Static configurations and nonlinear waves in rotating nonuniform self-gravitating fluids

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The equilibrium states and low-frequency waves in rotating nonuniform self-gravitating fluids are studied. The effect of a central object is included. Two-dimensional static configurations accounting for self-gravity, external gravity, and nonuniform rotation are considered for three models connecting the pressure with the mass density: thermodynamic equilibrium, polytropic pressure, and constant mass density. Explicit analytical solutions for equilibrium have been found in some cases. The low-frequency waves arising due to the vertical and horizontal fluid inhomogeneities are considered in the linear and nonlinear regimes. The relationship between the background pressure and mass density is supposed to be arbitrary in the wave analysis. It is shown that the waves considered can be unstable in the cases of polytropic pressure and constant mass density. The additional nonlinear term proportional to the product of the pressure and mass density perturbations, which is usually omitted, is kept in our nonlinear equations. There have been found conditions for this term to be important. Stationary nonlinear wave equations having solutions in the form of coherent vortex structures are obtained in a general form. The importance of involving real static configurations in the consideration of wave perturbations is emphasized.

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I. INTRODUCTION

An investigation of nonlinear dynamics of rotating geophysical and astrophysical fluids has great importance for understanding the creation of nonlinear structures of various scale sizes in our environment. Rotating objects are typical in the Universe: planets and their atmospheres, giant molecular and interstellar clouds, galaxies and clusters of galaxies [1–4]. In rotating systems, there may exist various complex phenomena, such as atmospheric and ocean vortices, eddies, the Jovian Great Red Spot, and Venusian "hot spots" [5–13]. Coherent vortices may be formed in rotating self-gravitating objects with extended mass distribution [14–17]. Laboratory experiments with a rotating fluid help to better understand the natural phenomena [18,19]. Enhanced zonal flows in a rotating fluid have been observed in laboratory experiments [20–22] and numerical simulations [22,23]. A theoretical consideration of large scale zonal flow generation by low-frequency (in comparison with the Coriolis frequency) propagating wave modes in nonuniform rotating fluids has been carried out in Refs. [24,25].

The low-frequency waves and nonlinear wave structures in nonuniform rotating fluids may depend on the background gradients of pressure, mass density, Coriolis frequency, and so on. Therefore, it is important to know the static background configurations. For nonrotating self-gravitating charged fluids this problem has been, in particular, considered in Ref. [26] for the case of Cartesian one-dimensional symmetry. The equilibrium of rotating self-gravitating fluids depends also on the rotation frequency and the presence of a central mass. The study of two- and three-dimensional equilibriums and perturbations of such fluids is more adequate to real situations.

In the present paper we consider static configurations and low-frequency waves in rotating nonuniform self-gravitating fluids. The possible existence of a central object is also included. We take into account the nonuniformity of the azimuthal mass flow. The full system of the equations for a self-gravitating neutral fluid including the equation for the pressure is used. Thus, the relationship between the equilibrium pressure and mass density is arbitrary in our model. Two-dimensional static configurations accounting for self-gravity, external gravity, and nonuniform rotation are considered for three models connecting the pressure with mass density: thermodynamic equilibrium, polytropic pressure, and constant mass density. The low-frequency waves arising due to the vertical and horizontal fluid inhomogeneity are considered in the linear and nonlinear regimes. A self-consistent relationship between the pressure and mass density disturbances is used. We keep an additional nonlinear term proportional to the product of pressure and mass density perturbations, which is usually omitted, and show when this term is important for the evolution of perturbations. The equations describing the nonlinear steady states are obtained in a general form. These equations have solutions in the form of coherent vortex structures.

Our paper is organized as follows. In Sec. II we introduce the basic equations and study the various two-dimensional static configurations of a rotating gravitating fluid (one case is a three-dimensional one). The linear and nonlinear stages for the waves arising due to the vertical (along the rotation axis) inhomogeneity are considered in Sec. III with neglect of rotation. In Sec. IV the same procedure is carried out in the geostrophic approximation for waves arising due to horizontal inhomogeneity. The solutions of the stationary nonlinear equations are briefly discussed in Sec. V. In Sec. VI the results obtained are summarized.

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II. BASIC EQUATIONS: EQUILIBRIUM CONFIGURATIONS

We start with the following set of equations in the rotating reference frame:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = 2\mathbf{v}\times\mathbf{\Omega}_0 - \frac{\nabla p}{\rho} - \nabla\psi - \nabla U + \nabla\Phi_0 + \mu\nabla^2\mathbf{v}, \tag{1}$$

the momentum equation,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{v} = 0, \tag{2}$$

the continuity equation,

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p + \gamma p\,\nabla\cdot\mathbf{v} = 0, \tag{3}$$

the equation for the pressure, and

$$\nabla^2\psi = 4\pi G\rho, \tag{4}$$

the Poisson equation.

Here $\mathbf{v}$ is the fluid velocity, ρ is the mass density, p is the pressure, $\mathbf{\Omega}_0=\mathbf{z}\Omega_0$, Ω_0 is some angular frequency of the differential fluid rotation (see below), the unit vector $\mathbf{z}$ is directed along the vertical rotation axis z, ψ is the self-gravity potential, $U=-GM/R$ is the gravity potential of the central object having mass M, $R=(r_\perp^2+z^2)^{1/2}$, $\Phi_0=(1/2)\Omega_0^2 r_\perp^2$ is the potential of the centrifugal force, the index $\perp$ marks the direction across the z axis, $r_\perp$ is the distance from the rotation axis, z is the coordinate from the symmetry horizontal plane, μ is the kinematic viscosity, γ is the adiabatic constant, and G is the gravitational constant. We use the cylindrical coordinate system.

Let us first consider the background stationary states. Suppose that the stationary fluid velocity $\mathbf{v}_0$ (the index 0 here and below denotes the equilibrium value) is directed along the azimuthal direction and depends on the radial coordinate (the differential rotation): $\mathbf{v}_0=\mathbf{i}_\theta v_{0\theta}(r)$ (the index $\perp$ on $r_\perp$ here and below is omitted), where $\mathbf{i}_\theta$ is the unit vector along the azimuthal direction (θ is the azimuthal angle). Let $v_{0\theta}(r_0)$ be zero. Then $\Omega_0 r_0=V_0(r_0)$, where $V_0(r)$ is the fluid velocity in the rest reference frame. Taking into account the shear velocity $\mathbf{v}_0$ and neglecting the small viscosity effect on the background state, we obtain the stationary momentum equation (1) in the form

$$\frac{\nabla p_0}{\rho_0} = -\nabla\psi_0 - \nabla U + \Omega^2(r)\mathbf{r}, \tag{5}$$

where $\Omega(r)=\Omega_0+v_{0\theta}(r)/r=V_0(r)/r$. Let us apply the operator $\nabla\cdot$ to Eq. (5) and use Eq. (4) for the equilibrium values. The result is

$$\nabla\cdot\frac{\nabla p_0}{\rho_0} = -\omega_j^2 - \nabla^2 U + \nabla\cdot\Omega^2(r)\mathbf{r}, \tag{6}$$

where $\omega_j=(4\pi G\rho_0)^{1/2}$ is the Jeans frequency [27]. For convenience we have retained the second term on the right-hand side of Eq. (6), which is equal to zero for $R\neq 0$. We have one equation (6) and two variables p_0 and ρ_0. Therefore, it is necessary to set some additional equations of state. Below we consider three cases.

A. Thermodynamic equilibrium

Suppose that the temperature T_0 along the system is constant. Then we obtain from Eq. (6) [or from Eqs. (4) and (5)] the following equation in dimensionless form:

$$\nabla'^2\eta_0 = -\exp(\eta_0+\lambda). \tag{7}$$

Here $\eta_0=-\psi_0/c_{s0}^2=\ln(\rho_0/\rho_{00})-\lambda$, $\lambda=c_{s0}^{-2}W$, $W=-U+\int dr\,\Omega^2(r)r$, $\mathbf{r}=r_D\mathbf{r}'$, $r_D=c_{s0}/\omega_{j0}$, $c_{s0}=(p_0/\rho_0)^{1/2}$ is the sound velocity, $\omega_{j0}=(4\pi G\rho_{00})^{1/2}$, and ρ_{00} is a constant. Equation (7) at $\lambda=0$ coincides with the corresponding equation in Ref. [26] in the limit $\mathbf{B}_0=\mathbf{0}$ ($\mathbf{B}_0$ is the magnetic field).

The right-hand side of Eq. (7) is different from zero, if $\omega_j\neq 0$. Thus, the dependence of the value η_0 on coordinates arises due to the self-gravity. When $\omega_j=0$ the solution for η_0 is $\eta_0=0$, i.e., $\rho_0=\rho_{00}e^{\lambda}$.

We derive now the two-dimensional axisymmetric solution of Eq. (7) in the particular case when $\partial^2\eta_0/\partial z'^2 \gg \nabla_\perp'^2\eta_0$. We neglect here the dependence of λ (or U) on the coordinate z, considering the region $r^2\gg z^2$ and supposing that $\omega_{j0}^2\gg\Omega_k^2$ or $3\gg(R_c/r)^3(\rho_c/\rho_{00})$, where $\Omega_k=(GM/r^3)^{1/2}$ is the Kepler frequency, and R_c and ρ_c are the radius and the mass density of the central object, respectively. The last two inequalities are obtained from Eq. (5), and denote that the vertical stratification is determined by self-gravity. These inequalities are (not) needed, if the central object is (not) present. Then, the radial inhomogeneity enters into Eq. (7) parametrically. We find the solution of Eq. (7) as

$$\eta_0 = -2\ln\cosh\xi,$$

where $\xi=\xi(z,r)=(z/\sqrt{2}r_D)e^{\lambda/2}$. The solution for the fluid mass density has the form

$$\rho_0(z,r) = \rho_{00}\frac{e^{\lambda}}{\cosh^2\xi}. \tag{8}$$

The solution (8) may be applied in the limit $\omega_j\to 0$ ($\xi\to 0$) (see above). In the case $\lambda=0$ this solution coincides with the one-dimensional solution for ρ_0 obtained in Ref. [26]. For finite ω_j we can estimate from (8) the thickness of the layer Δz from the condition $\lambda-2\xi(\Delta z)=-1$. Thus, $\Delta z=2^{-1/2}r_D(1+\lambda)e^{-\lambda/2}$. We see that for $\lambda\leqslant 1$ the thickness is $\Delta z\sim r_D$, and the fluid layer becomes flat along the z axis for $\lambda\gg 1$ (we assume $\lambda>0$). The condition justifying the neglect of the transverse operator in Eq. (7) for axisymmetric solutions and finite ω_j has for the whole object ($z\sim\Delta z$) the following form: $3r_D^{-2}\gg(1+\lambda)|\partial\lambda/r\partial r+\partial^2\lambda/\partial r^2+(1/2)(\partial\lambda/\partial r)^2|=(1+\lambda)\nu$ (the bars $|\cdots|$ here and below denote the absolute value).

If the condition $\partial\psi_0/\partial r\gg\partial W/\partial r$ is satisfied [see Eq. (5)], we can find the axisymmetric solution of Eq. (7), which is a periodic one in the radial direction. In the region where one may ignore the curvature effect, we obtain an equation which has the form of one of the equations for coherent vortex structures [28]. In this case the solution of Eq. (7) for ρ_0 has the form

$$\rho_0(z,r) = \rho_{00}\left(\cosh\frac{kz}{\sqrt{2}r_D} + \sqrt{1-\frac{1}{k^2}}\cos\frac{kr}{\sqrt{2}r_D}\right)^{-2}, \tag{9a}$$

where $k \geqslant 1$ is an arbitrary constant. Such a solution can take place, in general, if a central mass is absent, or in regions where a fluid moves almost exactly with the Keplerian velocity.

Equation (7) allows also a three-dimensional nonaxisymmetric solution, which is elongated in the radial direction and periodic in the azimuthal direction (so-called spokes). Neglecting the radial part of the operator ∇'^2 in Eq. (7), we find

$$\rho_0(z,y,r) = \rho_{00}e^{\lambda}\left(\cosh k\xi + \sqrt{1-\frac{1}{k^2}}\cos k\zeta\right)^{-2}, \tag{9b}$$

where $\zeta=\zeta(y,r)=(y/\sqrt{2}r_D)e^{\lambda/2}$ [$y=r(\theta-\theta_0)$, where θ_0 is some azimuthal angle]. The conditions for neglecting the radial part in the operator ∇'^2 in addition to that given above can be written in the form $2 \gg |(1/2)(1+\lambda)(\partial\lambda/\partial r)^2-\nu|kyr_D$ and $5 \gg e^{\lambda/2}(\partial\lambda/\partial r)^2k^2y^2$. To obtain these inequalities we have used (as above) the equality $2k\xi(\Delta z)=1+\lambda$. These conditions can be satisfied for a narrow band in the azimuthal direction at $\partial\lambda/\partial r \neq 0$. Note that the density wave structures described by the formulas (9a) and (9b) seem to be similar to the standing density waves seen by Cassini in Saturn's rings [29].

B. Polytropic pressure

Now we take the relationship between the pressure and mass density in the form $p_0=C\rho_0^{\gamma_0}$, where $\gamma_0 \neq 1$ is the adiabatic constant for the static state and C is a constant. Then Eq. (6) may be written as

$$\nabla'^2\delta_0^{\gamma_0-1} = -\delta_0 + \nabla'^2\lambda_{\gamma_0}, \tag{10}$$

where $\delta_0=\rho_0/\rho_{00}$, $\mathbf{r}=r_{D\gamma_0}\mathbf{r}'$, $r_{D\gamma_0}=c_{s\gamma_0}/\omega_{j0}$, $c_{s\gamma_0}=[\gamma_0/(\gamma_0-1)]^{1/2}c_{s0}$, $c_{s0}=(p_{00}/\rho_{00})^{1/2}$, and $p_{00}=C\rho_{00}^{\gamma_0}$. Here the value λ_{γ_0} is $\lambda_{\gamma_0}=c_{s\gamma_0}^{-2}W$.

Equation (10) in the one-dimensional case (in the $\mathbf{z}$ direction) and with $W=0$ has been investigated numerically in Ref. [26]. It was obtained that for $\gamma_0>1$ a self-gravitating fluid has the finite extent $z_{\max}$. However, if $\gamma_0=2$, i.e., $T_0 \sim \rho_0$, and $\nabla\psi_0 \gg \nabla W$ when the self-gravity dominates (in this case the last term on the right-hand side of Eq. (10) can be neglected [see Eq. (5)]), we can find the exact two-dimensional axisymmetric analytical solutions of Eq. (10) for finite extent in the $\mathbf{z}$ direction:

$$\rho_0(z,r) = \rho_{00}J_0\left(\sqrt{1-k^2}\frac{r}{r_{D\gamma_0}}\right)\cos\frac{kz}{r_{D\gamma_0}} \tag{11}$$

for $k \leqslant 1$, and

$$\rho_0(z,r) = \rho_{00}K_0\left(\sqrt{k^2-1}\frac{r}{r_{D\gamma_0}}\right)\cos\frac{kz}{r_{D\gamma_0}} \tag{12}$$

for $k>1$. Here J_0 and K_0 are the zero-order Bessel functions of the first and second kind, respectively. We see from Eqs. (11) and (12) that when $k \ll 1$ we obtain a cylindrical object, for $k \to 1$ we have a disk, and for $k \sim 1$ or >1 the form of the fluid object is close to a ball. Under the conditions mentioned above, Eq. (10) has also a solution decreasing exponentially along the z axis,

$$\rho_0(z,r) = \rho_{00}J_0\left(\sqrt{1+k^2}\frac{r}{r_{D\gamma_0}}\right)\exp\left(-\frac{k|z|}{r_{D\gamma_0}}\right),$$

where $k>0$.

In the opposite case $\nabla\psi_0 \ll \nabla W$, when the effect of the central object and the rotation play the main role, the solution of Eq. (10) has the approximate form

$$\rho_0(z,r) \simeq \rho_{00}(1+\lambda_{\gamma_0}-c_{s\gamma_0}^{-2}\psi_0)^{1/(\gamma_0-1)},$$

where the potential $\psi_0 \ll c_{s\gamma_0}^2(1+\lambda_{\gamma_0})$ and is determined by the equation $\nabla^2\psi_0=\omega_{j0}^2(1+\lambda_{\gamma_0})^{1/(\gamma_0-1)}$.

C. Constant mass density

Here we suppose that the mass density of the object is constant: $\rho_0=\rho_{00}$. This model may be appropriate for a dense fluid with a sufficiently large temperature inhomogeneity. In this case the exact solution of Eq. (5) for the pressure (temperature) accounting for Eq. (4) is

$$p_0(z,r) = p_{00}\left[1-k\frac{z^2}{r_D^2}+\frac{1}{2}\left(k-\frac{1}{2}\right)\frac{r^2}{r_D^2}+\lambda\right], \tag{13}$$

where $k>0$ is an arbitrary constant. The object may have finite sizes (and, in particular, a disk form). In this case the boundary of the object is found from the equation $p_0(z,r,k)=0$. Note that temperature stratification analogous to the solution (13) at $\omega_{j0}=U=0$ and $\Omega=\text{const}$ was used in Ref. [12] for the Venusian atmosphere.

III. NONLINEAR WAVES DUE TO VERTICAL INHOMOGENEITY

A. General equations and conditions of consideration

A thin (in the vertical $\mathbf{z}$ direction) extended layer (disk) has typical inhomogeneity length along the z axis, $L_z=|\partial\ln\rho_0/\partial z|^{-1}$, much smaller than that in the horizontal direction, $L_\perp=|\partial\ln\rho_0/\partial r|$: $L_z \ll L_\perp$. In Sec. II we have found some possible static configurations, which can have the disk form [see expressions (8) and (11)–(13) in the corresponding limits]. If the vertical scale size of the object is determined by the self-gravity, and the central object (if it is present) does not play a role [recall that the corresponding condition is $\omega_{j0}^2 \gg -U/r^2$ ($r^2 \gg z^2$)], the value L_z for the isothermal model is $L_z=(r_D/\sqrt{2})e^{-\lambda/2}\coth\xi$. If $\xi \gg 1$ and $\lambda \leqslant 1$ we have $L_z \sim r_D$. In the case of the disk configuration (11) or (12) ($k \to 1$) $L_z=r_D|\cot(z/r_D)|$. If $z/r_D \sim 1$ we obtain $L_z \sim r_D$. And for the disk solution (13) ($k \to 1/2$, $\lambda<1$) it is $L_z \sim r_D$ for the coordinate $z \sim r_D$. The transverse inhomogeneity length $L_\perp$ depends also on the model used. For thermal equilibrium we have $L_\perp=|(1-\xi\tanh\xi)\partial\lambda/\partial r|^{-1}$. In the case $\gamma_0=2$ [the solutions (11) and (12)] it is $L_\perp \sim r_D(1-k^2)^{-1/2} \ll |\partial\lambda/\partial r|^{-1}$ (as long as $\partial\psi_0/\partial r \gg \partial W/\partial r$). Here we adopt, for estimation,

$J_{0,1} \sim K_{0,1} \sim 1$. When the mass density is constant we obtain for the case $k=1/2$ that $L_\perp \sim |\partial\lambda/\partial r|^{-1}$ ($z \sim r_D$, $\lambda < 1$). In rough form the condition $L_z \ll L_\perp$ for the cases given above may be written as $r_D|\partial\lambda/\partial r| \ll 1$. Note that the atmospheres of planets, as well as rotating disks in polar regions [12], may be considered as thin layers.

In this section we consider disturbances with the frequency proportional to the vertical fluid inhomogeneity. As long as the latter is sufficiently large for thin layers (disks), we adopt here the perturbation frequency ω to be larger than the rotation frequency $\Omega(r)$ [more exactly, $\omega^2 \gg (1/r^3)\partial(r^2\Omega)^2/\partial r$]. Thus, these waves are internal (acoustic) gravity waves involving the self-gravity. Note that the Earth's rotation for acoustic gravity waves in the atmosphere has been taken into account in Ref. [30]. In the following, the relationship between the background pressure and mass density is arbitrary. The perturbations are supposed to have a small but finite amplitude: $p(\rho,\psi)=p_0(\rho_0,\psi_0)+\delta p(\delta\rho,\delta\psi)$, where the values with the index δ are the perturbations, and $p_0(\rho_0,\psi_0) \gg \delta p(\delta\rho,\delta\psi)$. Fluid motion in the waves under consideration is almost incompressible, i.e., $\nabla\cdot\delta\mathbf{v} \simeq 0$, where $\delta\mathbf{v}$ is the velocity perturbation ($\nabla\cdot\delta\mathbf{v}$ is small, but finite). Therefore, we may neglect in Eqs. (2) and (3) the nonlinearities proportional to $\nabla\cdot\delta\mathbf{v}$ and keep only the convective nonlinearities. In other words, the condition $\delta\mathbf{v}\cdot\nabla\delta\rho(\delta p) \gg \delta\rho(\delta p)\nabla\cdot\delta\mathbf{v}$ is supposed to be satisfied. As an example, we consider two-dimensional perturbations in the vertical and radial directions elongated upon the azimuth. We neglect the influence of the curvature effect on perturbations, considering the regions that are further from the rotation axis than the radial wavelength of perturbations. Due to the weak compressibility we may introduce the stream function φ: $\delta v_x \simeq \partial\varphi/\partial z$ and $\delta v_z \simeq -\partial\varphi/\partial x$ (for convenience we have substituted r by x). Then, Eqs. (2) and (3) for $\delta\rho$ and δp take the form

$$\frac{\partial\delta\rho}{\partial t} + \{\rho_0,\varphi\} + \rho_0\nabla\cdot\delta\mathbf{v} + \{\delta\rho,\varphi\} = 0, \tag{14}$$

$$\frac{\partial\delta p}{\partial t} + \{p_0,\varphi\} + \gamma p_0\nabla\cdot\delta\mathbf{v} + \{\delta p,\varphi\} = 0. \tag{15}$$

Here the curly brackets denote

$$\{a,b\} = \frac{\partial a}{\partial x}\frac{\partial b}{\partial z} - \frac{\partial a}{\partial z}\frac{\partial b}{\partial x}.$$

From the momentum equation (1) we can obtain the equation for the vorticity $\nabla^2\varphi$. Differentiating the x component of Eq. (1) over z, the z component over x, subtracting the obtained equations one from the other, and using the stream function φ and the condition $\rho_0 \gg \delta\rho$, we find

$$\left(\frac{\partial}{\partial t} - \mu\nabla^2\right)\nabla^2\varphi = \left\{\frac{1}{\rho_0},\delta p\right\} + \left\{p_0,\frac{\delta\rho}{\rho_0^2}\right\} + \{\varphi,\nabla^2\varphi\} + \left\{\delta p,\frac{\delta\rho}{\rho_0^2}\right\}, \tag{16}$$

where $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial z^2$. We keep the nonlinear term proportional to $\{\delta p,\delta\rho\}$. Below we show that in a self-gravitating fluid this term for the waves under consideration can have the same order of magnitude as the ordinary term $\{\varphi,\nabla^2\varphi\}$. Differentiating further the x component of Eq. (1) over x, the z component over z, and summing up the obtained equations, we derive the last desirable equation

$$\left(\frac{\partial}{\partial t} - \mu\nabla^2\right)\nabla\cdot\delta\mathbf{v} + 2\left(\frac{\partial^2\varphi}{\partial x\partial z}\right)^2 - 2\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial z^2} = -\nabla\cdot\frac{1}{\rho_0}\nabla\delta p + \nabla\cdot\frac{\delta\rho}{\rho_0^2}\nabla p_0 - \omega_j^2\frac{\delta\rho}{\rho_0}, \tag{17}$$

where $\omega_j=(4\pi G\rho_0)^{1/2}$.

The system of Eqs. (14)–(17) is a closed system of nonlinear equations describing the internal gravity waves in a self-gravitating fluid with an arbitrary relationship between the background pressure and mass density. Below we consider the linear and nonlinear stages of these waves.

B. Linear stage

We can find from Eqs. (14)–(17) in the linear approximation the frequency of oscillations ω. We consider the short wavelength perturbations, for which the conditions $k_\perp L_\perp \gg k_z L_z \gg 1$ are satisfied, where $\mathbf{k}=(k_\perp,k_z)$ is the wave vector of oscillations. Accomplishing the Fourier transformation, we obtain $\omega(\omega+i\mu k^2)=\omega_b^2 k_\perp^2/k^2$, where $k^2=k_\perp^2+k_z^2$ and

$$\omega_b^2 = \frac{(c_s^2\partial\rho_0/\partial z - \partial p_0/\partial z)(k^2\partial p_0/\partial z - \omega_j^2\partial\rho_0/\partial z)}{\rho_0^2(k^2c_s^2-\omega_j^2)}. \tag{18}$$

Here $c_s=(\gamma p_0/\rho_0)^{1/2}$ is the sound speed. The frequency ω_b is the generalization of the Brunt-Väisälä frequency for a self-gravitating fluid. The background gradients of the pressure and mass density can be found from the solutions obtained in Sec. II. If we put $\omega_j=0$, $\partial p_0/\partial z=-g\rho_0$, and $T_0=$const, where g is the gravitational acceleration in the external field, we obtain for ω_b the well-known expression $\omega_b=(\gamma-1)^{1/2}g/c_s$. According to the condition $k_z \gg 1/L_z$ the sound frequency kc_s is larger than the Jeans frequency ω_j for $L_z \sim r_D$. Therefore, the formula (18) describes qualitatively the influence of self-gravity on the frequency of perturbations for wave numbers k_z ($\geq k_\perp$) up to $k_{z\,\min} \sim \omega_j/c_s$. We see from Eq. (18) that the waves can be unstable, if $\partial p_0/\partial z > c_s^2\partial\rho_0/\partial z$. This inequality is satisfied, for example, for the polytropic pressure, if $\gamma_0 > \gamma$, and for the case $\partial\rho_0/\partial z=0$ (see also Refs. [30,31], where the same waves are considered in the Earth's atmosphere). Note that the real frequency ω_b is approximately equal, $\omega_b \sim c_s/L_z \sim \omega_j$, for self-gravitating objects in the **z** direction.

Let us compare the nonlinear terms in Eq. (16). In the local approximation ($k_z L_z \gg 1$) and for $k^2c_s^2 \gg \omega_j^2$ we have from Eqs. (14) and (17) that $\delta p \ll c_s^2\delta\rho$ (the inelastic regime). In this case the first nonlinear term on the right-hand side of Eq. (16) is larger than the second one. But when $kc_s \to \omega_j$ we have $\delta p \sim c_s^2\delta\rho$, and both nonlinear terms have the same order of magnitude. Thus, in the last case the additional nonlinear term influences the evolution of perturbations. Note, however, that the global perturbations with $k \sim L_z^{-1}$ require special consideration.

C. Stationary nonlinear stage

Let us consider Eqs. (14)–(17) (under the same conditions as those in Sec. III B) for stationary nonlinear waves traveling with the velocity u along the x axis. All perturbations depend on variables $x-ut$ and z. Thus, $\partial/\partial t=-u\partial/\partial x$. In the limiting case $k^2c_s^2 \gg \omega_j^2$ we may put $\delta p \simeq 0$. Then, neglecting the viscous effect, we obtain from Eqs. (14)–(16)

$$\{\delta\rho - d_0(z), \varphi - uz\} = 0, \tag{19}$$

$$\{\varphi - uz, \nabla^2\varphi\} + \left\{p_0, \frac{\delta\rho}{\rho_0^2}\right\} = 0, \tag{20}$$

where $d_0(z)=\int dz\, \rho_0 \partial \ln p_0^{1/\gamma}/\partial z - \rho_0$. From Eq. (19) we have $\delta\rho = d_0(z)+f(\varphi-uz)$, where $f(\chi)$ is an arbitrary function. Substituting the last equality in Eq. (20), we find the solution for the vorticity $\nabla^2\varphi$

$$\nabla^2\varphi = g_0(z)\frac{df(\varphi-uz)}{d(\varphi-uz)} + F(\varphi-uz), \tag{21}$$

where $g_0(z)=\int dz\, \rho_0^{-2}\partial p_0/\partial z$, and $F(\chi)$ is an arbitrary function. Below we discuss briefly some possible solutions of Eq. (21).

IV. NONLINEAR WAVES DUE TO HORIZONTAL INHOMOGENEITY

A. General equations and conditions of consideration

In the previous section perturbed fluid motion was considered in the vertical plane. Here we suppose that the perturbed velocity is mainly in the horizontal plane. We study low-frequency perturbations having frequency proportional to the horizontal inhomogeneity of fluid. We assume the wave frequency to be smaller than the typical rotation frequency. Thus, we consider here the geostrophic approximation. As above, fluid motion is weakly compressible for low-frequency perturbations. For the transverse velocity we may introduce the stream function φ: $\delta v_x \simeq \partial\varphi/\partial y$, $\delta v_y \simeq -\partial\varphi/\partial x$, where x (y) is the radial (azimuthal) coordinate (the curvature effect for perturbations is neglected). However, we take also into account the vertical motion. It will be seen below that the perturbations under consideration are flutelike ones along the z axis. In the momentum equation (1) for the perturbed velocity $\delta\mathbf{v}$ the presence of the background shear velocity $v_{0\theta}(r)$ produces some terms due to the Reynolds stress term $\mathbf{v}\cdot\nabla\mathbf{v}$. These terms may be combined with the Coriolis term $\delta\mathbf{v}\times\mathbf{\Omega}_0$. As a result, we may substitute Ω_0 by $\Omega(r)$ for the radial projection of Eq. (1), and by $(1/2r)\partial(r^2\Omega)/\partial r$ for the azimuthal projection. Let it be $\delta v_z/\delta v_\perp \ll L_z/L_\perp$. Under this condition we may only take into account the horizontal derivatives for ρ_0 and p_0 in Eqs. (2) and (3). Keeping in these equations only convective nonlinearities and supposing that $\delta\mathbf{v}_\perp\cdot\nabla_\perp \gg \delta v_z\partial/\partial z$, we obtain for the evolution of $\delta\rho$ and δp Eqs. (14) and (15), where z must be substituted by y in the Poisson brackets, and $\partial/\partial t$ by $\partial/\partial t' = \partial/\partial t + v_{0\theta}\partial/\partial y$.

The equation for the vorticity $\nabla_\perp^2\varphi$, derived from Eq. (1) in the same way as Eq. (16), has the form

$$\left(\frac{\partial}{\partial t'} - \mu\nabla^2\right)\nabla_\perp^2\varphi = 2\{\Omega,\varphi\} + 2\Omega\nabla_\perp\cdot\delta\mathbf{v} + \left\{\frac{1}{\rho_0},\delta p\right\} + \left\{p_0,\frac{\delta\rho}{\rho_0^2}\right\} + \{\varphi,\nabla_\perp^2\varphi\} + \left\{\delta p,\frac{\delta\rho}{\rho_0^2}\right\}. \tag{22}$$

For simplicity, the rotation frequency in Eq. (22) is taken the same for the x and y components of Eq. (1) and equal to $\Omega(r)$. The equation for the vertical velocity δv_z is

$$\left(\frac{\partial}{\partial t'} - \mu\nabla^2\right)\delta v_z = -\frac{\partial\delta p}{\rho_0\partial z} - \frac{\partial\delta\psi}{\partial z} + \{\varphi,\delta v_z\}. \tag{23}$$

In this equation we do not take into account the vertical inhomogeneity of the medium. It is possible for a cylindrical geometry or for disks in the equatorial region (the condition $k_zL_z\delta p \gg c_s^2\delta\rho$ is also sufficient).

The last equation closing the system is found in the same manner as in Eq. (17). In the geostrophic approximation, when the Coriolis force is larger than the inertial and viscous forces, we have

$$2\nabla_\perp\cdot\Omega\nabla_\perp\varphi = -\nabla_\perp\cdot\frac{1}{\rho_0}\nabla_\perp\delta p - \nabla_\perp^2\delta\psi. \tag{24}$$

For these perturbations $\nabla_\perp\delta p \gg (\nabla_\perp p_0/\rho_0)\delta\rho$.

The system of Eqs. (14), (15), and (22)–(24) together with Eq. (4) represents a closed system for waves arising due to the horizontal inhomogeneity and the rotation of the object, and having a finite z dependence. The linear and stationary nonlinear stages are considered below.

B. Linear stage

In the local approximation $\mathbf{k} \gg \partial p_0(\rho_0)/p_0(\rho_0)\partial\mathbf{r}$ the linearized system of equations under consideration results in the following dispersion relation:

$$\omega' + i\mu k_\perp^2 + \frac{4\Omega^2\omega'}{k_\perp^2c_s^2-\omega_j^2} - \frac{4\Omega^2}{\omega'+i\mu k_\perp^2}\frac{k_z^2}{k_\perp^2} = 2c + 2\Omega\frac{a+b}{k_\perp^2c_s^2-\omega_j^2} - \frac{ab}{\omega'(k_\perp^2c_s^2-\omega_j^2)}. \tag{25}$$

Here $\omega'=\omega-k_yv_{0\theta}$, $a=k_ya_x-k_xa_y$, $b=k_yb_x-k_xb_y$, $k_\perp^2c = k_y\partial\Omega/\partial x - k_x\partial\Omega/\partial y$, where $\rho_0\mathbf{a}=c_s^2\nabla_\perp\rho_0-\nabla_\perp p_0$, $\rho_0\mathbf{b} = -\nabla_\perp p_0 + (\omega_j^2/k_\perp^2)\nabla_\perp\rho_0$. In Eq. (25) the condition $4\Omega^2 \gg \omega'^2$ is taken into account. We see from Eq. (25) that for $k_z\neq 0$ the contribution of the vertical movement to the dispersion relation can be significant. It follows from Eqs. (23) and (24) that $\delta v_z/\delta v_\perp \simeq 2\Omega k_z/\omega k_\perp$ (here and below we omit the prime). The contribution of the vertical velocity to the convective nonlinearities may be neglected, if $k_\perp^2 \gg (2\Omega/\omega)k_z^2$. Thus, the perturbations under consideration have short wavelengths in the horizontal direction ($k_\perp^2 \gg k_z^2$) as long as $k_z\neq 0$ for finite objects. Having in mind that for self-gravitating disks $k_{z\,\min}\sim L_z^{-1}\sim r_D^{-1}$, we obtain $k_\perp^2c_s^2 \gg \omega_j^2$. Note that the solutions of Eq. (25) $\omega\sim c$, $a+b$ describe waves of the Rossby type [5].

Equation (25) at $k_z=0$ and $\mathbf{a}=\mathbf{0}$ has been obtained in Ref. [25]. In this case Eq. (25) is of the first order over ω. However, in the general case $\mathbf{a}\neq\mathbf{0}$, so this equation has two branches of oscillations [for $k_z^2 \ll (\omega^2/4\Omega^2)k_\perp^2$ or $\mu=0$]. Omitting the terms proportional to k_z, μ, and supposing that $k_\perp^2 c_s^2 \gg 4\Omega^2$, we find the solution arising due to the last term on the right-hand side of Eq. (25): $\omega^2=-ab/k_\perp^2 c_s^2$. This solution is unstable, if $\nabla_\perp p_0 > c_s^2 \nabla_\perp \rho_0$. In order of magnitude $|\omega| \sim c_s/L_\perp$. Thus, this solution is analogous to that considered in Sec. III. The obtained solution satisfies the assumptions given above accounting for the finite k_z. However, the ordinary solutions connected with the first and second terms on the right-hand side of Eq. (25) do not satisfy the condition for neglecting the term proportional to k_z for disk configurations.

In Ref. [25] the relationship between δp and $\delta\rho$ has been used in the form $\delta p \simeq c_s^2 \delta\rho$. However, such a connection is only satisfied in particular cases. It follows from the corresponding linear equations that

$$\delta p \simeq c_s^2 \delta\rho \frac{2\Omega\omega - a\omega_j^2/k_\perp^2 c_s^2}{2\Omega\omega - a}$$

(in Ref. [25] $a=0$). Using the solutions for ω from Eq. (25) ($k_z=\mu=c=0$), we find this relation in the whole spectrum over $k_\perp$ (for $\omega_j \gg \Omega$ we consider $k_\perp c_s \gg \omega_j$). In the case $\omega_j \gg \Omega$ we have $\delta p \ll c_s^2 \delta\rho$ for $k_\perp c_s \gg \omega_j$ and $\delta p \sim c_s^2 \delta\rho$ for $k_\perp c_s \sim \omega_j$. If $\omega_j \ll \Omega$, then $\delta p \sim c_s^2 \delta\rho$ for $k_\perp c_s \ll \Omega$ and $\delta p \ll c_s^2 \delta\rho$ for $k_\perp c_s \gg \Omega$. We can also compare two nonlinear terms on the right-hand side of Eq. (22), using the linear connections δp and $\delta\rho$ with φ. In the case $\omega_j \gg \Omega$ we find that $\{\varphi, \nabla_\perp^2 \varphi\} \gg (\sim) \{\delta p, \delta\rho\}\rho_0^{-2}$ for $k_\perp c_s \gg (\sim)\omega_j$, and when $\omega_j \ll \Omega$ we obtain $\{\varphi, \nabla_\perp^2\varphi\} \gg (\sim, \ll)\{\delta p, \delta\rho\}\rho_0^{-2}$ for $k_\perp c_s \ll (\sim, \gg)\omega_j^2/\Omega$ and $k_\perp c_s \gg (\sim, \ll)\Omega$.

C. Stationary nonlinear stage

Suppose that all background parameters depend on the coordinate x only. As above, we consider here stationary nonlinear waves traveling along the **y** direction with velocity u. The vertical velocity and viscosity are not taken into account. In the case $k_\perp^2 c_s^2 \gg \omega_j^2$ we find from Eqs. (14), (15), (22), and (24) that

$$\{\delta\rho - h_0(x), \varphi + ux\} = 0, \tag{26}$$

$$\{\varphi + ux, \nabla_\perp^2 \varphi - 2\Omega + q_0(x)\} + \left\{ p_0 - 2\Omega\rho_0\varphi, \frac{\delta\rho}{\rho_0^2} \right\} = 0, \tag{27}$$

where the function $h_0(x)$ is determined by the equation

$$\frac{dh_0}{dx} = \rho_0 \frac{d}{dx} \ln(p_0^{1/\gamma}/\rho_0) + 2\Omega\rho_0 c_s^{-2} u,$$

and the function $q_0(x)$ is equal to $q_0(x)=\int dx(2\Omega/\rho_0) \times (dh_0/dx)$. From Eq. (26) we have $\delta\rho = h_0(x) + f(\varphi+ux)$, where $f(\chi)$ is an arbitrary function. Substituting $\delta\rho$ into Eq. (27), we obtain the following equation for the stream function:

$$\nabla_\perp^2 \varphi = 2\Omega(x) + s_0(x) + w_0(x)\frac{df(\varphi+ux)}{d(\varphi+ux)} + F(\varphi+ux), \tag{28}$$

where $F(\chi)$ is an arbitrary function. The functions $s_0(x)$ and $w_0(x)$ are determined by the equations

$$\frac{ds_0}{dx} = -\frac{4\Omega}{\rho_0^2}\frac{d\rho_0}{dx} h_0,$$

$$\frac{dw_0}{dx} = \frac{1}{\rho_0^2}\frac{dp_0}{dx} + \frac{2\Omega u}{\rho_0}.$$

Note that the last nonlinear term in Eq. (22) must be taken into account [see the second curly brackets in Eq. (27)]. The opposite case $k_\perp^2 c_s^2 \ll \omega_j^2$ we do not consider because $k_\perp \gg k_z$ or $k_\perp c_s \gg \omega_j$ here.

V. SOLUTIONS OF EQS. (21) and (28)

Equations (21) and (28) have a general form. Choosing the concrete functions f and F, one can obtain solutions in the form of various vortices: dipoles, tripoles, vortex chains. These solutions are well-known in the literature (see, for example, Refs. [32,33]), and, therefore, we do not discuss them here. The choice of the arbitrary functions imposes rigid restrictions on the background state [32]. As a rule, the vortex chains are considered to be transverse to the background gradients. However, the vortex chains are also possible along the inhomogeneity. Such an example has been investigated numerically in Ref. [34].

VI. DISCUSSION AND CONCLUSION

In the present paper we have considered the equilibrium and perturbed states of a rotating nonuniform self-gravitating fluid. A central object has also been included. Two-dimensional static configurations have been studied in cases of thermodynamic equilibrium, polytropic pressure, and constant mass density. Configurations have been found in the form of a disk, cylinder, ball, radial wave structure, and azimuthal spokes (three-dimensional case) depending on the parameters of the system.

The disk and ball configurations are typical in the Universe: protoplanetary disks, galaxies, stars, and so on. The solutions for equilibrium obtained in the present paper can be relevant, for example, for protoplanetary disks, some types of spiral galaxies, and ball star clusters. If a central mass is present, the solution (8) is not appropriate in the regions of the inner and outer radial boundaries of the disk, where the conditions of applicability can be violated. Equation (5) is solved, usually, by using the self-consistent field iterative method (see, for example, Ref. [35]). We believe that analytical solutions describing some limited cases are of interest and importance. Note also that the thin structures described by the solutions (9a) and (9b) seem to be similar to the standing density waves seen by Cassini in Saturn's rings [29].

Linear and stationary nonlinear stages of the waves with frequencies proportional to vertical (neglecting the rotation frequency) and horizontal (in the geostrophic approximation) inhomogeneities have been studied. Both these waves can be unstable in the cases of polytropic pressure and constant mass density. We have taken into account the additional nonlinear term $\{\delta p, \delta\rho\}$, which is usually neglected. By using the linear connections between δp and $\delta\rho$, the spectral ranges over $\mathbf{k}$ where this term is important have been found.

The stationary nonlinear equations describing the horizontal and vertical tubes have been obtained here in a general form. These equations have vortexlike solutions under a concrete choice of the arbitrary functions. The additional nonlinear term must be taken into account for the considered stationary nonlinear waves in the geostrophic approximation. Our stationary equations (21) and (28) are similar to those known in the literature (e.g., Refs. [32–34]). Such equations describe vortices observed in experiments. Some experimental evidence for fluids and plasmas can be found, for example, in Refs. [36,37]. At the present time vortices have been investigated numerically in stratified protoplanetary disks [38].

The results obtained in the present paper relate to real situations existing in experiments and the environment. For example, in the framework of Eq. (25) the well-known wave solutions connected with the first and second terms on the right-hand side of this equation do not satisfy the condition for neglecting the contribution of the vertical movement (the term proportional to k_z) for disk configurations. Thus, taking into account the real background states is of great importance in the analysis of perturbations.

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