Stiff three-frequency orbit of the hydrogen atom

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We study a stiff quasiperiodic orbit of the electromagnetic two-body problem of Dirac's electrodynamics of *point* charges. The delay equations of motion are expanded about circular orbits to obtain the variational equations up to nonlinear terms. The three-frequency orbit involves two harmonic modes of the variational dynamics with a period of the order of the time for light to travel the interparticle distance. In the atomic magnitude, these harmonic modes have a frequency that is fast compared with the circular rotation. The quasiperiodic orbit has three frequencies: the frequency of circular rotation (*slow*) and the two fast frequencies of two mutually orthogonal harmonic modes. Poynting's theorem gives a mechanism for a beat of the mutually orthogonal fast modes to cancel the radiation of the unperturbed circular motion by interference. The nonradiation condition for this destructive interference is that the two fast frequencies beat at the circular frequency. The resonant orbits have magnitudes in qualitative and quantitative agreement with quantum electrodynamics (QED), as follows: (i) the orbital angular momenta are integer multiples of Planck's constant to a good approximation, (ii) the orbital frequencies agree with a corresponding emission line of QED within a few percent on average, (iii) the orbital frequencies are given by a difference of two linear eigenvalues, viz., the frequencies of the mutually orthogonal fast modes, and (iv) the angular momentum of gyration of the variational motion about a resonant circular orbit is of the order of Planck's constant.

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I. INTRODUCTION

We study a stiff quasicircular orbit of the electromagnetic two-body problem of Dirac's electrodynamics with retarded-only fields [1], a dynamics with implicitly defined delay. The motivation is to understand this complex dynamics described by the delay equations for particle separations in the atomic magnitude [2]. We give an economical method to derive the variational equations of dynamics about circular orbits up to nonlinear terms. The particular harmonic solutions of the variational dynamics, with a period of the order of the time for light to travel the interparticle distance, are henceforth called the *ping-pong* (PP) modes. For atomic orbits, the frequency of the PP modes turns out to be much faster than the orbital frequency. These modes introduce a fast (*stiff*) time scale in the dynamics and are physically important for the particles to have the option to avoid radiating energy away. The PP modes for vibration along the orbital plane turn out to have *almost* the same frequency of PP modes for vibrations normal to the orbital plane, a remarkable quasidegeneracy that naturally produces beats with a frequency of the order of the orbital frequency. The quasiperiodic orbits of hydrogen have three frequencies: the slow frequency of the circular orbit plus the two fast frequencies corresponding to a planar PP mode and a perpendicular PP mode, as illustrated in Fig. 1. This special combination appears in a mechanism suggested by Poynting's theorem to cancel the radiation of the slow circular motion by interference with a beat oscillation of the two mutually orthogonal PP modes. We investigate the conditions for the nonlinear variational equations to accept such fast gyrating solutions about the circular orbit—i.e., the PP oscillations. After the fast dynamics is established, Poynting's theorem gives a necessary resonance condition to avoid radiative losses; viz., the two mutually orthogonal PP modes must beat at the orbital frequency. This resonance condition turns out to be satisfied precisely in the atomic magnitude. The fast gyration defines an angular momentum vector of the order of the orbital angular momentum of the unperturbed circular orbit. We stress that the point charges are not spinning about themselves, but rather gyrating about a guiding center that is moving along a slow circular orbit, as illustrated in Fig. 1. The stiff three-frequency orbits share several magnitudes with those of the hydrogen atom of quantum electrodynamics (QED) [3], with reasonable precision and qualitative detail. The circular frequency of a resonant orbit agrees with the corresponding line of

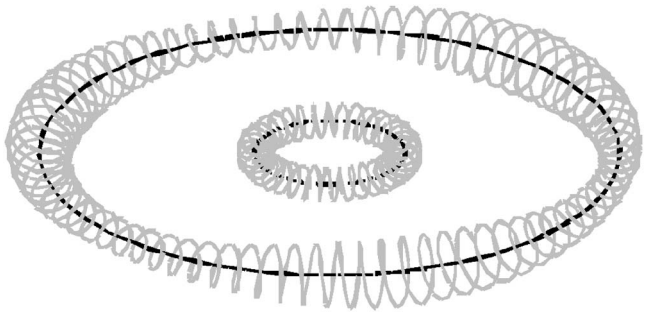

FIG. 1. Guiding-center circular orbit (dark lines) and particle trajectories gyrating about the guiding center (grey lines) for a three-frequency orbit *near* resonance. Illustrated is also the beat of the electron's gyration radius at *about* the orbital frequency. Drawing not to scale; the beat of the protonic gyration radius is not illustrated. Arbitrary units.

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QED within a few percent average deviation. There is also a large body of qualitative agreement with QED: (i) the resonant orbits have orbital angular momenta that are approximate integer multiples of a basic angular momentum (this basic angular momentum agrees well with Planck's constant), (ii) the angular momentum of gyration of the variational motion about the circular orbit is of the order of Planck's constant (this angular momentum of gyration is a vector that rotates at the orbital frequency), and (iii) the emitted frequency is given by a difference of two linear eigenvalues—i.e., the frequencies of the PP modes—analogously to the Rydberg-Ritz principle of QED.

The equations of motion of Dirac's electrodynamics of *point* charges [1] are briefly discussed in Appendix A. After Dirac's 1938 work [1], an early study of Eliezer [4–7] revealed the surprising result that an electron moving in an attractive Coulomb field can never fall into the center of force by radiating energy (henceforth called Eliezer's theorem). It was subsequently found that only scattering states are possible in any tridimensional motion with self-interaction in a Coulomb field [6,7]. Eliezer's theorem strongly suggests that a finite mass for the proton is essential for a physically meaningful dynamics in the electromagnetic two-body problem. When the proton has an infinite mass, there is an inertial frame where it rests at all times, and in this frame the protonic field on the electron is simply a Coulomb field—i.e., the dynamics of Eliezer's theorem. On the other hand, if the proton has a *finite* mass, such inertial frame does not exist and the equations of motion involve delay, because of the finite speed of light. A finite mass for the proton is what brings delay into the electromagnetic two-body dynamics, with its associated ping-pong phenomenon. The infinite-mass limit is a singular limit, because the equations of motion pass from delay equations to ordinary differential equations. This work is an attempt to put together what is lost in this singular limit where the PP modes disappear. We stress that the circular orbit is not an exact solution of the full equations of motion, so that we are not doing Lyapunov stability, but rather constructing particular solutions of the nonlinear variational equations—i.e., the three-frequency orbit.

The road map for this paper is as follows. In Appendix A we review the electrodynamics of point charges in a generalized setting that includes Dirac's theory as a special case and give the equations of motion of point charges in an intuitive form. A nonspecialist reader should start reading the paper from Appendix A. In Sec. II we define the PP modes and the quantities of the circular orbit, to be used as an approximate solution in Secs. III–V. In Sec. III we outline our economical method to derive the variational equations, a method that expands the implicit light-cone condition and uses the action formalism. In Sec. IV we derive the linearized variational equations and study the PP modes of tangent dynamics for vibrations along the orbital plane. This derivation is laborious and makes full use of our economical method plus the use of a symbolic manipulations software. We derive the linearized variational equations in the generalized electromagnetic setting of Appendix A, to compare with previously known results, but we stress that in Sec. V we use only Dirac's electrodynamics, the physically interesting special case. In Sec. V we give an application of Dirac's electrodynamics to atomic physics by discussing the existence of a three-frequency orbit involving mutually orthogonal PP oscillations of finite amplitude—i.e., a particular solution of the variational equations. We investigate the mechanism to cancel the radiation of the circular orbit by interference with a beat of two mutually orthogonal PP modes, a mechanism that starts to operate immediately after the PP dynamics is established. We discuss the necessary resonance condition of Poynting's theorem to avoid radiation—i.e., that the PP modes beat at the orbital frequency. We study the resonant orbits that are stabilized by this mechanism and compare their magnitudes with the magnitudes of QED. In this section we also give a second derivation of Poynting's resonance condition. This derivation averages the angular momentum of gyration over the fast timescale, yielding a vector rotating at a slow frequency. The rotating angular momentum produces a gyroscopic torque on the slow dynamics and introduces the same resonance condition of Poynting's theorem. In Appendix B we derive the tangent dynamics for oscillations perpendicular to the orbital plane, analogously to what is done in Sec. IV for the planar variational equations. These two derivations can be given separately up to the linear order, but the z and xy variational equations are otherwise coupled at higher order. The existence of fast harmonic solutions of finite amplitude is discussed in Appendix C. In Appendix C we also give a third derivation of the resonance condition, a derivation based on the detailed balance of the guiding-center dynamics. Last, in Sec. VI we put the conclusions and discussion.

II. CIRCULAR ORBIT

Our perturbation scheme takes the circular orbit as a first approximation. We use the index $i=1$ to indicate quantities of the electron and $i=2$ for the proton, with masses m_1 and m_2, respectively. We henceforth use a unit system where the speed of light is $c\equiv 1$, the electronic charge is $e_1=-e_2\equiv -1$, and the mass of the electron is $m_1\equiv 1$. In our unit system, the mass of the proton is given by $m_2=1824$, approximately. The circular orbit is illustrated in Fig. 2; the particles move in concentric circles and in diametral opposition at the same time of the inertial frame. The details of the familiar Coulombian circular orbit will be given now. The constant angular velocity is indicated by Ω, the interparticle distance of the light cone is r_b, and the angle that one particle turns while the light emanating from the other particle reaches it is $\theta \equiv \Omega r_b$. The delay angle θ is the natural independent parameter of the circular orbit, which turns out to be small for orbits of the atomic magnitude, $\theta \lesssim 10^{-2}$. For small θ, the interparticle distance of the light cone, r_b, is $O(\theta^2)$ close to the interparticle distance at the same time of the inertial frame, r_0. Because of this $O(\theta^2)$ approximation, the familiar Coulombian formulas with r_b replaced by r_0 yield the leading-order formulas in powers of θ. For example, the orbital frequency, Kepler's law, is given to leading-order in θ by

$$\Omega = \mu\theta^3 + \cdots , \tag{1}$$

where $\mu\equiv m_1m_2/(m_1+m_2)$ is the reduced mass (here and henceforth). In our unit system, $\mu=(1824/1825)\simeq 1$ for hy-

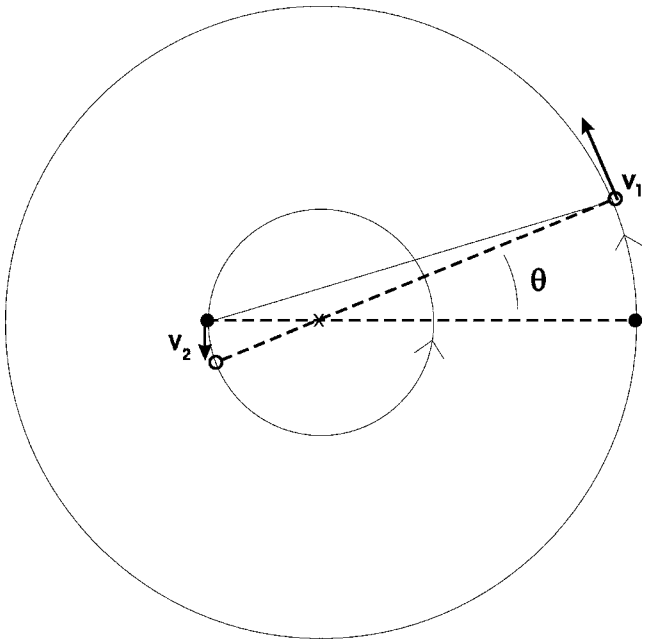

FIG. 2. Coulombian circular orbit with particles in diametral opposition at the same time of the inertial frame (open circles). Also indicated is the retarded position of both particles (solid circles) and the angle traveled during the light-cone time. Drawing not to scale. The circular orbit of the proton and the retardation angle have an exaggerated magnitude for illustrative purposes. Arbitrary units.

drogen. The interparticle distance of the light cone, r_b, is constant along the circular orbit and given, to leading order, by

$$r_b = \frac{1}{\mu\theta^2} + \cdots . \tag{2}$$

Using the radial equation of motion for the Coulombian orbit, one finds that the angular momentum of the circular orbit is given, to leading order in θ, by

$$l_z = \frac{1}{\theta} + \cdots . \tag{3}$$

The units of l_z in Eq. (3) are e^2/c, just that we are using a unit system where $e^2=c=1$. For orbits of the atomic magnitude, l_z as defined by Eq. (3) is of the order of the inverse of the fine-structure constant, $\alpha^{-1}=137.036$. Each particle travels a circular orbit with radius and scalar velocity defined by

$$r_1 \equiv b_1 r_b,$$

$$r_2 \equiv b_2 r_b, \tag{4}$$

and

$$v_1 = \Omega r_1 = \theta b_1,$$

$$v_2 = \Omega r_2 = \theta b_2, \tag{5}$$

for the electron and for the proton, respectively. For consistency with the definition of r_b as the exact interparticle distance of the light cone, the definition of b_1 and b_2 must include an $O(\theta^2)$ term: $b_1 \equiv (1+g\theta^2)m_2/(m_1+m_2)$ and $b_2 \equiv (1+g\theta^2)m_1/(m_1+m_2)$. Evaluating the interparticle distance of the light cone and equating it to r_b yields

$$r_b^2 = r_b^2[b_1^2 + b_2^2 + 2b_1 b_2 \cos(\theta)]. \tag{6}$$

Formula (6) becomes Eq. (3.1) of Ref. [8] after use of Eq. (5). Expanding Eq. (6) for small θ yields $g \simeq 0.5\mu/(m_1+m_2)$. The ratio $(b_1/b_2)=(m_2/m_1)$ is the Coulombian ratio, and the radii defined by Eq. (4) are $O(\theta^2)$ near the Coulombian radii. We henceforth call this the Coulombian circular orbit. As discussed in Appendix C, for Dirac's electrodynamics the circular orbit is only an approximate solution. Since we are doing perturbation, it suffices to use the above-defined Coulombian orbit as an approximate solution and let the perturbation scheme take care of the correction. In the action-at-a-distance theory [8,9], it turns out that the circular orbit *is* an exact orbit of the two-body problem. The action-at-distance electrodynamics is a special setting described by the equations of Appendix A with $\Gamma=-1/2$ and is used here only to cross-check the method of Sec. III.

The intuitive picture of a PP oscillation is a ping-pong game—i.e., the particles throwing a ball back and forth at the finite speed $c=1$—as a means to communicate changes in position. The PP modes have a period of the order of r_b/c, where r_b is given by Eq. (2). The order of magnitude of the PP frequency is

$$w_{PP} \sim \frac{2\pi}{r_b/c} = 2\pi\mu\theta^2, \tag{7}$$

where we used that $c=1$ in our unit system. To compare the PP frequency with the Coulombian orbital frequency (1), it is useful to express the PP frequency (7) as

$$w_{PP} \sim 2\pi\mu\theta^2 \equiv \frac{|\lambda|}{\theta}\Omega, \tag{8}$$

where $|\lambda|$ is a number of the order of 2π. For atomic orbits, $\theta \simeq 10^{-2}$, the PP frequency w_{PP} is larger than Ω by three orders of magnitude. We henceforth define the generalized frequency of a ping-pong normal mode by $\lambda\Omega/\theta$, so that a purely imaginary λ defines a harmonic oscillation and the eventual real part of λ defines a damping or a runaway. For small θ, the bouncing time for light to travel back and forth the interparticle distance is approximately $2r_b/c$, so that the phase shift of the fast oscillation during the bouncing time is $\Delta\Phi \equiv \mathrm{Im}[(\lambda\Omega/\theta), r_b/c]$. This phase shift evaluates to $\Delta\Phi = 2\mathrm{Im}(\lambda)$ if we use $c=1$ and $\Omega r_b=\theta$, and we shall see that it must be an integer multiple of 2π for the ping-pong solutions of Sec. V and of Appendix C.

III. VARIATIONAL EXPANSION ABOUT A CIRCULAR ORBIT

The variational equations are obtained by substituting a circular orbit plus a perturbation into the equations of motion of Appendix A—e.g., Eq. (A4)—and expanding in powers of the perturbation. Since these are complex equations, even after using a convenient coordinate system this derivation is

long. Our method uses the fact that these equations of motion are *almost* the Euler-Lagrange equations of a suitable Lagrangian, with the addition of the self-interaction force. After the variational equations of the Lagrangian sector are derived, we add in the variation of the terms due to the self-interaction. Readers should consult Appendix A for a brief review of the equations of motion of point charges and definition of terms such as the Lorentz force, the Lorentz-Dirac self-force, and the action integral. In this section we outline the procedure of substituting a circular orbit plus a perturbation into action (A11) of Appendix A and expanding the action up to a desired order. Minimization of this truncated action plus the expansion of the kinetic action yields the Lorentz sector of the variational equations. Since we perform the algebraic computations with a symbolic manipulations software, it is equally easy to derive the variational equations in a generalized electromagnetic setting that contains Dirac's electrodynamics as a special case, as explained in Appendix A. The calculation in the generalized setting has some interest of its own and provides a useful cross-check with other known calculations [2,11]. The generalized setting of Appendix A has an arbitrary constant Γ in the Green's function, and we stress that Γ will be set to *zero* in our application of Sec. V: i.e., we study the hydrogen orbit in Dirac's electrodynamics with retarded-only fields [1], a fundamental physical theory. A reader not interested in this generality can assume $\Gamma=0$ throughout the whole paper.

The variational equations for planar perturbations of circular orbits decuple from the equation for transverse perturbations up to linear order. It is convenient to write these planar variational equations using complex gyroscopic coordinates rotating at the frequency Ω of the unperturbed circular orbit. The coordinates (x_k, y_k) of each particle are defined by two complex gyroscopic coordinates η_k and ξ_k as

$$u_j \equiv x_k + iy_k \equiv r_b \exp(i\Omega t)[d_k + 2\eta_k],$$

$$u_j^* \equiv x_k - iy_k \equiv r_b \exp(-i\Omega t)[d_k + 2\xi_k], \tag{9}$$

where $k=1$ for electron and $k=2$ for proton and the real quantities $d_1 \equiv b_1$ and $d_2 \equiv -b_2$ are defined in Eqs. (4). Because x_k and y_k are real, we must have $\eta_k \equiv \xi_k^*$, but to obtain the variational equations it suffices to treat η_k and ξ_k as independent variables in Lagrangian (A12). Two quantities appear so often in the calculations that it is useful to name them: (i) The numerator of Lagrangian (A12), evaluated along the circular orbit, is constant and given by

$$C \equiv 1 + b_1 b_2 \theta^2 \cos(\theta), \tag{10}$$

having the same value for both retarded and advanced interactions, and (ii) the denominator of Lagrangian (A12), evaluated along the circular orbit and divided by r_b, is constant and defined by S as follows:

$$S \equiv 1 + b_1 b_2 \theta \sin(\theta), \tag{11}$$

having also the same value for both retarded and advanced interactions. For the stiff limit of Secs. IV and V we shall ignore the $O(\theta^2)$ corrections and set $C=S=1$. Here we derive the electron's equations of motion only [$j=1$ in Eq. (9)]. The equation for the proton is completely symmetric and can be obtained by interchanging indices 1 and 2. Notice that the exchange operation on the d's is $d_1 \Leftrightarrow -d_2$. This is because $d_1=b_1$ is defined positive while $d_2=-b_2$ is defined negative, so that at the same time the particles are in diametral opposition along the unperturbed circular orbit, as illustrated in Fig. 2.

The velocity of the electron at its time t_1 is calculated with Eq. (9) as

$$\dot{u}_1 \equiv v_{1x} + iv_{1y} \equiv \theta \exp(i\Omega t_1)[id_1 + 2i(\eta_1 - i\dot{\eta}_1],$$

$$\dot{u}_1^* \equiv v_{1x} - iv_{1y} \equiv \theta \exp(-i\Omega t_1)[-id_1 - 2i(\xi_1 + i\dot{\xi}_1)]. \tag{12}$$

The velocity of the proton at time t_2 can be obtained by interchanging indices 1 and 2 in Eq. (12), as explained above. Using the $c=\pm 1$ convention explained above Eq. (A13) of Appendix A, the quantities of particle 2 appearing in Lagrangian (A12) are evaluated at a time $t_2=t_{2c}$, of the light cone with the position of particle 1 at time t_1. The implicit light-cone condition must be expanded and solved by iteration, and for that it is convenient to define in each case the excess-lag function φ_c by

$$t_{2c} \equiv t_1 + \frac{r_b}{c} + \frac{\varphi_c}{\Omega}. \tag{13}$$

If the perturbation is zero, then $\varphi_+=\varphi_-=0$ and the dynamics is along the original circular orbit, where the light-cone lag is the constant r_b for the advanced case and $-r_b$ for the retarded case. We henceforth measure the evolution with the scaled-time parameter $\tau \equiv \Omega t_1$. The implicit definition of φ_c by the light-cone condition involves the position of particle 2 at either the advanced time t_{2+} or the retarded time t_{2-}, as defined by Eq. (9),

$$u_2(\tau + c\theta + \varphi_c) \equiv r_b \exp(i\Omega t_{2c})[d_2 + 2\eta_2(\tau + c\theta + \varphi_c)],$$

$$u_2^*(\tau + c\theta + \varphi_c) \equiv r_b \exp(-i\Omega t_{2c})[d_2 + 2\xi_2(\tau + c\theta + \varphi_c)], \tag{14}$$

as well as the velocity of particle 2 at the advanced or retarded position:

$$\dot{u}_2(\tau + c\theta + \varphi_c) \equiv i\theta \exp(i\Omega t_{2c})[d_2 + (\eta_2 - i\dot{\eta}_2) + 2\varphi_c(\dot{\eta}_2 - \ddot{\eta}_2)],$$

$$\dot{u}_2^*(\tau + c\theta + \varphi_c) \equiv -i\theta \exp(-i\Omega t_{2c})[d_2 + (\xi_2 + i\dot{\xi}_2) + 2\varphi_c(\dot{\xi}_2 + \ddot{\xi}_2)]. \tag{15}$$

To obtain the linearized variational equations we expand the equations of motion to first order in η_k and ξ_k, which are the Euler-Lagrange equations of the quadratic expansion of action (A11) in η_k and ξ_k. We must therefore carry all expansions up to the second order in the $\eta\xi$ coordinates. For example, the position of particle 2 is determined up to the second order by expanding the arguments of η_2 and ξ_2 of Eq. (14) in a Taylor series about the unperturbed light-cone for one order only as

$$u_2(\tau+c\theta+\varphi_c) \simeq r_b \exp(i\Omega t_{2c})\{d_2 + 2[\eta_2(\tau+c\theta) + \varphi_c \dot{\eta}_2(\tau+c\theta)]\},$$

$$u_2^*(\tau+c\theta+\varphi_c) \simeq r_b \exp(-i\Omega t_{2c})\{d_2 + 2[\xi_2(\tau+c\theta) + \varphi_c \dot{\xi}_2(\tau+c\theta)]\}. \quad (16)$$

In the following we find that φ_c is linear in η and ξ to leading order, so that the next term in expansion (16) would be a third order term, not needed for the linear variational equations. We henceforth indicate quantities of particle 2 evaluated at the *unperturbed* light cone by a subindex c—e.g., $\eta_{2c} \equiv \eta_2(\tau+c\theta)$ and $\xi_{2c} \equiv \xi_2(\tau+c\theta)$. Notice that the small parameter of expansion is the size of η and ξ, and we henceforth expand any quantity evaluated on the light cone in a Taylor series about the *unperturbed* light-cone, up to the order needed. For quasicircular orbits this method yields variational equations with a fixed delay. We stress that one should never expand the arguments in powers of θ; such an expansion yields ordinary differential equations and looses the PP modes. The perturbed light cone is expressed implicitly by the distance from the advanced or retarded position of particle 2 to the present position of particle 1, described in gyroscopic coordinates by the modulus of the complex number

$$u \equiv u_1(\tau) - u_2(\tau+c\theta+\varphi_c), \quad (17)$$

where $c=1$ describes the advanced light cone and $c=-1$ describes the retarded light cone. Using Eq. (16) to calculate u up to second order yields

$$u = r_b \exp(i\Omega t_{2c})\{D_c^* + 2[\exp(-ic\theta - i\varphi_c)\eta_1(\tau) - \eta_{2c} - \varphi_c \dot{\eta}_{2c}]\},$$

$$u^* = r_b \exp(-i\Omega t_{2c})\{D_c + 2[\exp(ic\theta + i\varphi_c)\xi_1(\tau) - \xi_{2c} - \varphi_c \dot{\xi}_{2c}]\}, \quad (18)$$

where we defined the following complex function of φ_c:

$$D_c \equiv b_2 + b_1 \exp(ic\theta + i\varphi_c). \quad (19)$$

At $\varphi_c=0$ (the unperturbed circular orbit), D_c has a unitary modulus, expressing the unperturbed light-cone condition (6). The right-hand side of Eq. (18) is a quadratic form times $\exp(i\Omega t_{2c})$, and in the action it appears multiplied by a counterrotating term—i.e., a quadratic form times $\exp(-i\Omega t_{2c})$—so that the product is independent of t_{2c}. Because of this rotational symmetry of action (A11), the following quadratic Gauge simplification can be applied directly to any quadratic rotating form. One can integrate by parts a quadratic term of the quantity—e.g., Eq. (18)—and disregard the boundary term. This gauge simplification yields a correct action up to the second order. For example, integrating by parts the quadratic terms in $\dot{\eta}_{2c}$ and $\dot{\xi}_{2c}$ on the right-hand side of Eq. (18) and disregarding the quadratic gauge yields

$$u = r_b \exp(i\Omega t_{2c})\{D_c^* + 2[(1 - i\varphi_c)\exp(-ic\theta)\eta_1(\tau) - (1 - \dot{\varphi}_c)\eta_{2c}]\},$$

$$u^* = r_b \exp(-i\Omega t_{2c})\{D_c + 2[(1 + i\varphi_c)\exp(ic\theta)\xi_1(\tau) - (1 - \dot{\varphi}_c)\xi_{2c}]\}, \quad (20)$$

where we have also expanded $\exp(i\varphi_c)$ up to linear order in φ_c, enough to give the correct quadratic action. Analogously, the velocity of particle 2 has the following expansion up to a quadratic gauge:

$$\dot{u}_2(\tau+c\theta+\varphi_c) \simeq i\theta \exp(i\Omega t_{2c})[d_2 + 2(\eta_{2c} - i\dot{\eta}_{2c})(1 - \dot{\varphi}_c)],$$

$$\dot{u}_2^*(\tau+c\theta+\varphi_c) \simeq -i\theta \exp(-i\Omega t_{2c}) \times [d_2 + 2(\xi_{2c} + i\dot{\xi}_{2c})(1 - \dot{\varphi}_c)]. \quad (21)$$

Using the above quantities, the numerator of Lagrangian (A12) can be calculated as

$$h_2 = (1 - \mathbf{v}_1 \cdot \mathbf{v}_{2c}) = \left(1 - \frac{1}{2}\dot{u}_1 \dot{u}_2^* - \frac{1}{2}\dot{u}_1^* \dot{u}_2\right), \quad (22)$$

and the denominator of Lagrangian (A12) can be calculated as

$$h_4 = r_{12c}\left(1 + \frac{\mathbf{n}_{12c} \cdot \mathbf{v}_{2c}}{c}\right) = r_b(1+\phi) + \frac{u\dot{u}_2^*}{2c} + \frac{u^*\dot{u}_2}{2c}. \quad (23)$$

In Eq. (23) we have introduced the scaled delay function ϕ by

$$\varphi_c \equiv c\theta\phi, \quad (24)$$

where the subindex under ϕ is omitted for simplicity of notation. To relate ϕ to the $\xi\eta$ perturbations we expand the implicit light-cone condition of the perturbed orbit up to quadratic order:

$$uu^* = \left| x_2\left(t_1 + \frac{r_b}{c} + \varphi_c\right) - x_1(t_1)\right|^2 = c^2(t_{2c} - t_1)^2 = \left(r_b + \frac{\varphi}{\Omega c}\right)^2. \quad (25)$$

The light-cone condition (25) is most simply expressed in terms of the scaled ϕ defined in Eq. (24). This expansion of Eq. (25) up to second order yields

$$\begin{aligned} C\phi^2 + 2S\phi = {} & 2[(b_1\xi_1 - b_2\xi_2) + (b_1\eta_1 - b_2\eta_2)] \\ & + 2[(b_2\xi_1 - b_1\eta_2)\exp(ic\theta) \\ & + (b_2\eta_1 - b_1\xi_2)\exp(-ic\theta)] \\ & + 4[\xi_1\eta_1 - \xi_1\eta_2\exp(ic\theta) - \xi_2\eta_1\exp(-ic\theta)] \\ & - 2\varphi[b_2(\dot{\xi}_2 + \dot{\eta}_2) + b_1\dot{\xi}_2\exp(-ic\theta) \\ & + b_1\dot{\eta}_2\exp(ic\theta)]. \end{aligned} \quad (26)$$

The solution to Eq. (26) up to first order in the $\xi\eta$ coordinates is

$$\begin{aligned} S\phi_{(1)} \equiv {} & [(b_1\xi_1 - b_2\xi_2) + (b_1\eta_1 - b_2\eta_2) \\ & + (b_2\xi_1 - b_1\xi_2)\exp(ic\theta) \\ & + (b_2\eta_1 - b_1\eta_2)\exp(-ic\theta)]. \end{aligned} \quad (27)$$

Last, as a check for the above calculations, in the following we derive the equations of motion for the circular orbit of the action-at-a-distance theory [8]. The Lagrangian of action (A11) for $\Gamma=-1/2$ is $\mathcal{L}\equiv(L_+ + L_-)/2$, and its expansion up to first order is

$$\tilde{\mathcal{L}} = 1 - \{[\theta^2(S-1)S + C^2][b_1 + b_2\cos(\theta)] + S[\theta\sin(\theta) - \theta^2\cos(\theta)]b_2\}\frac{(\eta_1+\xi_1)}{CS^2}, \tag{28}$$

where the tilde indicates that $\mathcal{L}$ was scaled by its value along the unperturbed circular orbit, $\tilde{\mathcal{L}}\equiv r_b S\ \mathcal{L}/C$. Scaling the kinetic Lagrangian with the same factor, and expanding to first order yields

$$\tilde{T}_1 = \frac{-r_b m_1 S}{C\gamma_1} + \frac{r_b\theta^2 m_1\gamma_1 S b_1}{C}(\eta_1+\xi_1), \tag{29}$$

where $\gamma_1\equiv 1/\sqrt{1-v_1^2}$ and v_1 is given by Eq. (5). The effective Lagrangian for particle 1 up to the linear order is

$$\tilde{L}^{(1)}_{eff} \equiv \tilde{T}_1 + \tilde{\mathcal{L}}. \tag{30}$$

Lagrangian (30) is a linear functional of ξ_1, independent of $\dot{\xi}_1$, so that the Euler-Lagrange equation for ξ_1 is simply $\partial\tilde{L}^{(1)}_{eff}/\partial\xi_1 = 0$—i.e.,

$$m_1 b_1 r_b\gamma_1\theta^2 S^3 = [C^2 + \theta^2 S(S-1)][b_1 + b_2\cos(\theta)] + S[\theta\sin(\theta) - \theta^2\cos(\theta)]b_2. \tag{31}$$

This is Eq. 3.2 of Ref. [8], and the equation for η_1 is the same condition by symmetry (this is actually the reason why the circular orbit is a solution [13]). The equation for particle 2 is obtained by interchanging indices 1 and 2 in Eq. (31), which yields Eq. 3.3 of Ref. [8]. In the next section we expand the action to second order to determine the linearized variational equations.

IV. LINEARIZED VARIATIONAL EQUATIONS

In this section we obtain the linear-order terms of the variational equations. For this we carry the expansion of the prior section to the quadratic order. The next term of expansion (29) of the kinetic Lagrangian of particle 1, calculated with Eq. (12), is

$$\tilde{T}_1 = -\frac{r_b S m_1}{C\gamma_1} + \frac{r_b\theta^2 m_1\gamma_1 S b_1}{C}(\eta_1+\xi_1) + \frac{S r_b\theta^2 m_1\gamma_1}{2C}[\gamma_1^2(\xi_1 + i\dot{\xi}_1 + \eta_1 - i\dot{\eta}_1)^2 - (\xi_1 + i\dot{\xi}_1 - \eta_1 + i\dot{\eta}_1)^2] + \cdots. \tag{32}$$

Disregarding the constant term, this kinetic Lagrangian of particle 1 has a quadratic form defined by two coefficients

$$\tilde{T}_1 = \frac{r_b\theta^2 m_1\gamma_1 S b_1}{C}(\eta_1+\xi_1) + M_1[\dot{\eta}_1\dot{\xi}_1 + i(\eta_1\dot{\xi}_1 - \xi_1\dot{\eta}_1) + \eta_1\xi_1] + \frac{\theta^2 G_1}{2}[\xi_1^2 + \eta_1^2 - \dot{\xi}_1^2 - \dot{\eta}_1^2], \tag{33}$$

where $M_1\equiv(1+\gamma_1^2)m_1\gamma_1 r_b\theta^2 S/C$ and $G_1\equiv(\gamma_1^2-1)m_1\gamma_1\theta^2 S/C$. We also need the solution of Eq. (26) to second order, $\phi=\phi_{(1)}+\phi_{(2)}$, where $\phi_{(1)}$ is given by Eq. (27) and $\phi_{(2)}$ is calculated by iteration to be

$$\begin{aligned} S\phi_{(2)} = {} & \varphi(ib_2\xi_1 - ib_1\eta_2 - b_1\dot{\eta}_2)\exp(ic\theta) \\ & + \varphi(ib_1\xi_2 - ib_2\eta_1 - b_1\dot{\xi}_2)\exp(-ic\theta) \\ & - \varphi b_2(\dot{\xi}_2 + \dot{\eta}_2) - \frac{1}{2}C[\phi_{(1)}]^2 \\ & + 2[\xi_1\eta_1 - \xi_1\eta_2\exp(ic\theta) - \xi_2\eta_1\exp(-ic\theta)]. \end{aligned} \tag{34}$$

Next we expand the numerator and denominator of Lagrangian (A12), Eqs. (22) and (23), up to the quadratic order. We also need the following quantities expanded up to second order: (i) the particle separation of the light cone, Eq. (18); (ii) the velocity of particle 1, Eq. (12); and (iii) the velocity of particle 2 at time t_{2c}, Eq. (21). This quadratic form can be greatly simplified by adding suitable gauge terms to it (essentially integration by parts), which yields the following normalized Lagrangian for particle 1:

$$\begin{aligned} L_c = {} & -\frac{i}{2}B_{21}(\eta_1\dot{\xi}_1 - \xi_1\dot{\eta}_1) + U_{11}\xi_1\eta_1 + \frac{1}{2}N_{11}\xi_1^2 + \frac{1}{2}N_{11}^*\eta_1^2 \\ & + R_c\xi_1\xi_{2c} + R_c^*\eta_1\eta_{2c} + P_c\xi_1\eta_{2c} + P_c^*\eta_1\xi_{2c} \\ & + \frac{Y_c}{2}(\xi_1\dot{\xi}_{2c} - \xi_{2c}\dot{\xi}_1) + \frac{Y_c^*}{2}(\eta_1\dot{\eta}_{2c} - \eta_{2c}\dot{\eta}_1) \\ & + \frac{\Lambda_c}{2}(\xi_1\dot{\eta}_{2c} - \eta_{2c}\dot{\xi}_1) + \frac{\Lambda_c^*}{2}(\eta_1\dot{\xi}_1 - \xi_2\dot{\eta}_1) + \frac{T_c}{2}\dot{\xi}_1\dot{\xi}_{2c} \\ & + \frac{T_c^*}{2}\dot{\eta}_1\dot{\eta}_{2c} + \frac{E_c}{2}\dot{\xi}_1\dot{\eta}_{2c} + \frac{E_c^*}{2}\dot{\eta}_1\dot{\xi}_{2c}. \end{aligned} \tag{35}$$

In Eq. (35) the coordinates of particle 2 appear evaluated in either the retarded or the advanced unperturbed light cone, as indicated by the subindex c. The coefficient of each normal-form binary is a function of m_1, m_2, and θ and is obtained by a gauge-invariant combination of derivatives; e.g., the coefficient B_{21} is the same in both retarded and advanced interactions and given by

$$B_{21} = i\left[\frac{\partial^2 L_c}{\partial\eta_1\,\partial\dot{\xi}_1} - \frac{\partial^2 L_c}{\partial\xi_1\,\partial\dot{\eta}_1}\right]. \tag{36}$$

The coefficients were evaluated with a symbolic manipulation software, starting directly from the effective Lagrangian and taking the necessary derivatives (in this way we avoid mistakes). The explicit functional dependences are not given here for brevity, but were checked at the various limits: Biot-Savart field, Coulomb interaction, Darwin interaction, etc.

The effective Lagrangian for particle 1 is composed of the partial Lagrangians (A12), evaluated at the advanced and the retarded light cones, as in Eq. (A11), plus the kinetic Lagrangian

$$L_{eff}^{(1)} = T_1 - \Gamma L_+ + (1+\Gamma)L_-. \tag{37}$$

The linearized Euler-Lagrange equation of Lagrangian (37) respect to ξ_1 is a linear function of the coordinates ξ_1, η_1, ξ_{2+}, η_{2+}, ξ_{2-}, and η_{2-}, as well as of their first and second derivatives:

$$\begin{aligned} l_{1\xi}(\xi_1,\eta_1,\xi_{2+},\eta_{2+},\xi_{2-},\eta_{2-}) &= -[(N_{11}+\theta^2 G)\xi_1 + \theta^2 G\ddot{\xi}_1] + [M_1(\ddot{\eta}_1 + 2i\dot{\eta}_1 - \eta_1) \\ &- U_{11}\eta_1 - iB_{21}\dot{\eta}_1] - (R_+^*\xi_{2+} + R_-^*\xi_{2-}) \\ &- (Y_+\dot{\xi}_{2+} + Y_-\dot{\xi}_{2-}) + (T_+\ddot{\xi}_{2+} + T_-\ddot{\xi}_{2-}) \\ &- (P_+\eta_{2+} + P_-\eta_{2-}) - (\Lambda_+\dot{\eta}_{2+} + \Lambda_-\dot{\eta}_{2-}) \\ &+ (E_+\ddot{\eta}_{2+} + E_-\ddot{\eta}_{2-}). \end{aligned} \tag{38}$$

The Euler-Lagrange equation of Lagrangian (37) with respect to η_1, ξ_2, and η_2 yields three more linear equations, which together with Eq. (38) form a system of four linear delay equations.

In the following we explain how to include self-interaction into Eq. (38). The Euler-Lagrange equation of the kinetic energy (33) with respect to ξ_1 can be expressed as

$$\begin{aligned} (1+\gamma_1^2)m_1\gamma_1(S/C)r_b^3\Omega^2(\ddot{\eta}_1 + 2i\dot{\eta}_1 - \eta_1) \\ = r_b^2(S/C)\frac{d}{dt}(m_1\gamma_1 r_b\dot{u}_1), \end{aligned} \tag{39}$$

where we replaced θ^2 by $r_b^2\Omega^2$ into the definition of M_1 given below Eq. (33). On the second line of Eq. (39) we recognize the variation of the complex momentum, $m_1\gamma_1 r_b\dot{u}_1 = p_{1x} + ip_{1y}$, multiplied by the factor $r_b^2(S/C)$. The variation of momentum is the force, so that to account for self-interaction we add to Eq. (38) the x component of force (A14) multiplied by the factor $r_b^2(S/C)$, plus i times the y component of force (A14) multiplied by $r_b^2(S/C)$—i.e.,

$$r_b^2(S/C)\frac{2}{3}(1+2\Gamma)r_b\frac{d^3u_1}{dt^3} \simeq \frac{2\theta^3}{3}(1+2\Gamma)\dddot{u}_1, \tag{40}$$

where the overdots represent derivatives with respect to scaled time $\tau \equiv \Omega t$. Using Eq. (40) with u_1 given by Eq. (9) and expanding up to the linear order yields the order-0 offensive force (C4) plus the linearization of the self-force in gyroscopic coordinates. The offensive force (C4) is the nonhomogeneous term of the variational equation and shall be dealt with in Appendix C. In this section we discard it and keep only the linear part of the variational equations. Notice that Eq. (39) came out naturally in the form of variation of momentum multiplied by r_b^2. This instructive normalization suggests that we scale the equations of motion with the unperturbed Coulombian attraction, $1/r_b^2$—i.e., that we scale forces with the size of the unperturbed attraction. This normalization is useful when discussing orders of magnitude, and it is used in Appendix C to discuss estimates with an intuition about physical orders of magnitude.

The linear variational equations for ξ_1, η_1, ξ_2, and η_2 form a set of four linear delay equations, a system that can be solved in general by Laplace transform [14]. In the following we focus on the planar normal-mode solutions to this linear system, with a *fast* ping-pong frequency $\lambda_{xy}\Omega/\theta$, a definition motivated by Eq. (8). The complex number λ_{xy} is so far arbitrary, but a *harmonic* oscillation is defined by an imaginary λ_{xy}. In the following we substitute $\xi_1 = A\exp(\lambda_{xy}\Omega t/\theta)$, $\eta_1 = B\exp(\lambda_{xy}\Omega t/\theta)$, $\xi_2 = C\exp(\lambda_{xy}\Omega t/\theta)$, and $\eta_2 = D\exp(\lambda_{xy}\Omega t/\theta)$ into the linearized equations and assume θ small and $|\lambda_{xy}|$ of order 1 or larger, as discussed below Eq. (8). This yields four homogeneous linear equations for A, B, C, and D, and a nontrivial solution exists only if the determinant vanishes. Using a symbolic manipulations software this determinant evaluates to

$$\begin{aligned} 1 - \frac{2(1+2\Gamma)\theta^2\lambda_{xy}}{3} + \frac{(1+2\Gamma)\theta^4\lambda_{xy}^2}{9} - \frac{2}{27}\frac{\mu}{M}(1+2\Gamma)^3\theta^6\lambda_{xy}^3 \\ + \cdots + \frac{\mu\theta^4}{M}\left(1 + \frac{7}{\lambda_{xy}^2} + \frac{5}{\lambda_{xy}^4}\right)[(1+2\Gamma)\sinh(2\lambda_{xy}) \\ - 2(1+2\Gamma+2\Gamma^2)\cosh^2(\lambda_{xy})] - 2\frac{\mu\theta^4}{M}\left(\frac{1}{\lambda_{xy}} + \frac{5}{\lambda_{xy}^3}\right) \\ \times[2(1+2\Gamma)\cosh^2(\lambda_{xy}) - (1+2\Gamma+2\Gamma^2)\sinh(2\lambda_{xy})] = 0, \end{aligned} \tag{41}$$

where $M \equiv m_1 + m_2$. In our unit system, the total mass of hydrogen is $M = 1825$. For Dirac's retarded-only electrodynamics, Eq. (41) with $\Gamma = 0$, we obtain the following planar normal-mode condition for ping-pong modes (the stiff limit):

$$\begin{aligned} \left(1 + \frac{2}{\lambda_{xy}} + \frac{7}{\lambda_{xy}^2} + \frac{10}{\lambda_{xy}^3} - \frac{5}{\lambda_{xy}^4} + \cdots\right)\left(\frac{\mu\theta^4}{M}\right)\exp(-2\lambda_{xy}) \\ = 1 - \frac{2}{3}\theta^2\lambda_{xy} + \frac{1}{9}\theta^4\lambda_{xy}^2 + \cdots. \end{aligned} \tag{42}$$

Comparing Eq. (42) with Eq. (B19) of Appendix B, we find that they agree up to terms of order $(1/\lambda)$ and $(\theta^2\lambda)$, a quasidegeneracy phenomenon that exists only in three cases: (i) $\Gamma = 0$, Dirac's theory with retarded-only interactions; (ii) $\Gamma = -1$, a nonphysical advanced-only interactions theory; and (iii) $\Gamma = -1/2$, the dissipative Fokker theory of Ref. [2] and the action-at-a-distance electrodynamics [11]. This discriminating degeneracy is an interesting curiosity, and in this paper we disregard the two other dynamics that exhibit the quasidegeneracy phenomenon: (ii) and (iii). In the next section we study a three-frequency orbit of hydrogen in Dirac's electrodynamics with retarded-only fields ($\Gamma = 0$), the physically sound choice to describe hydrogen in nature.

V. THREE-FREQUENCY ORBIT

As discussed in Sec. IV, in Dirac's electrodynamics there is a remarkable quasidegeneracy of the perpendicular and planar tangent dynamics. In the large-$|\lambda|$ limit, both Eq. (42) and Eq. (B19) of Appendix B reduce to

$$\left(\frac{\mu\theta^4}{M}\right)\exp(-2\lambda)=1, \tag{43}$$

henceforth called the degenerate stiff limit. The value of (μ/M) for hydrogen in Eq. (43) is a small factor of about (1/1825). Equation (43) requires that λ have a negative real part given by $\mathrm{Re}(\lambda)\equiv-\sigma\equiv-\ln(\sqrt{M/\mu\theta^4})$. For the atomic magnitude $\theta^{-1}\sim 137.036$, the value of σ is about $\sigma\simeq 15.0$. On the other hand, the imaginary part of λ can be an arbitrary multiple of π, so that the general solution to Eq. (43) is

$$\lambda=-\sigma+\pi q i, \tag{44}$$

where $i\equiv\sqrt{-1}$ and q is an arbitrary integer. Notice that the real part of λ is always negative, so that the tangent dynamics about the circular orbit is stable in the stiff limit. The unfolding of the degeneracy comes with the terms of order $1/\lambda^2$ and $\theta^4\lambda^2$, as found in Eq. (42) and Eq. (B19) of Appendix B. The exact roots of Eqs. (42) and (B19) near the limiting root (44) are defined, respectively, by

$$\lambda_{xy}(\theta)\equiv-\sigma_{xy}+\pi q i+i\epsilon_1,$$

$$\lambda_z(\theta)\equiv-\sigma_z+\pi q i+i\epsilon_2, \tag{45}$$

where $\epsilon_1(\theta)$ and $\epsilon_2(\theta)$ are real $O(\theta)$ numbers. The three-frequency orbit is formed from an initial circular orbit as follows: (i) Initially, the self-force (C4) dissipates energy, essentially the radiation of the circular orbit. Some of the radiated energy is absorbed *directly* by the PP oscillations. The slow guiding-center circular motion may also lose radius to account for some of the energy loss, spiraling in for a small number of turns. (ii) After the PP modes absorb enough energy, their amplitudes grow and the PP oscillations become neutrally stable, which is illustrated in Fig. 1. As discussed in Appendix C, at a finite PP amplitude, a harmonic solution to the nonlinear variational equations exists—i.e., the σ's of Eq. (45) vanish—so that the λ's become purely imaginary. This balancing is achieved at relatively small amplitudes—i.e., near the circular orbit. And (iii) The radiation of the PP modes starts to interfere with the orbital radiation if the orbit is a resonant one. In the following we discuss a surprisingly simple stabilization mechanism that operates *after* the fast harmonic oscillations are established near the guiding-center orbit.

We henceforth assume that the three-frequency orbit of Fig. 1 is defined by the following combination of a planar PP mode and a perpendicular PP mode:

$$x_k+iy_k\equiv r_b\exp(i\Omega t)[d_k+2\xi_k^*],$$

$$x_k-iy_k\equiv r_b\exp(-i\Omega t)[d_k^*+2\xi_k],$$

$$z_k\equiv r_b Z_k, \tag{46}$$

with

$$\xi_k=\frac{R_k^{xy}}{2}\exp[(\pi q i+i\epsilon_1^\rho)\Omega t/\theta],$$

$$Z_k=\mathrm{Re}\{R_k^z\exp[(\pi q i+i\epsilon_2^\rho)\Omega t/\theta]\}, \tag{47}$$

where R_k^{xy} and R_k^z are, respectively, the amplitude of the planar PP mode and the amplitude of the perpendicular PP mode ($k=1$ for the electron and $k=2$ for the proton). Notice that we chose the same q for the two perpendicular oscillations of Eq. (47), so that q cancels and the difference between the frequencies of Eq. (47) defines a beat in a slow time scale. The far-electric field of the electron, at a far distance r_∞, is obtained by exchanging indices and disregarding the first term on the right-hand side of Eq. (A6),

$$\mathbf{E}_1=-\frac{\mathbf{n}\times(\mathbf{n}\times\mathbf{a}_{1-})}{(1-\mathbf{n}\cdot\mathbf{v}_{1-})^3 r_\infty}+\frac{\mathbf{n}\times(\mathbf{v}_{1-}\times\mathbf{a}_{1-})}{(1-\mathbf{n}\cdot\mathbf{v}_{1-})^3 r_\infty}, \tag{48}$$

where we multiplied Eq. (A6) by -1 to account for the electronic charge. In Eq. (48) , the unit vector $\mathbf{n}$ points from the retarded position of the electron to the observation point at infinity, while $\mathbf{v}_{1-}$ and $\mathbf{a}_{1-}$ are, respectively, the Cartesian velocity and acceleration of the electron (particle 1). The Poynting flux is proportional to $|\mathbf{E}_1|^2$, so that the common denominator $(1-\mathbf{n}\cdot\mathbf{v}_{1-})^3$ on the right-hand side of Eq. (48) can be factored off $|\mathbf{E}_1|$ and is henceforth ignored. For the radiation of a circular orbit of atomic magnitude, the second term on the right-hand side of Eq. (48) is small and can be disregarded. This is the dipole approximation of atomic physics, valid when the interparticle distance is smaller than the radiated wavelength. This approximation *is not valid* for the three-frequency orbit of Fig. 1 because of the short wavelength of the PP modes. For the far field of the three-frequency orbit of Fig. 1, the second term on the right-hand side of Eq. (48) must be kept and is henceforth called the spin-radiation field. Because of this quadratic spin-radiation field, the vector product of the mutually orthogonal PP modes (46) beats at a slow frequency (cosine times cosine averages to cosine of the difference). The spin-radiation field of the electron, $\mathbf{E}_{PP}$, averaged over the fast time scale and assuming $|R_1^z|\simeq|R_1^{xy}|\simeq\rho_1$in Eq. (47), has the amplitude

$$|\mathbf{E}_{PP}|=\frac{\mu\theta^2\rho_1^2\pi^3q^3}{r_\infty}\cos\left(\left[1+\frac{(\epsilon_2^\rho-\epsilon_1^\rho)}{\theta}\right]\Omega t\right). \tag{49}$$

The oscillation of Eq. (49) is determined by the frequency difference (beat) of the mutually orthogonal PP modes of orbit (47),

$$\Delta w_{PP}\equiv\left[1+\frac{(\epsilon_2^\rho-\epsilon_1^\rho)}{\theta}\right]\Omega. \tag{50}$$

Because of the quasidegeneracy property, ϵ_2^ρ and ϵ_1^ρ are $O(\theta)$ quantities, so that Δw_{PP} in Eq. (50) defines a beat of the order of the guiding-center frequency Ω. The first term of the right-hand side of Eq. (48), averaged over the fast time scale, yields the usual dipole field $\mathbf{E}_d$ of the unperturbed circular motion, with magnitude

$$|\mathbf{E}_d|=\frac{\mu\theta^4}{r_\infty}\cos(\Omega t). \tag{51}$$

Poynting's theorem gives a necessary condition for $\mathbf{E}_{PP}$ to destruct $\mathbf{E}_d$ by interference; viz., both fields must oscillate at the same frequency. This yields the resonance condition

TABLE I. Quantum number k of the QED transition $k+1 \rightarrow k$, numerically calculated angular momenta $l_z = \theta^{-1}$ in units of e^2/c, orbital frequencies in atomic units $(137\theta)^3 = 137^3\theta^2(\epsilon_2 - \epsilon_1)$, the corresponding kth circular line of QED in atomic units, $w_{QED} \equiv (1/2)[1/k^2 - 1/(k+1)^2]$, and the integer q of Eqs. (45) and (47).

k	$l_z = \theta^{-1}$	$137^3\theta^2(\epsilon_2 - \epsilon_1)$	w_{QED}	q
1	185.99	3.996×10^{-1}	3.750×10^{-1}	7
2	307.63	8.831×10^{-2}	6.944×10^{-2}	9
3	475.08	2.398×10^{-2}	2.430×10^{-2}	11
4	577.99	1.331×10^{-2}	1.125×10^{-2}	12
5	694.77	7.667×10^{-3}	6.111×10^{-3}	13
6	826.22	4.558×10^{-3}	3.685×10^{-3}	14
7	973.12	2.790×10^{-3}	2.406×10^{-3}	15
8	1136.27	1.752×10^{-3}	1.640×10^{-3}	16
9	1316.44	1.127×10^{-3}	1.173×10^{-3}	17
10	1514.40	7.403×10^{-4}	8.678×10^{-4}	18
11	1730.93	4.958×10^{-4}	6.600×10^{-4}	19
12	1966.77	3.379×10^{-4}	5.136×10^{-4}	20
13	2222.70	2.341×10^{-4}	4.076×10^{-4}	21

$$\Omega = \Delta w_{PP}. \tag{52}$$

Notice that at resonance, the three-frequency orbit defined by Eqs. (46) and (47) has only *two* independent frequencies [16], because of relation (52) between its frequencies. In the gyroscopic coordinates (47), the resonant orbit has only one frequency, and in Appendix C such resonant orbit is called a harmonic solution. Use of Eq. (52) with Eq. (50) yields

$$\epsilon_2^\rho - \epsilon_1^\rho = 0. \tag{53}$$

In the following we approximate ϵ_1^ρ and ϵ_2^ρ, respectively, by the ϵ_1 and ϵ_2 of the linear problem, as defined by Eq. (45), assuming that essentially the real parts of the linear eigenvalues (45) are modified by the finite-amplitude correction (C3), while the imaginary parts acquire only a small correction. According to this approximation, we expand the small correction in powers of θ and take the first term

$$\epsilon_2 - \epsilon_1 \simeq \epsilon_2^\rho - \epsilon_1^\rho + p\theta + \cdots. \tag{54}$$

This approximation is justified in Appendix C and the linear coefficient p should depend on the radius of gyration, $\rho_1 \leqslant 1/\pi q$. Condition Eq. (53) together with Eqs. (50) and (1) gives Ω as

$$\Omega = \mu\theta^2(\epsilon_2 - \epsilon_1) + (1-p)\theta^3 + \cdots, \tag{55}$$

a Rydberg-Ritz-like formula. The resonant frequency Ω is expressed by an eigenvalue difference in another way: the exact difference between the frequencies of the orthogonal PP modes, Eq. (52). The PP frequencies should be rigorous linear eigenvalues of the tangent dynamics of the three-frequency orbit, but here we consider only the neighborhood of the circular orbit, and keep to approximation (55).

We henceforth set $p=1$ into Eq. (54) as a qualitative approximation, yielding

$$\epsilon_1 - \epsilon_2 + \theta = 0, \tag{56}$$

which has the solutions listed in Table I. The root searching problem of Eq. (56) is well posed, and for each integer q there is a single solution θ to Eqs. (56), (42), and (B19); i.e., θ is quantized by the integer q. According to QED, circular Bohr orbits have maximal angular momenta and a radiative selection rule ($\Delta l = \pm 1$) restricts the decay from level $k+1$ to level k only; i.e., circular orbits emit the first line of each spectroscopic series (Lyman, Balmer, Ritz-Paschen, Brackett, etc.), the fourth column of Table I. We have solved Eq. (56) together with Eqs. (42) and (B19) using a Newton method in the complex λ plane. The numerically calculated angular momenta θ^{-1} are given in Table I, along with the orbital frequency in atomic units $(137^3\Omega)/\mu = 137^3\theta^2(\epsilon_2 - \epsilon_1)$ and the QED first frequency of each spectroscopic series.

Table I illustrates the fact that a resonance among PP modes predicts magnitudes precisely in the atomic scale, as first discovered in Ref. [2]. In Ref. [2] we had to jump the integer q by 20 units for a complete quantitative and qualitative agreement with the Bohr atom. The qualitative agreement achieved by Table I is superior in this respect, but the correspondence is still not perfect for $q < 7$, after which q increases one by one, in qualitative agreement with QED. The angular momenta in the second row of Table I should be compared with the QED values—i.e., $\theta^{-1} = 137.036k$. In Table II we give the anomalous roots for $q < 7$. Notice that the angular momenta are still of the order of Planck's constant, but the orbital frequencies do not correspond to any line of hydrogen. Since the approximation at Eq. (54) uses an expansion in θ, we should expect it to fail for the largest values of θ, as it did.

In the following we give a second derivation of Poynting's resonance condition (52). The angular momentum of

TABLE II. Numerically calculated angular momenta $l_z=\theta^{-1}$ in units of e^2/c, orbital frequencies in atomic units $(137\theta)^3=137^3\theta^2(\epsilon_2-\epsilon_1)$, and the values of the integer q of Eqs. (45) and (47).

$l_z=\theta^{-1}$	$137^3\theta^2(\epsilon_2-\epsilon_1)$	q
93.26	3.16	1
51.06	1.93	2
57.26	1.36	3
76.14	5.82	4
103.92	2.29	5
140.37	0.929	6

gyration along the orbital plane, calculated with Eq. (47) and averaged over the fast time scale is

$$l_x+il_y=\mu r_b^2|R_1^{xy}||R_1^z|\frac{\pi q\Omega}{\theta}b_1\exp[i(\epsilon_2^\rho-\epsilon_1^\rho+\theta)\Omega t/\theta]$$

$$=\frac{\pi q}{\theta^2}|R_1^{xy}||R_1^z|\exp[i(\epsilon_2^\rho-\epsilon_1^\rho+\theta)\Omega t/\theta], \quad (57)$$

where we used Eqs. (1) and (2). The angular momentum vector of Eq. (57) rotates at the same frequency Δw_{PP} of Eq. (50), the ping-pong beat. Since angular momentum carries inertia, its rotation produces a gyroscopic torque on the orbit, so that the guiding-center motion should display an oscillation at the frequency Δw_{PP} of Eq. (50). The derivation of Eq. (57) assumed that the guiding-center orbit is a circular orbit of frequency Ω; therefore, we must have $\Omega=\Delta w_{PP}$. In Appendix C we give a third derivation of resonance (52) by averaging the electronic spin-radiation field over the fast time scale. The resonance is needed for the averaged spin-radiation field to rotate at the guiding-center frequency, so that it can participate in the guiding-center dynamics.

VI. CONCLUSIONS AND DISCUSSION

We stress that the QED condition that θ be of the order of the fine-structure constant was never used in the calculations. The PP frequencies were calculated for *arbitrary* θ, and it was the resonance condition of Poynting's theorem alone, a *nonradiation condition*, that calculated θ in the atomic region. The quasidegeneracy phenomenon was already found in Ref. [2], and we expect it to be a universal property of electromagnetic dynamics with delay, as follows. The corrections to the limiting form (43) are controlled by powers of $1/\lambda$ and $\theta^2\lambda$, so that the resonance condition essentially poses a relation $\theta^2\lambda\propto 1/\lambda$, which yields $\theta\cong c/|\lambda|=c/(\pi q)$, in agreement with Table I, with Ref. [2], and with the Bohr atom of QED [3]. The finite-amplitude corrections discussed in Appendix C introduce again powers of $\rho\simeq 1/|\lambda|$, so that the qualitative behavior should persist.

In Ref. [17] we argued that a stable orbit should emit at a neutrally stable frequency of its tangent dynamics. The physical process near the stable orbit is that the two terms on the right-hand side of Eq. (48) do not compensate exactly, so that there is a net radiation *at* the resonant frequency. Moreover, the guiding motion of the three-frequency orbit is not exactly a Coulombian circular orbit, because the perturbative scheme should correct the Coulombian approximation significantly, especially for the first states, where $\sup(\rho)\simeq 1/\pi q$ is larger. The orbital frequencies of Table I are only an approximation to the emitted frequencies, an approximation that should be worse for the low values of q, again in agreement with QED. More generally, according to Kurtzweil's *small delays do not matter* theorem [18], small-θ orbits should be solutions of a limiting *ordinary* differential equation obtained by setting all delays to zero. This theorem yields that the limiting orbits should be the Coulombian orbits. In agreement with this, the gyration radius of Eq. (C8) is a smaller fraction of the orbital radius for larger values of q, the $\sup(\rho)\to 1/\pi q$ limit of Appendix C. We call this limit the Kurtzweil *correspondence limit*.

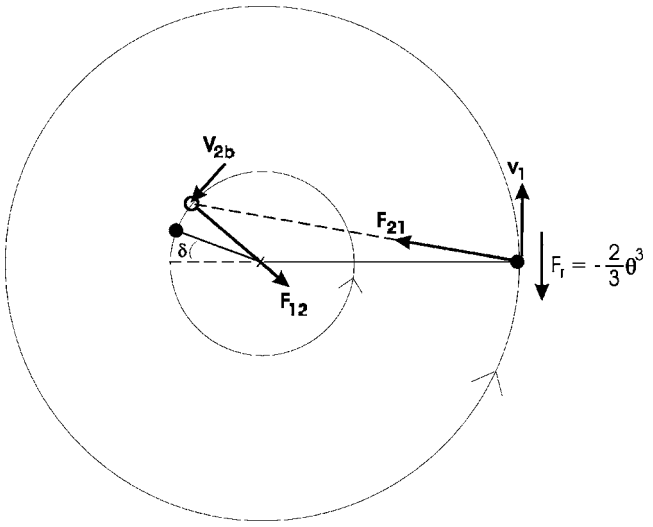

FIG. 3. Perturbed guiding-center orbit, with particles no longer in diametral opposition at the same time. Positions at the same time are the solid circles; the angular distance at the same time is $(\pi-\delta)$. At the retarded position of the proton (open circle), the averaged force F_{12} is perpendicular to the guiding-center velocity of the proton. At resonance, the averaged force F_{12} rotates at the guiding-center frequency. Protonic orbit and self-force have an exaggerated magnitude for illustrative purposes. Arbitrary units.

For θ in the atomic magnitude, there is a nontrivial stabilization mechanism involving PP modes, a mechanism that selects discrete orbits by a resonance. The ping-pong modes form a channel that can interfere and absorb the energy radiated by the orbital motion, a dynamics that involves a resonance between fast and slow time scales. A multiscale solution, *without assuming* that the guiding-center motion is circular, is still to be worked out. The multiscale solution involves a general *guiding-center* slow motion *plus* a fast PP gyration of small amplitude. After the fast dynamics is balanced locally, the multiscale method should yield differential equations for the guiding-center orbit by a Fredholm alternative [19–21]. Qualitatively, we expect the guiding-center equations of motion to accept richer orbits than illustrated in Figs. 1 and 3. The other possible orbital topologies are (i) gyration plane perpendicular to the normal connecting the particles, a dumbbell-like spinning, (ii) fast gyration re-

stricted to the orbital plane, and (iii) nonplanar guiding-center motion, like a p orbital. The intuition about the dynamics should follow the general guidelines learned from analyzing the circular-guiding-center three-frequency orbits of Figs. 1 and 3, as summarized in the following. For a resonant orbit, the electronic spin-radiation force, averaged over the fast timescale, yields an $O(1)$ slowly rotating force at the proton that perturbs the protonic equilibrium position along the guiding-center orbit. The protonic attraction cancels the electronic self-interaction, so that the guiding-center motion does not dissipate (any guiding-center motion). The averaging of the electronic spin-radiation force involves a resonance, and this is the reason for discrete orbits from the perspective of the equations of motion for the guiding center. We derived the resonance condition in three different ways here: (i) In Sec. V we derived the resonance condition using Poynting's theorem; (ii) still in Sec. V, we derived the same resonance condition using general arguments about the gyroscopic torque; and last (iii) in Appendix C, we derived the resonance condition from the detailed balancing of the guiding-center dynamics.

In the process of emission, the sharp line is emitted when the dynamics is approaching the stable orbit, as discussed in Ref. [17]. This long time scale is to be compared with the time of spontaneous decay of QED for hydrogen: about 10^6 orbital periods or 10^{-10} seconds. On the other hand, the dynamics has a fast time scale, the frequency of the PP modes, of about $\pi q/\theta \simeq 1000$ times the orbital frequency. This fast frequency is about 10^{20} Hz and resonates with the x-ray frequencies used in the Compton effect [22], an interesting coincidence. The ping-pong frequencies of hydrogen have the magnitude of the zitterbewegung frequency of Dirac's relativistic version of Schroedinger's equation [23]. In modern Aharonov-Bohm experiments [24], the ballistic electron passes at a manometric distance from the electrons inside the solenoid, such that the PP frequencies, which fall as $1/r$, can be even at x rays. The magnetic field interacts with the PP oscillations of the solenoid electrons, which in turn influences the ballistic electron. This fact that electrons can play *ping-pong-at-a-distance* with frequencies up to x-ray suggests the need of an x-ray shielding for the solenoid of the Aharonov-Bohm experiments—e.g., a layer of lead.

The three-frequency orbit solves several conundrums of the hydrogen atom, paradoxes that were created by *guessing* that the equations of classical electrodynamics would accept nonstiff planetarylike orbits at every scale. The ping-pong phenomenon is a nontrivial feature that is not present in the ordinary differential equations of planetary dynamics. The qualitative agreements with QED are listed in the following: (i) The resonant orbits are quantized by integers that appear naturally because of the delay, and the orbital frequencies agree reasonably with the Bohr circular lines. (ii) The angular momenta of the resonant orbits turn out to be approximate integer multiples of Planck's constant. (iii) The emitted frequencies are given by a difference of two eigenvalues of a linear operator, the frequencies of the PP modes, analogously to the Rydberg-Ritz combinatorial principle of QED. (iv) The averaged angular momentum of gyration is of the order of the electronic spin angular momentum of QED [26,27]. The approximation for the gyration angular momentum is not satisfactory yet, though.

We exhibited a new orbit of the hydrogen atom of Dirac's electrodynamics of *point* charges with retarded-only fields. The Lyapunov stability of this orbit should be further investigated. Because we are dealing with a physical theory, the equations should be sufficiently robust to allow some kind of numerical integration [28]. The stability of the three-frequency orbit poses a linear set of delay equations, a dynamical system that needs an initial function as the initial condition, just like Schrödinger's equation. It would be interesting to learn if this linear operator has a self-adjoint Fredholm alternative [19,20], like Schrödinger's equation does. The frequency of PP modes is proportional to the inverse of the interparticle distance, which turns out to be proportional to the electrostatic potential energy (an accidental analogy, because these are different phenomena). The potential energy is thought to be the reason why the $1/r$ appears in Schrödinger's equation, and it is interesting to notice that a linearized equation for PP modes has the same generic form. The ping-pong modes appear universally in electromagnetic many-body dynamics because of the delay, a phenomenon that has been so far overlooked. The interference mechanism of Poynting's theorem can produce orbits that do not radiate, if a resonance is satisfied. This resonance turns out to be satisfied precisely in the atomic magnitude, a surprise that makes this dynamics interesting for theoretical physics. We hope that our preliminary findings serve to motivate and guide further studies of this dynamics, and there is much to be settled yet.

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APPENDIX A: ELECTRODYNAMICS OF POINT CHARGES

In 1947 Eliezer generalized Dirac's covariant subtraction of infinities [10]. In this generalized electrodynamics of point charges [10], the field produced by the point charge is supposed to be the retarded field plus an intrinsic free field G:

$$F^{\nu}_{\mu} = F^{\nu}_{\mu,ret} + G^{\nu}_{\mu}. \tag{A1}$$

The self-free-field G used by Eliezer in Ref. [10] is finite along the particle's trajectory and vanishes when the particle is at rest,

$$G^{\nu}_{\mu} = \Gamma(F^{\nu}_{\mu,ret} - F^{\nu}_{\mu,avd}), \tag{A2}$$

where Γ is a constant [10]. This generalized electromagnetic setting is henceforth called the Eliezer setting (ES). In Eliezer's theory, the electron and the proton of a hydrogen atom, of charges -1 and $+1$, respectively, have the following equations of motion [10]:

$$m_1 \dot{\boldsymbol{v}}_{\mu 1} = \frac{2}{3}(1+2\Gamma)[\ddot{\boldsymbol{v}}_{\mu 1} - \|\dot{\boldsymbol{v}}_1\|^2 \boldsymbol{v}_{\mu 1}] - [F^{\nu}_{\mu 1,in} + (1+\Gamma)F^{\nu}_{\mu 2,ret} - \Gamma F^{\nu}_{\mu 2,adv}]\boldsymbol{v}_{\nu 1},$$

$$m_2 \dot{\boldsymbol{v}}_{\mu 2} = \frac{2}{3}(1+2\Gamma)[\ddot{\boldsymbol{v}}_{\mu 2} - \|\dot{\boldsymbol{v}}_2\|^2 \boldsymbol{v}_{\mu 2}] + [F^{\nu}_{\mu 2,in} + (1+\Gamma)F^{\nu}_{\mu 1,ret} - \Gamma F^{\nu}_{\mu 1,adv}]\boldsymbol{v}_{\nu 2}, \tag{A3}$$

where the $\boldsymbol{v}_{\mu i}$ stand for the four components ($\mu=1,2,3,4$) of the Minkowski velocity of particle i, double ventricle bars stand for the Minkowski norm scalar product and the overdots indicate derivatives respect to the proper time of each particle. Equation (A3) includes an external field $F^{\nu}_{\mu,in}$ produced by the other charges of the universe *at* each particle's trajectory, which vanishes for the isolated electromagnetic two-body problem. The first term on the right-hand side of Eq. (A3) is the sourceless combination of half of the retarded Lienard-Wiechert self-potential minus half of the advanced Lienard-Wiechert self-potential, all multiplied by $(1+2\Gamma)$ and evaluated at the particle itself [10]. This term generalizes Dirac's self-interaction force [1]. The last two terms on the right-hand side of the first line of Eq. (A3) define the interaction with the retarded Lienard-Wiechert potential of the proton at the electronic position, $F^{\nu}_{\mu 2,ret}$, and the interaction with the advanced Lienard-Wiechert potential of the proton at the electronic position, $F^{\nu}_{\mu 2,adv}$. The ES has four interesting limits: (i) $\Gamma=0$ is Dirac's electrodynamics with retarded-only fields [1], (ii) $\Gamma=-1$ is a nonphysical theory with advanced-only interactions, (iii) $\Gamma=-1/2$ is the action-at-a-distance electrodynamics [29] (notice that the self-interaction vanishes), and (iv) the limit when Γ tends to $-1/2$ from above is the dissipative Fokker setting of Ref. [2], with a charge renormalization controlled by $(1+2\Gamma)$.

For the nonspecialist reader, in the following we write the equations of motion of Dirac's electrodynamics, $\Gamma=0$, in the intuitive form of physics textbooks. For the isolated hydrogen atom, the spatial component of the electronic equation of motion, the first line of Eq. (A3), multiplied by $\sqrt{1-|\mathbf{v}_1|^2}$, yields

$$m_1 \frac{d}{dt_1}\left(\frac{\mathbf{v}_1}{\sqrt{1-|\mathbf{v}_1|^2}}\right) = \mathbf{F}_1 - \mathbf{E}_2 - \mathbf{v}_1 \times \mathbf{B}_2, \tag{A4}$$

where $\mathbf{v}_1$ is the Cartesian electronic velocity [12,25]. The equation for the proton is obtained exchanging the indices in Eq. (A4) and multiplying the last two terms on the right-hand side of Eq. (A4) by -1, to account for the positive protonic charge. The first term on the right-hand side of Eq. (A4) is the force $\mathbf{F}_1$ of the electronic fields on the point electron itself and is called the Lorentz-Dirac self-force,

$$\mathbf{F}_1 = \frac{2}{3}\gamma_1^2\{\dot{\mathbf{a}}_1 + \gamma_1^2(\mathbf{v}_1\cdot\dot{\mathbf{a}}_1)\mathbf{v}_1 + 3\gamma^2(\mathbf{v}_1\cdot\mathbf{a}_1) \times[\mathbf{a}_1 + \gamma_1^2(\mathbf{v}_1\cdot\mathbf{a}_1)\mathbf{v}_1]\}, \tag{A5}$$

where $\gamma_1 \equiv 1/\sqrt{1-|\mathbf{v}_1|^2}$ and $\mathbf{a}_1$ and $\dot{\mathbf{a}}_1$ stand for the Cartesian electronic acceleration and time derivative of the Cartesian electronic acceleration, respectively (cf. p. 116 of Ref. [25]). In Eq. (A5) a dot between two Cartesian vectors indicates scalar product and a dot over a vector indicates a derivative respect to laboratory time t_1. The second term on the right-hand side of Eq. (A4) is the electric force—i.e., the electronic charge $e=-1$ times the electric field of the proton acting on the electron, $\mathbf{E}_2$. In our unit system with $c=1$, the retarded electric field of the proton, of charge +1, is given by the Lienard-Wiechert formula [15]

$$\mathbf{E}_2 = \frac{\mathbf{n}_{12b}-\mathbf{v}_{2b}}{\gamma_{2b}^2(1-\mathbf{n}_{12b}\cdot\mathbf{v}_{2b})^3 r_{12b}^2} + \frac{\mathbf{n}_{12b}\times[(\mathbf{n}_{12b}-\mathbf{v}_{2b})\times\mathbf{a}_{2b}]}{(1-\mathbf{n}_{12b}\cdot\mathbf{v}_{2b})^3 r_{12b}}, \tag{A6}$$

where $\mathbf{v}_{2b}$ and $\mathbf{a}_{2b}$ are, respectively, the Cartesian velocity and Cartesian acceleration of the proton at the retarded time t_{2b} and $\gamma_{2b} \equiv 1/\sqrt{1-|\mathbf{v}_{2b}|^2}$. In Eq. (A6), the unit vector $\mathbf{n}_{12b}$ connects the retarded protonic position to the present electronic position and r_{12b} is the interparticle distance along the retarded light-cone. The advanced fields are obtained by replacing $\mathbf{v}_{2b}$ by $-\mathbf{v}_{2b}$ in Eq. (A6) and are not used here, since we deal we Dirac's retarded-only theory. The third term on the right-hand side of Eq. (A4) is the magnetic force—i.e., the electronic charge times the vector product of the electronic velocity $\mathbf{v}_1$ by the magnetic field of the proton $\mathbf{B}_2$. The retarded magnetic field of the proton is given by the Lienard-Wiechert formula [15]

$$\mathbf{B}_2 = \mathbf{n}_{12b}\times\mathbf{E}_2. \tag{A7}$$

For the advanced magnetic field, formula (A7) includes a minus sign, so that the Poynting flux $\mathbf{E}_2\times\mathbf{B}_2$ calculated with the purely advanced fields of an *unperturbed circular orbit* is an incoming energy flux. The combination $-(\mathbf{E}_2+\mathbf{v}_1\times\mathbf{B}_2)$ is usually called the Lorentz force, not to be confused with the Lorentz-Dirac self-force (A5). The electric and magnetic fields (A6) and (A7) depend on the retarded position, velocity, and acceleration of the proton, and this is where delay enters in the dynamics. Equation (A5) can be rearranged as

$$\mathbf{F}_1 = \frac{2}{3}\gamma_1^2 U_1\{\dot{\mathbf{a}}_1 + 3\gamma_1^2(\mathbf{v}_1\cdot\mathbf{a}_1)\mathbf{a}_1\}, \tag{A8}$$

where we introduced the dyadic matrix $U_1 \equiv I + \gamma_1^2\mathbf{v}_1\mathbf{v}_1^t$, a nonsingular matrix whose inverse is $U_1^{-1} = I - \mathbf{v}_1\mathbf{v}_1^t$. The left-hand side of Eq. (A4)—i.e., the variation of the momentum—can be expressed using U_1 as

$$m_1 \frac{d}{dt_1}\left(\frac{\mathbf{v}_1}{\sqrt{1-|\mathbf{v}_1|^2}}\right) = m_1\gamma_1 U_1\mathbf{a}_1. \tag{A9}$$

Using Eqs. (A8) and (A9), the equation of motion (A4) can be expressed as

$$m_1\mathbf{a}_1 = \frac{2}{3}\gamma_1\{\dot{\mathbf{a}}_1 + 3\gamma_1^2(\mathbf{v}_1\cdot\mathbf{a}_1)\mathbf{a}_1\} - \sqrt{1-|\mathbf{v}_1|^2}U_1^{-1} \times(\mathbf{E}_2+\mathbf{v}_1\times\mathbf{B}_2), \tag{A10}$$

where $\mathbf{E}_2$ and $\mathbf{B}_2$ are the electric and magnetic fields of the proton, respectively, and we have canceled the invertible matrix U_1 and a power of γ_1. Equation (A10) has a familiar

Newtonian-like form, but it is still the full relativistic equation. In the low-velocity limit, the first term of the right-hand side of Eq. (A10) reduces to the third derivative of the position multiplied by 2/3, which is called the Abraham-Lorentz-Dirac self-force.

There exists an action formalism for the Lorentz-force sector of Eq. (A4)—i.e., Eq. (A4) without the first term on the right-hand side. We give a general action including the parameter Γ solely for the calculations and cross-checking of Secs. III and IV. A reader interested only in Dirac's theory can set $\Gamma=0$. This general action, involving the instantaneous position and velocity of particle 1, is composed of a kinetic term plus the electromagnetic action

$$\Theta = -\Gamma \int \frac{(1-\mathbf{v}_1\cdot\mathbf{v}_{2a})}{r_{12a}(1+\mathbf{n}_{12a}\cdot\mathbf{v}_{2a})}dt_1 + (1+\Gamma)\int \frac{(1-\mathbf{v}_1\cdot\mathbf{v}_{2b})}{r_{12b}(1-\mathbf{n}_{12b}\cdot\mathbf{v}_{2b})}dt_1. \quad \text{(A11)}$$

In Eq. (A11), $\mathbf{v}_1$ stands for the Cartesian velocity of particle 1 at time t_1, while $\mathbf{v}_{2a}$ and $\mathbf{v}_{2b}$ stand for the Cartesian velocities of particle 2, respectively, at the advanced time t_{2a} and at the retarded time t_{2b}. The vector $\mathbf{n}_{12a}$ in Eq. (A11) is a unit vector connecting the advanced position of particle 2 to the position of particle 1 at its present time t_1, while unit vector $\mathbf{n}_{12b}$ connects the retarded position of particle 2 to the present position of particle 1 at time t_1. Still in Eq. (A11), r_{12a} and r_{12b} indicate the interparticle distance along the advanced and retarded light cones, respectively. To derive the equations of motion of particle 1, one needs to add the usual kinetic action to Eq. (A11), $K_1 \equiv -\int m_1\sqrt{1-|\mathbf{v}_1|^2}dt_1$ (the integral of the kinetic Lagrangian). In Ref. [12] it is shown that formal minimization of the sum of action (A11) with the kinetic action yields the equations of motion of particle 1 suffering the electromagnetic fields of particle 2—i.e., Eq. (A4) without the self-interaction term. Each integrand of the right-hand side of Eq. (A11) is a familiar electromagnetic Lagrangian

$$L_c \equiv \frac{(1-\mathbf{v}_1\cdot\mathbf{v}_{2c})}{r_{12c}[1+(\mathbf{n}_{12}\cdot\mathbf{v}_{2c})/c]} \equiv -(V_c - \mathbf{v}_1\cdot\mathbf{A}_c), \quad \text{(A12)}$$

where V_c and $\mathbf{A}_c$ are the Lienard-Wiechert scalar potential and the Lienard-Wiechert vector potential, respectively. We introduced the quantity $c=\pm 1$ in Eq. (A12) to indicate if the interaction is along the advanced or retarded light cone. Equation (A12) with $c=1$ indicates that particle 2 is in the future, while $c=-1$ indicates that particle 2 is in the past. The quantities of particle 2 in Eq. (A12) are to be evaluated at the time t_{2c} defined implicitly by

$$t_{2c} = t_1 + \frac{r_{12c}}{c}, \quad \text{(A13)}$$

where r_{12+} is the distance along the advanced light cone and r_{12-} is the distance along the retarded light cone. According to Eq. (A13), the time lag along the advanced light cone is r_{12+} and the time lag along the retarded light cone is $-r_{12-}$.

The shortest way to obtain the equations of motion of Dirac's electrodynamics in any given coordinate system is to start from the Euler-Lagrange equations of action (A11) plus the relativistic kinetic action. This yields the dynamics without self-interaction of each particle suffering the electromagnetic fields of the other particle. The self-force can be added to the Euler-Lagrange equations, watching carefully for the correct multiplicative factor. The stiff limit of PP modes is determined by the largest-order derivative appearing in the linearized equations of motion. In this limit, the contribution of the self-interaction to the linearized dynamics about a circular orbit is given by the Abraham-Lorentz-Dirac self-force

$$\mathbf{F}_{rad} = \frac{2}{3}(1+2\Gamma)\dot{\mathbf{a}}. \quad \text{(A14)}$$

The contribution of the other nonlinear terms, which become important at a finite distance from the circular orbit, is discussed in Appendix C. The electrodynamics of point charges is discussed at length in Refs. [10,25], while the Lienard-Wiechert potentials, Lienard-Wiechert fields, Poynting's theorem, and the physics of electrodynamics are found in numerous textbooks—e.g., Refs. [12,15].

APPENDIX B: VARIATIONAL EQUATIONS ALONG THE $\hat{z}$ DIRECTION

In this appendix we derive the linearized variational equations for displacements perpendicular to the orbital plane, henceforth called the $\hat{z}$ direction. Since $z_k=0$ is an exact solution of the equations, in this appendix we perform a Lyapunov stability analysis. The variational dynamics along the z direction is uncoupled from the planar variational equations up to the linear order. In the same way of Sec. IV, we expand to second order the implicit light-cone condition, Eq. (25), and action (A11). The Cartesian coordinates of a transversely perturbed circular orbit are defined as

$$x_k + iy_k \equiv r_b d_k \exp(i\Omega t),$$

$$x_k - iy_k \equiv r_b d_k^* \exp(-i\Omega t),$$

$$z_k \equiv r_b Z_k, \quad \text{(B1)}$$

where $k=1$ for the electron and $k=2$ for the proton, Z_k is the transverse perturbation, $d_1 \equiv b_1$ and $d_2 \equiv -b_2$ are defined in Eq. (4), and Ω is the orbital frequency (1). In the following we calculate the delay function φ_c of Eq. (13) by expanding the light-cone time t_{2c} about the constant lag r_b up to the second order in Z_1 and Z_2. The distance r_{12c} in Eq. (A13) is evaluated from the position of particle 1 at time t_1 to the position (B1) of particle 2 at time t_{2c}. Using t_{2c} defined by Eq. (A13) and orbit (B1), this implicit distance $r_{12c}=|t_2-t_1|$ evaluates to

$$r_{12c}^2 \equiv r_b^2(1+\phi)^2 = r_b^2[b_1^2+b_2^2+2b_1b_2\cos(\varphi+\theta c)] + r_b^2(Z_1-Z_{2c})^2, \quad \text{(B2)}$$

where we expressed φ_c in terms of the scaled function ϕ of Eq. (24). The Z variations decuple from the planar variations because powers of Z always appear squared, so that there is no mixed linear term of Z times a linear perturbation of the

planar coordinate in Eq. (B2). Expanding Eq. (B2) up to second order and rearranging yields

$$\phi^2 + 2S\phi = (Z_1 - Z_{2c})^2. \tag{B3}$$

Equation (B3) is a quadratic equation for ϕ, and the regular solution is given up to second order by

$$\phi = \frac{1}{2S}(Z_1 - Z_{2c})^2. \tag{B4}$$

The coordinate Z_2 appears evaluated at the advanced and retarded time in Eq. (B4), and to obtain the action up to quadratic terms it is sufficient to keep the first term $Z_2(\tau_1 + c\theta + \varphi) \simeq Z_2(\tau_1 + c\theta) \equiv Z_{2c}$. Using (B1) to calculate the numerator of Lagrangian (A12) yields

$$\begin{aligned}(1 - \mathbf{v}_1 \cdot \mathbf{v}_{2c}) &= 1 + \theta^2 \cos(\theta + c\varphi) b_1 b_2 - \theta^2 \dot{Z}_1 \dot{Z}_{2c} \\ &\approx C - \theta^2 (S-1)\phi - \theta^2 \dot{Z}_1 \dot{Z}_{2c},\end{aligned} \tag{B5}$$

while the denominator of Lagrangian (A12) evaluates to

$$\begin{aligned}r_{12c}\left(1 + \frac{\mathbf{n}_{12c} \cdot \mathbf{v}_{2c}}{c}\right) &= r_b[1 + \phi + \theta c b_1 b_2 \sin(\theta c + \varphi_c) \\ &\quad + \theta c (Z_1 - Z_{2c})\dot{Z}_{2c}].\end{aligned} \tag{B6}$$

Notice that the quadratic term $Z_{2c}\dot{Z}_{2c}$ on the right-hand side of Eq. (B6) can be dropped because it represents an exact gauge that does not affect the Euler-Lagrange equations. We also expand the argument of the sine function of the right-hand side of Eq. (B6) until the linear term in φ_c, so that the quadratic approximation to Eq. (B6) is

$$r_{12c}\left(1 + \frac{\mathbf{n}_{12c} \cdot \mathbf{v}_{2c}}{c}\right) \approx r_b[S + C\phi + \theta c Z_1 \dot{Z}_{2c}], \tag{B7}$$

where the equivalence sign $\approx$ henceforth means equivalent up to a gauge term of second order. Even if a quadratic gauge term appears in the denominator, in an expansion up to quadratic order it would still produce a gauge and therefore it can be dropped directly from the denominator. The Lagrangian of action (A11) expanded up to second order is

$$\begin{aligned}\mathcal{L}_\Gamma \approx \left(\frac{C}{r_b S}\right)\Bigg\{&(1+\Gamma)\Bigg[1 - \theta^2 C S^2 \dot{Z}_1 \dot{Z}_{2-} - \frac{C^2 S}{2}(Z_1 - Z_{2-})^2 \\ &+ \theta C^2 S Z_1 \dot{Z}_{2-}\Bigg] - \Gamma\Bigg[1 - \theta^2 C S^2 \dot{Z}_1 \dot{Z}_{2+} - \frac{C^2 S}{2}(Z_1 - Z_{2+})^2 \\ &- \theta C^2 S Z_1 \dot{Z}_{2+}\Bigg]\Bigg\}.\end{aligned} \tag{B8}$$

We henceforth disregard $O(\theta^2)$ corrections and substitute $C = S = 1$ in the coefficients of Lagrangian (B8). Last, we need the kinetic Lagrangian along orbit (B1) as

$$T_1 = -m_1\sqrt{1 - v_1^2} = -\frac{m_1}{\gamma_1}\sqrt{1 - \gamma_1^2 \theta^2 \dot{Z}_1^2}, \tag{B9}$$

where the overdot means a derivative with respect to the scaled time τ, $\gamma_1^{-1} \equiv \sqrt{1 - v_1^2}$, and we have used that $\Omega r_b = \theta$. Equation (B9) expanded up to second order is

$$T_1 = \left(\frac{1}{r_b}\right)\left\{-\frac{r_b m_1}{\gamma_1} + \frac{\epsilon_1}{2}\dot{Z}_1^2 + \cdots\right\}, \tag{B10}$$

where $\epsilon_1 \equiv r_b^3 m_1 \gamma_1 \Omega^2$. Using Eqs. (1) and (2), we obtain $\epsilon_1 = m_1/\mu = M/m_2$. The equation of motion for particle 1, without the self-interaction term, is the Euler-Lagrange equation of the quadratic Lagrangian

$$L_{eff}^{(1)} = T_1 + \mathcal{L}_\Gamma. \tag{B11}$$

The Lagrangian sector of the equation for Z_1 is

$$\begin{aligned}\epsilon_1 \ddot{Z}_1 = &-[Z_1 - (1+\Gamma)Z_{2-} + \Gamma Z_{2+}] + \theta(\Gamma \dot{Z}_{2+} + (1+\Gamma)\dot{Z}_{2-}) \\ &+ \theta^2[\Gamma \ddot{Z}_{2+} - (1+\Gamma)\ddot{Z}_{2-}].\end{aligned} \tag{B12}$$

Notice that the left-hand side of Eq. (B12) can be written as

$$\epsilon_1 \ddot{Z}_1 = r_b^3 m_1 \gamma_1 \Omega^2 \ddot{Z}_1 = r_b^2 \frac{dp_z}{dt}, \tag{B13}$$

which is the force along the z direction multiplied by r_b^2. As explained above Eq. (40), to account for self-interaction we must add to Eq. (B12) the Abraham-Lorentz-Dirac self-force (A14) multiplied by r_b^2:

$$r_b^2 \mathbf{F}_{rad} = \frac{2}{3}(1 + 2\Gamma)\theta^3 \dddot{Z}_1. \tag{B14}$$

The full linearized variational equation for Z_1 is

$$\begin{aligned}\epsilon_1 \ddot{Z}_1 = &\frac{2}{3}(1+2\Gamma)\theta^3 \dddot{Z}_1 - [Z_1 + \Gamma Z_{2+} - (1+\Gamma)Z_{2-}] \\ &+ \theta[\Gamma \dot{Z}_{2+} + (1+\Gamma)\dot{Z}_{2-}] + \theta^2[\Gamma \ddot{Z}_{2+} - (1+\Gamma)\ddot{Z}_{2-}].\end{aligned} \tag{B15}$$

The linearized equation for Z_2 is obtained by interchanging Z_1 by Z_2 and ϵ_1 by ϵ_2 in Eq. (B15). [Comparing Eq. (B15) to Eq. (30) of Ref. [2] we find that Eqs. (29) and (30) of Ref. [2] are both missing a θ^3 factor in front of the $\dddot{Z}_1$ term, which is a typo. After Eq. (30), the other equations of Ref. [2] have the self-interaction included correctly].

A ping-pong normal mode is obtained by substituting $Z_1 = A\exp(\lambda_z \Omega t/\theta)$ and $Z_2 = B\exp(\lambda_z \Omega t/\theta)$ into Eq. (B15) and the corresponding linearized equation for the proton equations. Again we use a general λ_z, but a harmonic solution needs a purely imaginary λ_z. Setting the determinant to zero,

$$\begin{vmatrix} 1 + \dfrac{M\lambda_z^2}{m_2\theta^2} - \dfrac{2}{3}(1+2\Gamma)\lambda_z^3 & G(\theta,\lambda_z) \\ G(\theta,\lambda_z) & 1 + \dfrac{M\lambda_z^2}{m_1\theta^2} - \dfrac{2}{3}(1+2\Gamma)\lambda_z^3 \end{vmatrix} = 0, \tag{B16}$$

where $G(\theta,\lambda_z) \equiv [1 - (1+2\Gamma)\lambda_z - \lambda_z^2]\cosh(\lambda_z) - [(1+2\Gamma) - \lambda_z - (1+2\Gamma)\lambda_z^2]\sinh(\lambda_z)$. The stiff-limit is when $|\lambda_z|$ is large, and we should keep in mind that the hyperbolic functions in $G(\theta,\lambda_z)$ can acquire a large magnitude [11]. Multiplying determinant (B16) by $\mu\theta^4/(M\lambda_z^4)$ we obtain

$$1-\frac{2}{3}(1+2\Gamma)\theta^2\lambda_z+\frac{4}{9}\frac{\mu}{M}\theta^4\lambda_z^2 - \frac{\mu\theta^4}{M}\left\{\left[1-\frac{1}{\lambda_z^2}+\frac{(1+2\Gamma)}{\lambda_z}\right]\cosh(\lambda_z) - \left[\frac{1}{\lambda_z}+(1+2\Gamma)\left(1-\frac{1}{\lambda_z^2}\right)\right]\sinh(\lambda_z)\right\}^2=0, \tag{B17}$$

up to small $O(\theta^2)$ terms. The stiff-mode condition defined by Eq. (B17) with $\Gamma=-1/2$ is Eq. (33) of Ref. [2]—i.e.,

$$1-\frac{2}{3}\theta^2\lambda_z+\frac{4\mu}{9M}\theta^4\lambda_z^2-\frac{\mu\theta^4}{M}\left[\left(1-\frac{1}{\lambda_z^2}\right)\cosh^2(\lambda_z) - \frac{1}{\lambda_z}\sinh(2\lambda_z)\right]^2=0. \tag{B18}$$

In Ref. [2] there is another typo in passing from Eq. (33) to Eq. (34); Eq. (34) is missing a bracket that should start after the $(\mu\theta^4/M)$ factor and close at the end of Eq. (34). The stiff limit in Dirac's theory with retarded-only fields ($\Gamma=0$) is

$$1-\frac{2}{3}\theta^2\lambda_z+\frac{4\mu}{9M}\theta^4\lambda_z^2-\frac{\mu\theta^4}{M}[1+\exp(-2\lambda_z)] \times\left(1+\frac{2}{\lambda_z}-\frac{1}{\lambda_z^2}-\frac{1}{\lambda_z^3}+\frac{1}{\lambda_z^4}\right)=0, \tag{B19}$$

and the appearance of the negative exponential is related to the retardation only. Comparing Eq. (B17) to Eq. (41) we find that the quasidegeneracy phenomenon exists only for (i) $\Gamma=0$, i.e., Dirac's theory; (ii) $\Gamma=-1$, a nonphysical advance-only case; and (iii) $\Gamma=-1/2$, the action-at-a-distance electrodynamics [29] and the dissipative Fokker dynamics of Ref. [2].

APPENDIX C: THE PING-PONG SOLUTIONS

In this appendix we discuss the existence of a harmonic orbit such as given by Eq. (47) and illustrated in Figs. 1 and 3. In the following we show that an orbit such as Eq. (47) exists in the limit where $\rho_k\equiv\sup(|R_k^z|,|R_k^{xy}|)\to 1/|\lambda|$, a limit where the stiff gyration approaches the speed of light. In this limit, the dominant field of the particle is the far-electric field, Eq. (48). The near-electric field is the first term on the right-hand side of Eq. (A6), which vanishes because $\gamma_k\to\infty$. The far-magnetic interaction is the next-to-leading term in size and is disregarded in the following. We also disregard the self-interaction, Eq. (A5), because its contribution to the force *normal* to the velocity is smaller. The equation of motion for the proton is obtained by exchanging indices in Eq. (A10), and the leading fast dynamics *normal* to the fast velocity is

$$m_2\mathbf{a}_2=-\sqrt{1-|\mathbf{v}_2|^2}U_2^{-1}\left\{\frac{\mathbf{n}\times[(\mathbf{n}\times\dot{\mathbf{v}}_{1-})-(\mathbf{v}_{1-}\times\dot{\mathbf{v}}_{1-})]}{(1-\mathbf{n}\cdot\mathbf{v}_{1-})^3 r_{21b}}\right\}, \tag{C1}$$

where underscore minus indicates that particle 1 is in the past light cone. The unit vector $\mathbf{n}$ points from the retarded position of particle 1, which is not indicated with underscore to avoid an overloaded notation. For the PP modes, the fact that the electronic coordinates on the right-hand side of Eq. (C1) are evaluated at $t_{1-}\simeq t_2-r_b$ makes *a lot* of difference, because the PP modes execute complete periods during this time lag. Since the gyration amplitude is small compared with r_b, we henceforth assume that the distance r_{21b} along the retarded light cone on the right-hand side of Eq. (C1) is constant and given by r_b. Along the orbit of Fig. 1, the fast gyration can be parallel to the normal, so that $(1-\mathbf{n}\cdot\mathbf{v}_{1-})^3$ becomes arbitrarily small, and we henceforth approximate the denominator of the right-hand side of Eq. (C1) by $(1-|\lambda|\rho_1)^3 r_b$. We also approximate the square root on the left-hand side of Eq. (C1) by $(1-|\lambda|^2\rho_2^2)^{1/2}$. The stiff limit is obtained by substituting orbit (47) with $\rho_k=\sup(|R_k^z|,|R_k^{xy}|)$ into Eq. (C1) and taking the Fourier component of Eq. (C1) along the PP frequency $w_{PP}\simeq\pi q\Omega/\theta$. We also multiply Eq. (C1) by r_b^2, so that the order-of-magnitude balancing posed by the Fourier-transformed version of Eq. (C1) is

$$\frac{m_2}{\mu\theta^2}\rho_2\simeq\frac{(1-|\lambda|^2\rho_2^2)^{1/2}}{(1-|\lambda|\rho_1)^3}\rho_1\exp(-\lambda). \tag{C2}$$

The equation of motion for the electron is obtained by interchanging indices in Eq. (C2). A salient feature of Eq. (C2) is that λ must be purely imaginary and multiple of π, $\lambda=\pi qi$, so that the ρ's can be real in Eq. (C2); i.e., the phase shift $2\,\mathrm{Im}(\lambda)$ must be a multiple of 2π. Considering the fraction on the right-hand-side of Eq. (C2) as a given number, Eq. (C2) and the corresponding equation for the electron are two linear homogeneous equations for ρ_1 and ρ_2. Equating the determinant of this homogeneous system to zero and disregarding smaller terms yields

$$\frac{\mu\theta^4\exp(-2\lambda)}{M}\simeq\frac{(1-|\lambda|\rho_1)^{5/2}(1-|\lambda|\rho_2)^{5/2}}{\sqrt{(1+|\lambda|\rho_1)(1+|\lambda|\rho_2)}}. \tag{C3}$$

Equation (C3) gives the main finite-amplitude correction to Eq. (43), to which it reduces for $\rho_1=\rho_2=0$. The finite-ρ corrections cancel the σ's, which was the justification for the approximation explained above Eq. (54). When $\sup(\rho_1,\rho_2)$ approaches $1/|\lambda|$ from below, the right-hand side of Eq. (C3) matches the $O(\theta^4)$ left-hand side, so that Eq. (C3) accepts a purely harmonic solution with $\lambda=\pi qi$, as we wanted to demonstrate here.

After solving for the ping-pong oscillation, the dynamics at the slow frequency is the next-to-leading of the multiscale solution. For the electronic motion, Eq. (A10) has a zero-order dissipative force along the unperturbed orbit, as illustrated in Fig. 3. This offending force has a component along the electronic velocity that is given by

$$r_b^2F_r=r_b^2\frac{2}{3}\dddot{x}_1=-\frac{2e^2}{3c^3}\Omega^2r_b^2\dot{x}_1=-\frac{2}{3}\theta^3, \tag{C4}$$

where we multiplied the force by r_b^2, as in Eqs. (40) and (B14). The Lorentz force of the proton along the *unperturbed* circular orbit is almost normal to the electronic velocity and does not contribute much for the dissipation. On the equation of the proton, the main offending force at zero order is the

delayed interaction with the electron, instead of the much smaller protonic self-interaction. Using Page's series in the same way of Ref. [17], we find an offensive zero-order force against the protonic velocity of the same magnitude of force (C4). It turns out that it is impossible to cancel the zero-order force (C4) with the linear terms of the variational equations, which is shown by averaging the linearized equations over the orbital period, an averaging that yields a 4×4 linear system with no solutions. The balancing of the offensive zero-order force is nevertheless possible by using the resonance between ping-pong modes that is introduced at *quadratic* order by the spin-radiation term, as discussed below.

Multiplying the offensive force (C4) by the velocity along the circular orbit yields an $O(\theta^4)$ dissipated power. Again, the discussion of the dissipated power must start from the leading fast dynamics, as follows. Taking the scalar product of Eq. (A10) with $\mathbf{v}_1$ and disregarding the contribution of the protonic fields for the dissipated power yields

$$m_1(\mathbf{v}_1\cdot\mathbf{a}_1)=\frac{2}{3}\gamma_1\{(\mathbf{v}_1\cdot\dot{\mathbf{a}}_1)+3\gamma_1^2(\mathbf{v}_1\cdot\mathbf{a}_1)^2\}. \tag{C5}$$

The pair of brackets on the right-hand side of Eq. (C5) is multiplied by the possibly large factor γ_1, so that it must vanish in the large-$|\mathbf{v}_1|$ limit—i.e.,

$$(\mathbf{v}_1\cdot\mathbf{a}_1)^2\simeq-\frac{1}{3\gamma_1^2}(\mathbf{v}_1\cdot\dot{\mathbf{a}}_1). \tag{C6}$$

The offending force (C4), multiplied by the slow guiding-center velocity θ, gives only an $O(\theta^4)$ contribution to the right-hand side of Eq. (C5). The dominant contribution is given by the PP oscillation, of frequency $\Omega|\lambda|/\theta$, and defined by Eqs. (46) and (47). We henceforth replace the $\dot{\mathbf{a}}_1$ on the right-hand-side of Eq. (C6) by $-(\Omega^2|\lambda|^2/\theta^2)\mathbf{v}_1$. Multiplying Eq. (C6) by r_b^2 and using the fast component of orbit (47) with $\rho_1\equiv\sup(|R_1^z|,|R_1^{xy}|)$ yields

$$|\lambda|^2\rho_1^2\cos^2(\varkappa)\simeq\frac{|}{3}(1-\rho_1^2|\lambda|^2), \tag{C7}$$

where $\varkappa$ is the angle between the velocity and the acceleration. In the limit when $|\mathbf{v}_1|\to1$, this angle $\varkappa$ must be close to ninety degrees, since $|\mathbf{v}_1|$ must be less than one and $(\mathbf{v}_1\cdot\mathbf{a}_1)=(d/dt)(|\mathbf{v}_1|^2/2)$. Using Eq. (C7) together with $|\cos(\varkappa)|\leqslant1$ yields a *lower bound* for ρ_1:

$$\rho_1\geqslant\frac{1}{2|\lambda|}. \tag{C8}$$

Next-to-leading in the multiscale solution is the guiding-center dynamics—i.e., the equilibration of offending force (C4)—a balance that takes place *after* the fast dynamics is established as a harmonic oscillation with a radius given by Eq. (C8). In the following we give a third derivation of resonance (52) by averaging the equations of motion over the fast time scale. Due to the larger protonic mass, we henceforth assume that the protonic field at the electron averages essentially to the Coulombian field, a force that rotates at the guiding-center frequency. On the other hand, in the limit $\rho_1|\lambda|\to1$, the electronic field on the proton, averaged over the fast time scale, is significantly changed. Analogously to the derivation of Eq. (57), the electronic spin-radiation field, second term on the right-hand side of Eq. (C1), averaged over the fast time scale, rotates at the guiding-center frequency Ω *if* resonance (52) holds. This averaged field adds to the rotating electronic Coulombian field on the proton [not included in Eq. (C1)], so that the averaged attractive force on the proton deviates from the diametral direction. The proton repositions along the circular orbit, until its velocity becomes *perpendicular* to this perturbed centripetal force, as illustrated in Fig. 3. After repositioning, the angular distance between particles at the same time is *less* than 180°. The self-consistent repositioning stops when the Coulombian field of the proton acquires a component along the electronic velocity to balance the electronic self-interaction (C4). The perturbed equilibrium position along the guiding-center orbit is illustrated in Fig. 3. The situation of Fig. 3 is possible only at resonance, when the averaged spin-radiation force acting on the proton rotates at the guiding-center frequency. In Ref. [17] we also used a resonance between mutually orthogonal vibration modes. The present work goes beyond our simple estimates of the helium dynamics of Ref.[17]. Unlike the Coulombian orbits of helium [17], Coulombian circular orbits of hydrogen radiate in dipole, and we have seen here that the fast PP oscillations are essential to cancel this dipolar radiation, a complex dynamics that demands a multiscale solution. The PP modes are lost when the delay is expanded, as in Page's series of Ref. [17], a nontrivial manifestation of stiffness.

For θ in the atomic magnitude, the lower bound (C8) is a few percent of the interparticle distance; nevertheless, the distance $\rho_1 r_b$ is already some 1000 classical electronic radii, much larger than the radius of the fat Lorentz electron [27]. To improve on estimate (C8) one needs the relation between $\cos(\varkappa)$ and $(1-\rho_1^2|\lambda|^2)$, which is beyond the present work. To solve Eq. (C3) it suffices that either $|R_1^z|$ or $|R_1^{xy}|$ approaches the upper limiting value $1/|\lambda|$; the other R_k can take a much lower value—i.e., a solution exists already in the limit when just the electron rotates near the speed of light. As an estimate for the angular momentum of gyration, Eq. (57), we use $|R_1^z|$ and $|R_1^{xy}|$given by Eq. (C8), yielding

$$|l_{xy}|=\frac{1}{4\pi\theta^2 q}. \tag{C9}$$

Calculating the angular momentum (C9) with the first line of Table I gives $|l_{xy}|=213.5$. The angular momenta estimated by Eq. (C9) depend on q, but are in the correct order of magnitude. The electronic spin angular momentum of QED is independent of the orbital quantum number and given by $|s|=\sqrt{3}\hbar/2=118$.

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