Dynamic properties in a family of competitive growing models

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The properties of a wide variety of growing models, generically called X-RD, involving the deposition of particles according to competitive processes, such that a particle is attached to the aggregate with probability p following the mechanisms of a generic model X that provides the correlations and at random [random deposition (RD)] with probability $(1-p)$, are studied by means of numerical simulations and analytic developments. The study comprises the following X models: Ballistic deposition, random deposition with surface relaxation, Das Sarma–Tamboronea, Kim-Kosterlitz, Lai–Das Sarma, Wolf-Villain, large curvature, and three additional models that are variants of the ballistic deposition model. It is shown that after a growing regime, the interface width becomes saturated at a crossover time (t_{x2}) that, by fixing the sample size, scales with p according to $t_{x2}(p) \propto p^{-y}$ $(p>0)$, where y is an exponent. Also, the interface width at saturation (W_{sat}) scales as $W_{\text{sat}}(p) \propto p^{-\delta}$ $(p>0)$, where δ is another exponent. It is proved that, in any dimension, the exponents δ and y obey the following relationship: $\delta = y\beta_{\text{RD}}$, where $\beta_{\text{RD}}=1/2$ is the growing exponent for RD. Furthermore, both exponents exhibit universality in the $p \to 0$ limit. By mapping the behavior of the average height difference of two neighboring sites in discrete models of type X-RD and two kinds of random walks, we have determined the exact value of the exponent δ. When the height difference between two neighbouring sites corresponds to a random walk that after walking $\langle n\rangle$ steps returns to a distance from its initial position that is proportional to the maximum distance reached (random walk of type A), one has $\delta=1/2$. On the other hand, when the height difference between two neighboring sites corresponds to a random walk that after $\langle n\rangle$ steps moves $\langle l\rangle$ steps towards the initial position (random walk of type B), one has $\delta=1$. Finally, by linking four well-established universality classes (namely Edwards-Wilkinson, Kardar-Parisi-Zhang, linear [molecular beam epitaxy (MBE)] and nonlinear MBE) with the properties of type A and B of random walks, eight different stochastic equations for all the competitive models studied are derived.

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I. INTRODUCTION

The study and understanding of the properties of growing interfaces have attracted great interest. In fact, interfaces are ubiquitous in Nature and their study has opened a promising field of multidisciplinary research [1–4]. Interfaces naturally emerge in a wide variety of systems such as film growth by vapour deposition, chemical deposition, or molecular beam epitaxy [1,5], propagation of fire fronts [6], diffusion fronts [7], bacterial growth [8], solidification [9], propagation of reaction fronts in catalyzed reactions [10], electrodeposition and dissolution experiments [11], sedimentation [12], etc.

Models of growing interfaces may be defined and studied either by using discrete lattices or by means of continuous equations. Discrete models are defined by a set of rules that provide a detailed microscopic description of the evolution of the growing aggregate. In these models the interface is described by a discrete set $h(i,t)$ that represents the height of site i at time t. The interface has L^d sites, where L is the linear size and d is the dimensionality of the substrate. The interface of the aggregate is characterized through the scaling behavior of the interface width $W(L,t) \equiv \sqrt{1/L^d \sum_{i=1}^{L^d}[h(i,t)-\langle h(t)\rangle]^2}$. For this purpose, the Family-Vicsek phenomenological scaling approach [12,13] has proved to be very successful for the description of the dynamic evolution of growing interfaces. In fact, it may be expected that $W(L,t)$ would show the spatiotemporal scaling behavior given by [12,13] $W_{\text{sat}} \propto L^{\alpha}$ for $t \gg t_c$ and $W(t) \propto t^{\beta}$ for $t \ll t_c$, where $t_c \propto L^Z$ is the crossover time between these two regimes. The scaling exponents α, β, and $Z=\alpha/\beta$ are called roughness, growth, and dynamic exponents, respectively. Also, different models can be grouped into universality classes when they share the same scaling exponents.

In contrast to the microscopic details of the growing mechanisms of the interface, continuous equations focus on the macroscopic aspects of the roughness. Essentially, the aim is to follow the evolution of the coarse-grained height function $h(\mathbf{x},t)$ by using a well-established phenomenological approach that takes all the relevant processes that survive at a coarse-grained level into account. This procedure normally leads to stochastic nonlinear partial differential equations that, in general, may be written as follows [1,14–16]:

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = G_j\{h(\mathbf{x},t)\} + F + \eta(\mathbf{x},t), \qquad (1)$$

where the index j symbolically denotes different processes, $G_j\{h(\mathbf{x},t)\}$ is a local functional that contains the various surface relaxation phenomena and only depends on the spatial derivatives of $h(\mathbf{x},t)$ since the growth process is assumed to be determined by the local properties of the surface only. Also, F denotes the mean deposition rate and $\eta(\mathbf{x},t)$ is the deposition noise that determines the fluctuations of the incoming flux around its mean value F. It is usually assumed that the noise is spatially and temporally uncorrelated.

In order to establish the correspondence between a continuous growth equation and a discrete model one can apply at least three different methods: (i) to numerically simulate the model and compare the obtained scaling exponents with those of the corresponding continuous equation, (ii) to develop a set of plausibility arguments using physical principles, and (iii) to derive the continuous equation analytically starting from a given discrete model. There are few papers in the direction of the last method. For example, a systematic approach proposed by Vvedensky *et al.* [17], where the continuous equations can be constructed directly from the growth rules of some discrete models, based on the master equation description, has been applied successfully [17–20]. This procedure requires a regularization step, in which nonanalytic quantities are expanded and replaced by analytic approaches, e.g., the step function is approximated by a shifted hyperbolic tangent function expanded in a Taylor series. As pointed out by Předota and Kotrla [19], the choice of the regularization scheme for the step function is ambiguous. Thus, the coefficients entering in the derived continuum stochastic equation cannot be determined uniquely. Another method has shown the connection between the ballistic deposition discrete model and the Kardar-Parisi-Zhang (KPZ) equation in $d=(1+1)$ dimensions. However, this method is not successful in $d=(2+1)$ dimensions [21].

It is worth mentioning that most of the already mentioned progress in the understanding of the properties of interfaces has been achieved when the growth of the aggregate is due to one kind of particle only. In contrast, less attention has been drawn to the study of the dynamics of competitive growing processes. It is well known that these competitive processes are significant to the growth of real materials in at least two different ways: (a) when the growing process involves two or more kinds of particles and (b) when deposition of a single kind of particle is considered, but such type of particle may undergo different growing mechanisms.

One example of case (a) arises from the deposition of alloys or systems with impurities, see, e.g., Refs. [22–27], and references therein. In this case, there may be different interactions between different kinds of particles causing the growing mechanisms to change [22–27]. Based on these ideas, Cerdeira *et al.* [22–25] have studied various models for binary systems involving competitive randomlike and ballisticlike deposition. Recently, the scaling behavior of a two-component surface-growth model has been studied by Kotrla *et al.* [27]. This study addresses the relationship between kinetic roughening and phase ordering in a $(1+1)$-dimensional single-step solid-on-solid model with Ising-like interactions between two components.

On the other hand, considering the deposition of one kind of particle [case (b)], Pellegrini *et al.* [28,29] have studied a ballistic model of surface growth that considers "sticky" and "sliding" particles. The model interpolates between a standard ballistic model when only sticky particles are deposited (with probability $P=1$) and a completely restructured ballistic model for $P=0$ when only unrestricted sliding particles are allowed to become attached to the sample. Using this model Pellegrini *et al.* [28,29] have given evidence of a roughening transition in dimensions $d=3$ and $d=4$, while such kind of transition is no longer observed in $d=2$.

In a related context of competitive growing processes, we have also studied two competitive growth models in $(1+1)$, $(2+1)$, and $(3+1)$ dimensions [30–32]. In the first discrete growth model, namely, the RDSR-RD model, the same types of particles are aggregated according to the rules of random deposition with surface relaxation (RDSR) with probability p and according to the rules of random deposition (RD) with probability $(1-p)$ [30]. In the second discrete growth model, namely, the BD-RD model, particles are aggregated according to the rules of ballistic deposition (BD) with probability p and according to the rules of random deposition (RD) with probability $(1-p)$ [31].

For both the RDSR-RD and the BD-RD models the saturation process of the interface width depends sensitively on p: saturation takes place at longer times for smaller values of p, while the final width of the interface is smaller for larger p values. Furthermore, in both models, three different regimes and two corresponding crossovers can easily be observed. For short times, say $t<t_{x1}$, the random growth of the interface is observed (i.e., the RD process dominates). At this stage, correlations have not been developed yet and $W(t) \propto t^{\beta_{\mathrm{RD}}}$ $(t<t_{x1}, \beta_{\mathrm{RD}}=1/2)$ holds. During an intermediate time regime, say $t_{x1}<t<t_{x2}$, correlations develop since the RDSR (BD) process now dominates leading to $W(t) \propto t^{\beta_{\mathrm{RDSR}}}$ $[W(t) \propto t^{\beta_{\mathrm{BD}}}]$. At a later stage, for $t>t_{x2}$, correlations can no longer grow due to the geometrical constraint of the lattice size and saturation is observed. The saturation value of the interface width $W_{\mathrm{sat}}(L,p)$ and the characteristic crossover time t_{x2} behave as [30–32]

$$W_{\mathrm{sat}}(L,p) \propto L^{\alpha_X} p^{-\delta} \quad (p>0) \tag{2}$$

and

$$t_{x2}(L,p) \propto L^{Z_X} p^{-y} \quad (p>0), \tag{3}$$

respectively. Here, δ and y are exponents and $X \equiv$ RDSR or BD, depending on the model. On the other hand, one has that the crossover time t_{x1} also scales with p as t_{x2} does [see Eq. (3)].

In these previous studies we have shown that the exponents y and δ are independent of the dimensionality. For the RDSR-RD (BD-RD) model we have found that $\delta \approx 1(\approx 1/2)$ and $y \approx 2(\approx 1)$ [30–32]. Based on these numerical estimates we have conjectured the following exact values $\delta=1$ and $y=2$ for RDSR-RD, and $\delta=1/2$ and $y=1$ for BD-RD. Very recently, this early conjecture has proved to be correct by using an exact analysis [33]. Furthermore, these values allowed us to formulate another conjecture by stating that $\delta=y/2$ for both models [30–32].

Also, we have shown that the stochastic representation of the RDSR-RD model is given by [30–32]

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = F + \nu_0 p^2 \nabla^2 h(\mathbf{x},t) + \eta(\mathbf{x},t), \tag{4}$$

while for the BD-RD model one has

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = F + \nu_0 p \nabla^2 h(\mathbf{x},t) + \frac{\lambda p^{3/2}}{2}[\nabla h(\mathbf{x},t)]^2 + \eta(\mathbf{x},t), \tag{5}$$

where ν_0 plays the role of an effective surface tension, λ represents the lateral growth, and $\eta(\mathbf{x},t)$ is the uncorrelated white-noise term.

Within this context, the aim of this work is to perform a systematic study of a wide variety of competitive growth models of the type X-RD, with X=Das Sarma–Tamboronea (DST) [34], Kim-Kosterlitz (KK) [35], Lai-Das Sarma (LDS) [36,37], Wolf-Villain (WV) [38], large curvature (LC) [39,40], RDSR [12], BD [1], BD1, BD2, and BD3. The last three ballistic deposition-motivated models are variants of the BD model that will be described in detail below. The study is focused on the behavior of the exponents y and δ as well as on the derivation of the stochastic equations of competitive models. For this purpose the manuscript is organized as follows. In Secs. II and III the relevant properties of y and δ are addressed and the exact relationship between both exponents is derived, respectively. Subsequently, in Sec. IV numerical results obtained by means of Monte Carlo computer simulations covering all competitive models listed above are presented. After that, in Sec. V, the exact values for the exponents y and δ are obtained for all the studied models. Section VI is devoted to the derivation of the stochastic equations for different competitive models while our conclusions are stated in Sec. VII.

II. THE RELATIONSHIP BETWEEN δ AND y

In previous papers [30–32] we have shown that for the RDSR-RD and the BD-RD models, the values of the exponents δ and y are independent of the dimensionality of the substrate. Furthermore, our study leads us to conjecture a simple relationship between them, namely, $\delta = y/2$. Furthermore, we have also proposed and numerically tested the following phenomenological dynamic scaling ansatz for both the RDRS-RD and the BD-RD models [30–32]

$$W(t,L,p) \propto L^{\alpha_X} p^{-\delta} F\left(\frac{t}{L^{Z_X} p^{-y}}\right), \quad p>0, \quad t>t_{x1}, \quad L\to\infty, \tag{6}$$

where $X \equiv$ RDSR or BD depending on the model and F is a suitable scaling function. Now, if we restrict the previous ansatz by considering variations of p only (i.e., fixing L), Eq. (6) can be written as

$$W(t,p) \propto p^{-\delta} F^*(t/p^{-y}), \quad p>0, \quad t>0, \quad L\to\infty, \tag{7}$$

where $F^*(u)$ is a suitable scaling function such that (i) $F^*(u) \propto u^{\beta_{RD}}$ for $u \to 0$, (ii) $F^*(u) \propto u^{\beta_X}$ for u in the intermediate regimen, and (iii) $F^*(u)$=const for $u \gg 1$. It is worth mentioning that the ansatz given by Eq. (7) is valid for the three regimens ($t>0$) while the previous one, given by Eq. (6), only holds for regimes (ii) and (iii) with $t>t_{x1}$. This effect is due to the fact that by fixing the lattice size one also fixes the crossover time t_{x1} that depends on L. In order to check the validity of the new ansatz, Fig. 1 shows log-log plots of $W(t,L,p)p^{\delta}$ versus t/p^{-y} as obtained in d=1 dimensions, by using values of p within the range $0.01 \leq p \leq 0.64$, and for both models. Here, we have taken L=256 and L=512 for the RDSR-RD and the BD-RD models, respectively.

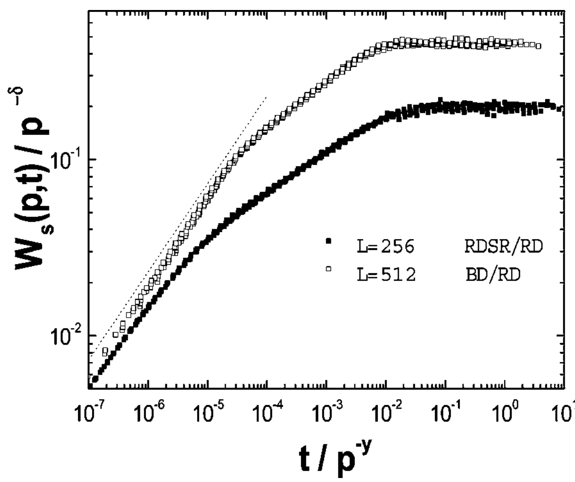

FIG. 1. Log-log plot of $W(t,L,p)/p^{-\delta}$ versus t/p^{-y} where X=RDRS, BD. Results obtained for different values of p $(0.01 \leq p \leq 0.64)$ and lattices of size L=256 for X=RDRS and L=512 for X=BD. The data corresponding to the BD-RD model have been shifted one decade to the left for the sake of clarity. The dotted line has slope β_{RD}=1/2 and has been drawn for the sake of comparison. More details in the text.

Considering the short-time regime $t<t_{x1}$, or equivalently $u \to 0$ in Eq. (7), we observed that the initial slope is independent of the considered model and it is given by β_{RD}=1/2, since the RD process dominates the early stages of growth. Consequently, within the short-time regime, Eq. (7) can be written as

$$W(t,p) \propto p^{-\delta}(t/p^{-y})^{\beta_{RD}}, \quad p>0, \quad t<t_{x1}, \quad L\to\infty, \tag{8}$$

and, according to the results shown in Fig. 1, this relation has to be independent of p. So, this is true only if

$$\delta = y\beta_{RD}. \tag{9}$$

This result strongly suggests that the factor 1/2 already found in the relationship between δ and y [30–32] is just β_{RD}. It is also worth mentioning that β_{RD} is independent of the dimensionality of the substrate, and therefore one should expect that Eq. (9) would also hold in any dimension.

III. NUMERICAL EVIDENCE ON NEGLIGIBLE FINITE-SIZE CORRECTIONS TO THE VALUES OF THE EXPONENT δ

Usually, it is observed that the numerical values of the scaling exponents (α, β, and Z) undergo systematic deviations when the size of the lattices used in the simulations is changed. Figure 2 shows log-log plots of $W_{sat}(L,p)/L^{\alpha_{BD}}$ versus p as obtained for lattices of different size and taking

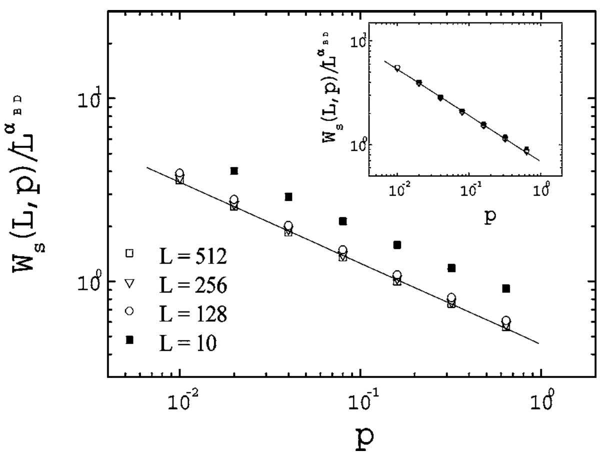

FIG. 2. Log-log plots of $W_{\text{sat}}(L,p)L^{-\alpha_{\text{BD}}}$ versus p obtained for lattices of different size, as indicated in the figure, and assuming $\alpha_{\text{BD}}=1/2$. The full line has slope $\delta=0.45$ and corresponds to the best fit of the data. The inset shows the same scaled plot but obtained assuming $\alpha'_{\text{BD}}=0.43$ for $L=512$, 256, 128 and $\alpha'_{\text{BD}}=0.46$ for $L=10$. Again the full line with slope $\delta=0.45$ corresponds to the best fit of the data.

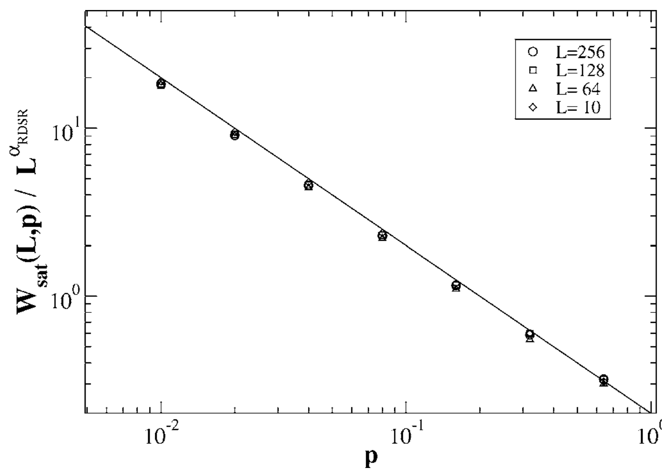

FIG. 3. Log-log plots of $W_{\text{sat}}(L,p)/L^{\alpha_{\text{RDSR}}}$ versus p obtained for lattices of different size, as indicated in the figure, and assuming $\alpha_{RDSR}=1/2$. The full line has slope $\delta=0.97$ and corresponds to the best fit of the data.

the BD-RD model. In this figure, we have added the data corresponding to $L=10$ to the results already shown in Ref. [31]. Using the exact value $\alpha_{\text{BD}}=\frac{1}{2}$, straight lines are observed, in agreement with Eq. (2), and the best fit of the data gives the slope $\delta\cong 0.45\pm 0.01$. However, a rather small systematic deviation of the data, according to the size of the lattice, is observed: the larger the lattice, the smaller the ordinate. This behavior is due to high-order corrections to scaling that we have neglected in Eq. (2). On the other hand, using the roughness exponent obtained by fitting our data, namely, $\alpha'_{\text{BD}}=0.43\pm 0.05$ for $L=512$, 256, 128, and $\alpha'_{\text{BD}}=0.46\pm 0.05$ for $L=10$, it is possible to achieve an excellent data collapsing, as shown in the inset of Fig. 2. In this case, the slope obtained by means of a least-squares fit is also $\delta\cong 0.45\pm 0.01$. Summing up, all results shown in Fig. 2 point out that the scaling ansatz given by Eq. (2) holds for the BD-RD model. Furthermore, we would like to emphasize that the systematic shift of the data observed in Fig. 2 does not affect the slope of the power law, and consequently the exponent δ is almost independent of the lattice size (up to $L=10$ in Fig. 2).

Figure 3 shows log-log plots of $W_{\text{sat}}(L,p)/L^{\alpha_{\text{RDSR}}}$ versus p as obtained for the RDSR-RD model using lattices of different size. In this figure, we have also added the data corresponding to $L=10$ to the results already shown in Ref. [30]. Using the exact value $\alpha_{\text{RDSR}}=\frac{1}{2}$ for $L=256,128,64$ and $\alpha_{\text{RDSR}}=0.46$ for $L=10$, straight lines are observed, in agreement with Eq. (2), and the best fit gives the slope $\delta\cong 0.97\pm 0.01$.

So, these results show that the values of δ are not appreciably affected by systematic deviations even when extremely small lattices are used (up to $L=10$ in the examples of Figs. 2 and 3). It is worth mentioning that for both models we have arrived at the same conclusion in higher dimensions. More explicitly, we have performed numerical simulations by using $L\times L=6\times 6$ in $d=(2+1)$ and $L\times L\times L=6\times 6\times 6$ in $d=(3+1)$ and we have found $\delta\sim 0.45$ and $\delta\sim 0.97$ independently of the dimensionality, for the BD-RD and RDSR-RD models, respectively. (These results are not shown here for the sake of space.)

The lack of appreciable finite-size effects in the values of δ (up to $L=10$) is a rather surprising result. It should be noticed that a similar behaviour is also found in the exponent β_{RD} for the random deposition model, and it is due to the lack of correlations between different columns of the aggregate. So, although we have not a convincing explanation for the behavior of δ in our case, we think that it may be related in a nontrivial way to the decorrelation induced by the fraction $(1-p)$ of particles that are deposited according to the random deposition rules, which are expected to play a relevant role precisely for $p\rightarrow 0$.

IV. SYSTEMATIC EVALUATION OF THE EXPONENT δ FOR A FAMILY OF COMPETITIVE MODELS

In this section we take advantage of the almost negligible dependence of the values of the exponent δ on the lattice size, as discussed in the previous section, in order to evaluate it by using small lattices for a wide family of competitive models called X-RD. This family of models is defined such that particles of the same type are aggregated according to the rules of a generic discrete model X with probability p and according to the rules of RD with probability $(1-p)$.

We expect that for this family of models, the dynamics of the RD process would play the same role as in the cases of the RDSR-RD and the BD-RD models. That is, RD causes the slowing down of correlations among particles. So, we also expect that Eqs. (2) and (3) would be valid, but the values of the exponents α and Z entering in the scaling relationships have to be those of the X model that introduces the correlations among particles. Furthermore, we expect that the general relationship given by Eq. (9) will also hold, allowing us to evaluate the exponent y after the determination of δ.

Figure 4 shows log-log plots of W_{sat} versus p, in $d=1$, as obtained for the family of models of the type X-RD where

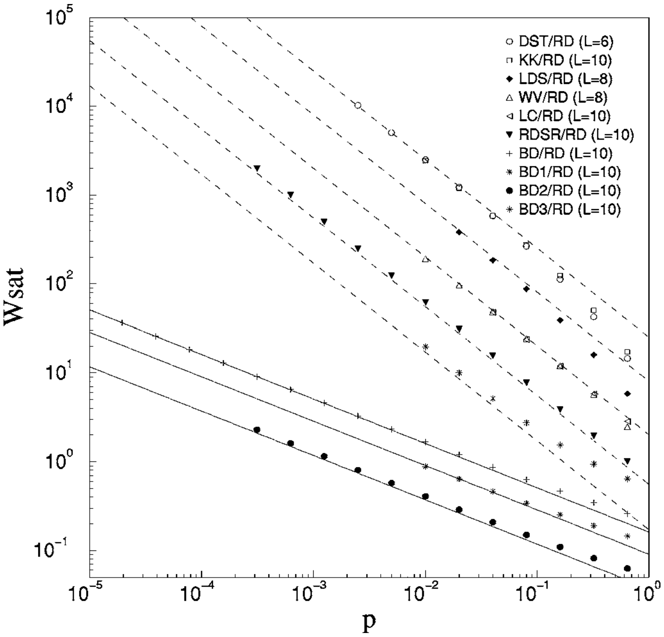

FIG. 4. Log-log plot of W_{sat} versus p for the following competitive models in dimension $d=1$: DST-RD for $L=6$, KK-RD for $L=10$, LDS-RD for $L=8$, WV-RD for $L=8$, LC-RD for $L=10$, RDSR-RD for $L=10$, BD-RD for $L=10$, BD1 for $L=10$, BD2 for $L=10$, BD3 for $L=10$. Different plots have been shifted vertically for the sake of clarity.

X=Das Sarma–Tamboronea (DST) [34], Kim-Kosterlitz (KK) [35], Lai–Das Sarma (LDS) [36], Wolf-Villain (WV) [38], large curvature (LC) [39], RDSR [12], BD [1], BD1, BD2, BD3. The last three models are variants of the BD model and the difference between them is due to the rules used for the sticking of the particles that are shown schematically in Fig. 5. In fact, for the BD1 model, when the height of a selected site for the deposition of a particle is lower than that of a neighboring site, the particle becomes stuck at half height between the selected site and the highest neighbouring site. For the case of the BD2 model, when the height of the site selected to deposit a particle is lower than that of a neighboring site, the particle becomes stuck to the highest neighboring site but the actual deposition height is decreased by one lattice unit. Finally, for the BD3 model, when the selected site for the deposition of a particle is lower than a neighboring site, the particle sticks leaving a single hole in

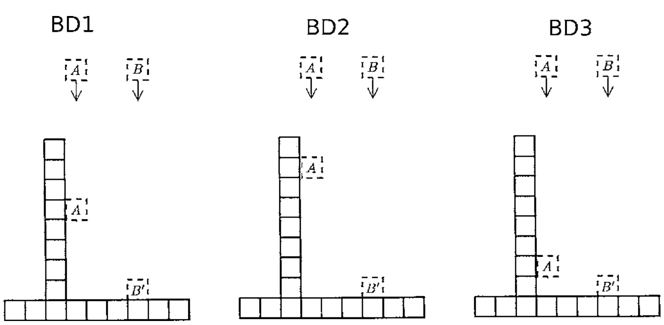

FIG. 5. Schematic view of the deposition of particles in the BD1, BD2, and BD3 models. More details in the text.

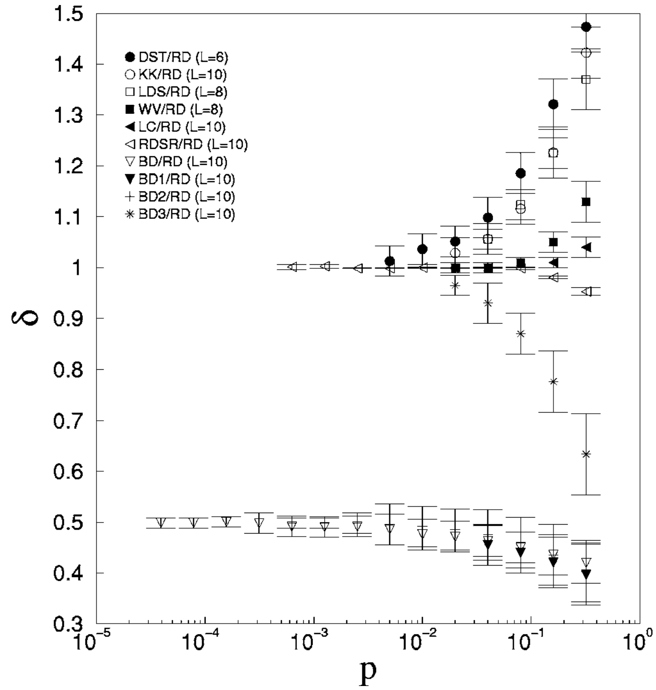

FIG. 6. Log-linear plot of δ versus p for the following competitive models in dimension $d=1$: DST-RD for $L=6$, KK-RD for $L=10$, LDS-RD for $L=8$, WV-RD for $L=8$, LC-RD for $L=10$, RDSR-RD for $L=10$, BD-RD for $L=10$, BD1 for $L=10$, BD2 for $L=10$, and BD3 for $L=10$.

the selected column, so after deposition the height of the selected site becomes enlarged by two lattice units. (See also Fig. 5.)

As shown in Fig. 4, the behavior of δ exhibits two relevant features. On the one hand, the obtained values of δ are independent of the universality class of the competitive model that introduces the correlations to the aggregate. So, models belonging to different universality classes may have the same value of δ, while models belonging to the same universality class may have different values of δ. On the other hand, the exponents δ obtained in the $p\rightarrow 0$ limit can only assume two possible values that are very close to $\delta=1/2$ and $\delta=1$. Further support to this statement is given in Fig. 6 that shows plots of the effective value of the exponent δ obtained for the family of X-RD models. So, we conjectured that in the $p\rightarrow 0$ limit the exact values for δ should be $1/2$ or 1, depending on the model. Furthermore, due to the large number of models considered, it would not be surprising that the exponent δ for all models of the type X-RD may assume one of the already found values.

V. ANALYTICAL CALCULATION OF THE EXACT VALUES OF δ IN THE $p\rightarrow 0$ LIMIT

In order to evaluate the exact values of δ we have used three concepts linking the interface width (W) to the height difference between two neighbouring sites $[h(i)-h(i+1)]$ in a competitive growth model.

First, we take into account that from the statistical point of view, it is easy to show that if the interface width stops its growth (i.e., by reaching a saturation value), then the height difference between two neighboring sites must stop its growth too. Secondly, we considered that if the growing stage of the interface width can be described by a power law of the type $W(t)\propto t^b$, then one also has that $|h(i,t)-h(i+1,t)|\propto t^b$. Finally, if the dependence of the interface width on p is given by another power law, namely, $W(p)\propto p^b$, then the relationship $|h(i,p)-h(i+1,p)|\propto p^b$ must also hold. It is worth mentioning that the inverse relationships are also valid in all cases.

So, based on these statements we studied the behavior of $|h(i)-h(i+1)|$ in order to obtain the relevant properties of the interface width. The simplest example for the application of the above-discussed concepts is provided by the RD model. Here, the height difference between two neighboring sites due to the deposition of a single particle can only be increased or reduced by one lattice unit. So, this height difference corresponds to the displacement of a typical random walk. This fact allows us to conclude that if $h(i,t=0)-h(i+1,t=0)=0$ (as usual), then $\langle[h(i,t)-h(i+1,t)]^2\rangle\propto t$ and using this result we conclude that $W(t)\propto t^{1/2}$, in agreement with the fact that $\beta_{RD}=1/2$.

Considering a competitive growth model, let us call "RD particles" and "X particles" those particles that are deposited according to RD rules and X rules, respectively. Since in the $p\to 0$ limit the deposition probability of RD particles is larger than that of X particles, the deposition process in an X-RD model can be thought as cycles involving the deposition of n_i RD particles followed by a single X particle (n_i is the number or RD particles in the cycle i). Also, it is clear that from the statistical point of view, the evolution through this kind of cycles is equivalent to the evolution of any hypothetic model such that the same number $\langle n\rangle$ of RD particles followed by an X particle is deposited during each cycle, where $\langle n\rangle$ are the average number of RD particles given by

$$\langle n\rangle=\sum_{n=0}^{\infty} n(1-p)^n p=\frac{1-p}{p}. \tag{10}$$

Since the statistical evolution of the height difference $\langle h(i)-h(i+1)\rangle$ is related to the saturation width of the interface according to

$$W_{sat}(p)\propto \sqrt{(\langle[h(i)-h(i+1)]^2\rangle)}, \tag{11}$$

our aim is to show that the behavior of $W_{sat}(p)$ can also be derived from the statistical knowledge of $[h(i)-h(i+1)]$ as a function of the deposition cycles.

On the one hand, it is straightforward to show that for models such as BD-RD, BD1-RD, and BD2-RD, the behavior of $h(i)-h(i+1)$ in each cycle corresponds to a random walk that after $\langle n\rangle$ steps returns either (i) to its initial position, (ii) to a point placed at a distance that is just half of the maximum distance reached from its initial position, or (iii) to a neighboring site of its initial position. So, these types of competitive models $\langle h(i)-h(i+1)\rangle$ correspond to a random walk that in each cycle walks $\langle n\rangle$ steps and after that returns to a site placed at a certain distance (from the starting point) that is proportional to the maximum distance reached from its initial position, a being the proportionality constant. Hereafter, this kind of random walk is called "type A." Now, when the number of cycles is large enough the saturation of the interface width is expected to occur and consequently one has

$$\langle[h(i)-h(i+1)]^2\rangle=\langle n\rangle\sum_{j=1}^{\infty} a^j=\langle n\rangle\frac{a}{1-a}, \tag{12}$$

where j is the number of cycles. So, using Eq. (10) it follows that Eq. (12) is equivalent to $\langle[h(i)-h(i+1)]^2\rangle\propto 1/p$ ($p\to 0$), so that Eq. (11) gives

$$W(t)\propto p^{-1/2}. \tag{13}$$

On the other hand, it is easy to show that for the BD3-RD, TDS-RD, LDS-RD, KK-RD, WV-RD, LC-RD, and RDSR-RD models, the behavior of $\langle h(i)-h(i+1)\rangle$ in each cycle corresponds to a random walk that after $\langle n\rangle$ steps returns to a site placed at a distance $\langle l\rangle$ from the maximum distance reached during the walk (we call this random walk "type B"). So, after the first cycle one has

$$\langle\{[h(i)-h(i+1)]_1\}^2\rangle=(\sqrt{\langle n\rangle}-l)^2, \tag{14}$$

then after the second cycle it follows that

$$\langle\{[h(i)-h(i+1)]_2\}^2\rangle=[\sqrt{\langle n\rangle+(\sqrt{\langle n\rangle}-l)^2}-l]^2, \tag{15}$$

and so on. The analytic form of $\langle[h(i)-h(i+1)]^2\rangle$ when the number of cycles is large enough can be obtained by following a simple procedure: Since after reaching saturation $\langle h(i)-h(i+1)\rangle$ stops growing, in the next cycle after saturation the increment of $\langle h(i)-h(i+1)\rangle$ due to the deposition of $\langle n\rangle$ RD particles is equal to the effect caused by the subsequent X particle. As we have already discussed, the effect of the additional X particle is just to move towards its initial point l step. So, if before the deposition of the new particle one had $\langle[h(i)-h(i+1)]^2\rangle=c^2$ (where c is a constant), then after that deposition one already has $\langle[h(i)-h(i+1)]^2\rangle=(c-l)^2$. Then, the difference between them is given by $\langle\{[h(i)-h(i+1)]^2\}_{before}\rangle-\langle\{[h(i)-h(i+1)]^2\}_{after}\rangle\sim 2l\langle[h(i)-h(i+1)]\rangle$, and it has to be equal to the effect of the $\langle n\rangle$ RD particles. So,

$$\langle[h(i)-h(i+1)]\rangle\propto\langle n\rangle, \tag{16}$$

and using Eq. (11) one obtains

$$W_{sat}(p)\propto p^{-1}. \tag{17}$$

Therefore, we conclude that for all models that can be mapped into random walks of types A and B one has that $\delta=1/2$ [see Eq. (13)] and $\delta=1$ [see Eq. (17)], respectively. These results provide an independent, more general, confirmation of the exact values of the exponents obtained very recently for the RDSR-RD and BD-RD models [33]. Of course, by using Eq. (9) one can also obtain the exact value of the exponent y for any model that can be mapped into the two types of random walks already considered.

TABLE I. Summary of stochastic equations corresponding to models belonging to four different universality classes such that the competitive process can be represented by random walks of type A and B. More details in the text.

Universality class	Random walk A	Random walk B
Edwards-Wilkinson	$\partial h(\mathbf{x},t)/\partial t=\nu_0 p\boldsymbol{\nabla}^2 h(\mathbf{x},t)+\eta(\mathbf{x},t)$	$\partial h(\mathbf{x},t)/\partial t=\nu_0 p^2\boldsymbol{\nabla}^2 h(\mathbf{x},t)+\eta(\mathbf{x},t)$
Kardar-Parisi-Zhang	$\partial h(\mathbf{x},t)/\partial t=\nu_0 p\boldsymbol{\nabla}^2 h(\mathbf{x},t)+\lambda p^{3/2}[\nabla h(\mathbf{x},t)]^2+\eta(\mathbf{x},t)$	$\partial h(\mathbf{x},t)/\partial t=\nu_0 p^2\boldsymbol{\nabla}^2 h(\mathbf{x},t)+\lambda p^3[\boldsymbol{\nabla} h(\mathbf{x},t)]^2+\eta(\mathbf{x},t)$
Linear MBE	$\partial h(\mathbf{x},t)/\partial t=-Kp\boldsymbol{\nabla}^4 h(\mathbf{x},t)+\eta(\mathbf{x},t)$	$\partial h(\mathbf{x},t)/\partial t=-Kp^2\boldsymbol{\nabla}^4 h(\mathbf{x},t)+\eta(\mathbf{x},t)$
Nonlinear MBE	$\partial h(\mathbf{x},t)/\partial t=-Kp\boldsymbol{\nabla}^4 h(\mathbf{x},t)+\lambda_1 p^{3/2}\boldsymbol{\nabla}^2[\nabla h(\mathbf{x},t))^2+\eta(\mathbf{x},t)$	$\partial h(\mathbf{x},t)/\partial t=-Kp^2\nabla^4 h(\mathbf{x},t)+\lambda_1 p^3\boldsymbol{\nabla}^2[\boldsymbol{\nabla} h(\mathbf{x},t)]^2+\eta(\mathbf{x},t)$

VI. PHENOMENOLOGICAL STOCHASTIC GROWTH EQUATIONS

It is well known that the stochastic growth equations describing the BD-RD and the RDSR-RD models are the KPZ (see Eq. (5)) and the Edwards-Wilkinson (EW) [see Eq. (4)] equations, respectively. So, both models belong to the same universality class as that of the model that introduces the correlations among particles, namely, the X model with X =BD and X=RDSR [30–33]. Also, using scaling arguments on p and the values of δ and y we have found that for the BD-RD model the parameter p appears in the linear and nonlinear terms of the stochastic equation, taking the form $\nu(p)=\nu_0 p$ and $\lambda(p)=\lambda p^{3/2}$ [see Eq. (5)], while for the RDSR-RD model the parameter p appears as a factor of the form $\nu(p)=\nu_0 p^2$ [see Eq. (4)] [30–33].

The scaling argument on p implies that, assuming the interface $h(\mathbf{x},p,t)$ to be self-similar, on rescaling the coordinate p according to

$$p \rightarrow p' \equiv cp \tag{18}$$

and the height according to

$$h \rightarrow h' \equiv c^{-\delta}h, \tag{19}$$

one should obtain an interface that is statistically indistinguishable from the original one. Since the interface roughness depends on time t as well, one should have

$$t \rightarrow t' \equiv c^{-y}t. \tag{20}$$

So, for any competitive model belonging to the X-RD family we can propose a stochastic growth equation similar to its X model equation where the parameter p appears in the linear and nonlinear terms. After that, by using the scaling arguments on p and the exact values of δ and y, it is possible to obtain the exact dependence of the prefactors on p for any stochastic equation. This systematic procedure and the results of the previous section allows us to conclude that the behavior of $[h(i)-h(i+1)]$ quantitatively determines the exact dependence of the stochastic equation on p.

Summing up, by using this procedure we have found that in the cases of BD1-RD, BD2-RD, and BD-RD models, the KPZ stochastic growth equation is given by

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = F + \nu_0 p\boldsymbol{\nabla}^2 h(\mathbf{x},t) + \frac{\lambda p^{3/2}}{2}[\boldsymbol{\nabla} h(\mathbf{x},t)]^2 + \eta(\mathbf{x},t). \tag{21}$$

On the other hand, for the KK-RD and BD3-RD models, the KPZ stochastic growth equation is given by

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = F + \nu_0 p^2\boldsymbol{\nabla}^2 h(\mathbf{x},t) + \frac{\lambda p^{3}}{2}[\boldsymbol{\nabla} h(\mathbf{x},t)]^2 + \eta(\mathbf{x},t). \tag{22}$$

Also, for the case of LDS-RD, WV-RD, LC-RD, and RDSR-RD models, the EW stochastic growth equation is given by

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = F + \nu_0 p^2\boldsymbol{\nabla}^2 h(\mathbf{x},t) + \eta(\mathbf{x},t), \tag{23}$$

while for the case of the DST-RD model, the stochastic growth equation is given by

$$\frac{\partial h}{\partial t} = -Kp^2\boldsymbol{\nabla}^4 h(\mathbf{r},t) + \lambda_1 p^3\boldsymbol{\nabla}^2[\boldsymbol{\nabla} h(\mathbf{r},t)]^2 + \eta(\mathbf{r},t). \tag{24}$$

Finally, in Table I we have summarized the list of all possible stochastic equations resulting for competitive deposition models belonging to four different universality classes when the competitive process can be mapped into the two types of random walks already considered.

VII. CONCLUSIONS

We have studied a wide family of competitive growth models (generically called X-RD models) where particles of the same type are aggregated according to the rules of a generic discrete model X with probability p and according to the rules of random deposition (RD) with probability $(1-p)$.

First, we have focussed our study on the properties of the exponents related to the interface width $W_{\text{sat}} \propto p^{-\delta}$ and the characteristic crossover time $t_{x2} \propto p^{-y}$. We have shown that both exponents are not independent and one has that the exact relationship $\delta = y\beta_{\text{RD}}$ ($\beta_{\text{RD}}=1/2$) holds for the BD-RD and RDSR-RD models. However, we expect that the above relationship would hold for any competitive growth model of the type X-RD.

Also, we have found that the values of the exponent δ do not significantly depend on the finite size of the sample. This property has allowed us to systematically study competitive growth models using lattices of modest size. This study shows that δ exhibits universality and its values, in the limit $p\rightarrow 0$, are restricted to either $\delta=1/2$ or $\delta=1$, depending on the model considered. Furthermore, by using a correspondence between two neighboring sites in the discrete model $\{[h(i)-h(i+1)]\}$ and two types of random walks, we have determined the exact values of the exponent δ. When the height difference between two neighbor sites corresponds to a random walk of type A that in each cycle walks $\langle n\rangle$ steps and after that returns to a point at a distance proportional to its initial position, one has $\delta=1/2$ and consequently $y=1$. On the other hand, when the height difference between two neighboring sites corresponds to a random walk of type B that after $\langle n\rangle$ steps moves towards its initial position l steps, we have found that $\delta=1$ and $y=2$.

Finally, using the exact values for the exponents δ and y, as well as a scaling argument on p, we have derived the stochastic growth equations for the whole family of competitive models studied. So, we conclude that the properties of the height difference at saturation $[h(i)-h(i+1)]$ in the discrete model, which determines the behavior of W_{sat}, have allowed us to quantitatively determine the exact dependence on p of the coarse-grained stochastic equations.

It is worth mentioning that the derivation of this type of coarse-grained stochastic equations, based on the growing rules of the corresponding microscopic models, is an interesting challenge in the field of modern Statistical Physics. So, we expect that the relationships between microscopic parameters and stochastic equations obtained and discussed in this work will stimulate studies of this field.

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