus and ν is the Poisson ratio. Consequently, the stress components that enter the crack-surface boundary conditions are given by

$$\sigma_{yy} = -\partial \Phi / \partial y + y \partial^2 \Phi / \partial y^2, \quad (4)$$

$$\sigma_{yx} = y \partial^2 \Phi / \partial y \partial x, \quad (5)$$

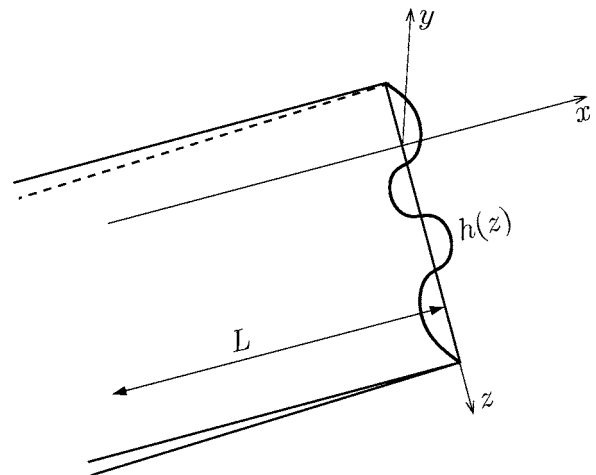


FIG. 1. Schematic of the problem of a half-plane crack on $y=0$ in an infinite body. The average penetration of the crack front in the x direction is L . The straight reference front in the z direction and the perturbation $h(z)$ around it are also shown.

$$\sigma_{yz} = y \partial^2 \Phi / \partial y \partial z, \quad (6)$$

which satisfy the shear traction free boundary conditions on the crack plane. Thus, the problem of loading on the crack faces is one of finding a function Φ satisfying

$$\Delta \Phi(x, y, z) = 0, \quad (7)$$

having vanishing derivatives at infinity, and generating stress σ and opening gap Δu on $y=0$ given by

$$\sigma = -\partial \Phi / \partial y|_{y=0}, \quad \Delta u = -[4(1-\nu^2)/E] \Phi|_{y=0^+}, \quad (8)$$

respectively. Defining $h(z)$ as the position of the crack front, we may write the boundary conditions as

$$\Phi(x, 0, z) = 0, \quad x > h(z), \quad (9)$$

$$\frac{\partial \Phi}{\partial y}(x, 0, z) = p(x, z), \quad x < h(z), \quad (10)$$

$$\frac{\partial \Phi}{\partial x}(x, y, z), \frac{\partial \Phi}{\partial y}(x, y, z) \rightarrow 0, \quad (x^2 + y^2) \rightarrow \infty, \quad (11)$$

where $p(x, z)$ is the normal pressure that loads the crack faces. The problem cannot be solved explicitly without specifying $p(x, z)$, but it is known from classical fracture mechanics analysis that solutions to it exhibit characteristic square-root stress singularities [16]. The harmonic function Φ generating such a singularity necessarily has the form given by

$$\Phi(x, 0^+, z) \sim -\frac{2K(z)}{\sqrt{2\pi}} \sqrt{-X} - \frac{4A(z)}{3\sqrt{2\pi}} (-X)^{3/2}, \quad (12)$$

where $X \equiv x - h(z) \rightarrow 0^-$. The function $K(z)$ is given by

$$K(z) = K_l(z)[1 + h'^2(z)]^{1/4}, \quad (13)$$

where $K_l(z)$ is the local stress intensity factor, which is defined with respect to a coordinates system lying in the plane perpendicular to the crack front at the location $x=h(z)$ and extending into the y direction [16]. The second term in Eq. (12) corresponds to the next order in the expansion of the stress field in the vicinity of the crack front, which is proportional to $\sqrt{h(z)-x}$. The parameter $A(z)$ has the dimension of the stress intensity factor over length.

As a first step, we will consider that K does not depend on z through the position of the crack front with respect to the x direction, i.e. $K(z, h(z))=K(z)$. This condition (to be relaxed later) is of course restrictive, but it will help to construct the full perturbation analysis. This simplification consists explicitly in assuming that a straight crack front will have the same stress intensity factor—wherever it is on the x axis. The real stress intensity factor will be found by relaxing this constraint in a similar way as done in [11].

The piecewise boundary conditions (9), (10), (12) motivate a change into a coordinate system on the crack front, i.e., from (x, y, z) to $(X \equiv x - h(z), y, z)$ [8]. We may then write Eq. (7) as

$$(1 + h'^2) \frac{\partial^2 \Phi}{\partial X^2} - h'' \frac{\partial \Phi}{\partial X} - 2h' \frac{\partial^2 \Phi}{\partial z \partial X} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0, \quad (14)$$

where the prime denotes the derivative with respect to z . Now, we construct an expansion in powers of h , which accounts for the perturbation of the crack front. Without the loss of generality, we write the expansion in the following way:

$$\Phi = \phi_0 + \left(\phi_1 + h \frac{\partial \phi_0}{\partial X} \right) + \left(\phi_2 + h \frac{\partial \phi_1}{\partial X} + \frac{h^2}{2} \frac{\partial^2 \phi_0}{\partial X^2} \right), \quad (15)$$

$$K(z) = K_0(z) + K_1(z) + K_2(z), \quad (16)$$

where the subscripts indicate the order of the perturbation expansion. The advantage of this way of writing the perturbation expansion is that it simplifies the equations for the zeroth-, first-, and second-order problem. A direct substitution of the expansion (15) into the equilibrium equation (14) yields

$$\frac{\partial^2 \phi_i}{\partial X^2} + \frac{\partial^2 \phi_i}{\partial y^2} + \frac{\partial^2 \phi_i}{\partial z^2} = 0, \quad (17)$$

with $i=0, 1, 2$. To complete the formulation of the problem, we need to specify the boundary conditions of each order of the expansion. These are given by

$$\phi_i = 0, \quad y = 0, \quad X > 0, \quad (18)$$

$$\frac{\partial \phi_0}{\partial y} = p(X, z), \quad \frac{\partial \phi_1}{\partial y} = \frac{\partial \phi_2}{\partial y} = 0, \quad y = 0, \quad X < 0, \quad (19)$$

$$\frac{\partial \phi_i}{\partial X}(X, y, z), \frac{\partial \phi_i}{\partial y}(X, y, z) \rightarrow 0, \quad (X^2 + y^2) \rightarrow \infty. \quad (20)$$

The expansion of Eq. (12) to the second order in h yields

$$\phi_0(X, 0^+, z) \sim -\frac{2K_0(z)}{\sqrt{2\pi}} \sqrt{-X} - \frac{4A_0(z)}{3\sqrt{2\pi}} (-X)^{3/2}, \quad (21)$$

$$\phi_1(X, 0^+, z) \sim -\frac{K_0(z)h(z)}{\sqrt{-2\pi X}} - \frac{2K_1(z)}{\sqrt{2\pi}} \sqrt{-X} - \frac{4A_1(z)}{3\sqrt{2\pi}} (-X)^{3/2}, \quad (22)$$

$$\begin{aligned} \phi_2(X, 0^+, z) \sim & -\frac{K_0(z)h^2(z)}{4\sqrt{2\pi}(-X)^{3/2}} - \frac{K_1(z)h(z)}{\sqrt{-2\pi X}} \\ & - \frac{2[K_2(z) + A_1(z)h(z)]}{\sqrt{2\pi}} \sqrt{-X} \end{aligned} \quad (23)$$

for $X \rightarrow 0^-$. It is assumed that the perturbation terms induced by $A_0(z)$ as given in Eq. (21) are negligible compared to those induced by $K_0(z)$. Indeed, dimensional analysis shows that $A_0(z) \sim K_0(z)/L$, where L is the geometrical length scale induced by the tractions $p(x, z)$ or by the average length of the crack plane in the x direction (see Fig. 1). This length scale is large compared to the characteristic scale of the per-

turbation $h(z)$. Therefore, we will neglect the corresponding contributions in the following. However, the contribution proportional to $(-X)^{3/2}$ of the first order as shown in Eq. (22) should be taken into account, because it depends on $K_0(z)$ and, thus, contributes to the stress intensity factor term of the second-order problem.

The zeroth-order problem cannot be solved without specifying the loading $p(x; z)$. However, this is not needed for solving the first- and second-order problems, which will depend on the stress intensity factor $K_0(z)$ only. However, if one takes into account, in the perturbation analysis, the contributions of the parameter $A_0(z)$, the resolution of the zeroth-order problem becomes necessary [8]. Let us decompose $\phi_i(X, y, z)$ into Fourier modes in the z axis,

$$\phi_i(X, y, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}_i(x, y, p) e^{ipz} dp, \quad (24)$$

and use polar coordinates (r, θ) in the (X, y) plane. The solutions for $\hat{\phi}_1$ and $\hat{\phi}_2$ are readily given by

$$\hat{\phi}_1(r, \theta, p) = -\frac{\hat{H}_1(p)}{\sqrt{2\pi r}} e^{-|p|r} \sin(\theta/2), \quad (25)$$

$$\begin{aligned} \hat{\phi}_2(r, \theta, p) = & \frac{\hat{L}_2(p)}{4\sqrt{2\pi r}^{3/2}} (1 + |p|r) e^{-|p|r} \sin(3\theta/2) \\ & - \frac{\hat{H}_2(p)}{\sqrt{2\pi r}} e^{-|p|r} \sin(\theta/2). \end{aligned} \quad (26)$$

These forms satisfy the bulk equations (17) and the boundary conditions (18)–(20). The conditions (22) and (23) are then satisfied if the functions $\hat{H}_1(p)$, $\hat{H}_2(p)$, and $\hat{L}_2(p)$ are given by

$$\hat{H}_1(p) = \int_{-\infty}^{\infty} K_0(z) h(z) e^{-ipz} dz, \quad (27)$$

$$\hat{H}_2(p) = \int_{-\infty}^{\infty} \hat{K}_1(p') \hat{h}(p-p') \frac{dp'}{2\pi}, \quad (28)$$

$$\hat{L}_2(p) = \int_{-\infty}^{\infty} \hat{H}_1(p') \hat{h}(p-p') \frac{dp'}{2\pi}. \quad (29)$$

Identifying the stress intensity factors at each order as given by Eqs. (22) and (23) one finds

$$\hat{K}_1(p) = -\frac{1}{2} |p| \hat{H}_1(p), \quad (30)$$

$$\hat{K}_2(p) = -\frac{1}{16} \int_{-\infty}^{\infty} (6p'^2 + p^2 - 4|p||p'|) \hat{H}_1(p') \hat{h}(p-p') \frac{dp'}{2\pi}. \quad (31)$$

Before performing the inverse Fourier transform of these quantities, let us generalize these results to the case where

the stress intensity factor depends on the location of the crack front in the x direction.

Until now, we have supposed that the stress intensity factor does not depend on the mean location of the crack front. This is not true in general for quasistatic cracks, which should be at equilibrium, and for which the condition $dK/dL < 0$ must be satisfied. The decomposition of the perturbation follows from Rice's approach [11,16]. First, we locate the straight crack front on which the perturbation is performed at the position $[L+h(z)]$. The stress intensity factor at the leading order is then given by $K_0[z, L+h(z)]$ and the location of any point of the curved front is taken with reference to this position. It is clear that the perturbation expansion of the stress intensity factor will include contributions of the form $hh''K_0$, hdK_0/dL , and $h^2d^2K_0/dL^2$. However, one should neglect them because the terms induced by $A_0(z)$, which introduce contributions of the same order, were already neglected. Therefore, the perturbation expansion with respect to (h/L) will be at the leading order. Let us focus on the stress intensity factor $K_I[z, L+h(z)]$ as given by Eq. (13), and write

$$K_I(z) = K_{I0}(z) + K_{I1}(z) + K_{I2}(z) + O\left(h^3, \frac{h}{L}\right), \quad (32)$$

where the L dependence has been omitted. Therefore, using Eqs. (13), (30), and (31) and performing inverse Fourier transforms, we find that $K_{I0}(z) = K_0(z)$, and

$$K_{I1}(z) = PV \int_{-\infty}^{\infty} K_0(z') \frac{h(z') - h(z)}{(z' - z)^2} \frac{dz'}{2\pi}, \quad (33)$$

$$\begin{aligned} K_{I2}(z) = & -\frac{1}{8} K_0(z) h'^2(z) + PV \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_0(z') \\ & \times \frac{[h(z') - h(z)][h(z'') - h(z)]}{(z' - z)^2 (z'' - z)^2} \frac{dz' dz''}{2\pi 2\pi}, \end{aligned} \quad (34)$$

Finally, when $K_0(z)$ is independent of z , the expansion to the second order in h and to the leading order in (h/L) of the mode I stress intensity factor is simplified into

$$\begin{aligned} \frac{K_I(z)}{K_0} = & 1 - \frac{1}{8} h'^2(z) \\ & + PV \int_{-\infty}^{\infty} \frac{h'(z')}{z' - z} \left[1 + PV \int_{-\infty}^{\infty} \frac{h'(z'')}{z'' - z'} \frac{dz''}{2\pi} \right] \frac{dz'}{2\pi}, \end{aligned} \quad (35)$$

$$\begin{aligned} \frac{\hat{K}_I(p)}{K_0} = & 2\pi \delta(p) - \frac{1}{2} |p| \hat{h}(p) + \frac{1}{8} \int_{-\infty}^{\infty} [2|p||p'| \\ & + p'(p-p')] \hat{h}(p') \hat{h}(p-p') \frac{dp'}{2\pi}. \end{aligned} \quad (36)$$

Let us emphasize again that for the study of the crack-front stability, this perturbation expansion is incomplete, because the (h/L) contributions have been omitted. This statement is true even for a linear stability analysis. An example

of the importance of such contributions is given by the linear stability analysis of the peeling-induced crack front in a confined elastic film [8], where the (h/L) terms do rule the stability of the crack front. From a conceptual point of view, these terms are important to keep contact with the experiments [4], because a quasistatic moving crack front will always stop ($dK/dL < 0$), unless the applied force is increased. Indeed, the experimental realizations for the study of crack-front roughness use the large length scale L in order to make the interface moving, by applying an increasing opening in a cantilever beam configuration. We believe that such effects are also present in the wetting experiments, where the contact line is displaced by pulling off the substrate. In such conditions, the roughening of the interface results from a competition between the microscopic pinning effects and the destabilizing effects of the macroscopic driving.

We now propose an equation for the motion of a planar crack in a heterogeneous material. The present approach is very similar to the one introduced by Gao and Rice [13–15]. We write the equation of motion for the moving crack front as a stochastic partial differential equation by using two main ingredients—the irreversibility of the crack-front propagation and the nonlinear effects. We refer to $h(z)$ as the fluctuating part of the interface, so that by definition, the real location of the interface is given by $L+h(z)$, and L is its average. First, we expect a contribution of the form $[K_I(h) - K_c(z, h)]$, where the perturbative calculations to second order for $K_I(h)$ are given above, and $K_c(z, h)$ is some random toughness describing the heterogeneity of the material. Then, the irreversibility of the fracture process implies that the crack-front motion is possible only at locations of $h(z)$ where the stress intensity factor is larger than the local toughness $K_I(h) > K_c(z, h)$. This results in a term like $\Theta[K_I(h) - K_c(z, h)]$ where $\Theta(\cdot)$ is the Heaviside function. Finally, since the crack propagation is locally normal to the interface

[16], one should include a KPZ-like term of the form $\sqrt{1+h'^2}(z)$. So a possible form, where the velocity is taken to be proportional to the difference $(K_I - K_c)$ is given by

$$\frac{\partial h}{\partial t} \propto \sqrt{1+h'^2}(K_I(h) - K_c(z, h))\Theta(K_I - K_c). \quad (37)$$

This is a highly nonlinear stochastic partial differential equation, even if just second-order terms are taken. Clearly, the presence of the Heaviside function complicates the treatment. In this equation, properties of the noise term need to be specified, and should be generically described by short-range correlations.

To summarize, we derived the second-order variation in the stress intensity factor of a tensile crack with a curved front propagating in a brittle material. We pointed out that for linear stability analysis one has to take into account the contributions coming from the large scales, and so the complete resolution of a given problem must be fully performed for that purpose. Finally, we proposed an equation of motion of the planar crack fronts in heterogeneous media that contains both the irreversibility of the propagation of the crack front and the nonlinear effects. We suggest that the proposed equation can be useful in studying the roughening of propagating crack fronts. In particular, we expect that the nonlocal character of the nonlinear term (in contrast to the local KPZ nonlinearity) is likely to change the universality class of the original equation obtained at first order. Finally, the perturbation method introduced in this study can be generalized without major difficulties to higher orders.

This work was supported by the EEC PatForm Marie Curie action (E.K.). We thank A. Boudaoud for fruitful discussions. Laboratoire de Physique Statistique de l'École Normale Supérieure is associated with the CNRS (Contract No. UMR 8550) and Universities Paris VI and Paris VII.

-
- [1] J. Fineberg and M. Marder, *Phys. Rep.* **313**, 2 (1999); M. Adda-Bedia, *J. Mech. Phys. Solids* **3**, 227 (2005).
- [2] B. Cotterell and J. R. Rice, *Int. J. Fract.* **16**, 155 (1980).
- [3] M. Adda-Bedia and Y. Pomeau, *Phys. Rev. E* **52**, 4105 (1995).
- [4] J. Schmittbuhl and K. J. Måløy, *Phys. Rev. Lett.* **78**, 3888 (1997); A. Delaplace, J. Schmittbuhl, and K. J. Måløy, *Phys. Rev. E* **60**, 1337 (1999).
- [5] D. S. Fisher, *Phys. Rep.* **301**, 113 (1998).
- [6] E. Bouchaud *et al.*, *J. Mech. Phys. Solids* **50**, 1703 (2002).
- [7] J. Schmittbuhl, A. Hansen, and G. G. Batrouni, *Phys. Rev. Lett.* **90**, 045505 (2003); J. Schmittbuhl, A. Hansen, and G. G. Batrouni, *ibid.* **92**, 049602 (2004); M. J. Alava and S. Zapperi, *ibid.* **92**, 049601 (2004).
- [8] M. Adda-Bedia and L. Mahadevan *Proc. R. Soc. London* (to be published).
- [9] A. Prevost, E. Rolley, and C. Guthmann, *Phys. Rev. B* **65**, 064517 (2002); S. Moulinet, C. Guthmann, and E. Rolley, *Eur. Phys. J. E* **8**, 437 (2002).
- [10] R. Golestanian and E. Raphaël, *Phys. Rev. E* **67**, 031603 (2003); P. Le Doussal, K. J. Wiese, E. Raphaël, and R. Golestanian, *Phys. Rev. Lett.* **96**, 015702 (2006).
- [11] J. R. Rice, *J. Appl. Mech.* **54**, 571 (1985).
- [12] K. P. Meade and L. M. Keer, *J. Elast.* **14**, 79 (1984).
- [13] H. Gao and J. R. Rice, *J. Appl. Mech.* **56**, 828 (1989).
- [14] S. Ramanathan and D. S. Fisher, *Phys. Rev. B* **58**, 6026 (1998).
- [15] S. Roux, D. Vandembroucq, and F. Hild, *Eur. J. Mech. A/Solids* **22**, 743 (2003); Y. Charles *et al.*, *J. Mech. Phys. Solids* **52**, 1651 (2004).
- [16] J. B. Leblond, *Mécanique de la Rupture Fragile et Ductile* (Hermes Science, Paris, 2003).
- [17] M. Kardar, G. Parisi, and Y. C. Zhang, *Phys. Rev. Lett.* **56**, 889 (1986).