PHYSICAL REVIEW E **73**, 026108 (2006)

Probabilistic dynamics of some jump-diffusion systems

Edoardo Daly* and Amilcare Porporato†
Department of Civil and Environmental Engineering & Nicholas School of the Environment and Earth Sciences, Duke University, Durham, North Carolina 27708, USA
(Received 29 August 2005; published 7 February 2006)

Some exact solutions to the forward Chapman-Kolmogorov equation are derived for processes driven by both Gaussian and compound Poisson (shot) noise. The combined action of these two forms of white noise is analyzed in transient and equilibrium conditions for different jump distributions and additive Gaussian noise. Steady-state distributions with power-law tails are obtained for exponentially distributed jumps and multiplicative linear Gaussian noise. Two applications are discussed: namely, the virtual waiting-time or Takàcs process including Gaussian oscillations and a simplified model of soil moisture dynamics, in which rainfall is modeled as a compound Poisson process and fluctuations in potential evapotranspiration are Gaussian.

DOI: 10.1103/PhysRevE.73.026108 PACS number(s): 02.50.−r, 89.60.−k, 05.40.−a

I. INTRODUCTION

High-dimensional dynamical systems can be often studied in terms of a single representative variable and separating an internal deterministic dynamics from random noises accounting for external fluctuations [1,2]. The nature of such external forcings can be very different, according to their temporal regime and intensity distribution, and in some cases multiple forms of noise may coexist. A very typical case arises when fluctuations acting continuously in time are modeled using a Gaussian noise, which can be either additive or multiplicative [2–4], while random jumps that occur instantaneously are described by means of a compound Poisson process [5–8]. Such systems, which will be referred to as jump-diffusion processes, have been analyzed in various fields. In stochastic finance, for example, they have been used to describe the joint action of small, frequent transactions and rare, large movements of money. The same models apply in general to problems related to queuing theory and storage models [5,6]. In physics, they have been adopted to study mechanisms of noise-driven transport—e.g., ratchet-type models [7,9,10]. In neurophysiology, the membrane potential of a single neuron has been modeled by coupling an Ornstein-Uhlenbeck process (Gaussian noise), which defines the nerve-cell voltage from its resting level, to a renewal process (Poisson noise), which determines the spike trains of the action potential [11–13]. Jump-diffusion processes have also applications in population dynamics [14], in hydrology to model soil water balance (see [15] and Sec. IV B), and in the general theory of stochastic processes [16–18]. Despite their wide interest, however, few analytical solutions have been proposed so far.

The dynamics of the jump-diffusion system under analysis can be described using a Langevin equation

$$\dot{x} = f(x(t)) + g(x(t))\xi(t) + I(t), \qquad (1)$$

where $f(x)$ and $g(x)$ are deterministic functions of the state variable $x(t)$, $\xi(t)$ is a Gaussian δ-correlated noise of zero mean and intensity σ^2—i.e., $\langle \xi(t)\rangle = 0$ and $\langle \xi(t)\xi(u)\rangle = 2\sigma^2\delta(t-u)$—and $I(t)$ is a compound Poisson noise, defined as

$$I(t) = \sum_{k=1}^{N(t)} z_k \delta(t - t_k), \qquad (2)$$

where $N(t)$ is a Poisson counting process with frequency λ (e.g., the mean number of δ impulses per unit time) and $\{z_k\}$ are the jump sizes distributed according to a probability density function $b(z)$. When necessary, Eq. (1) will be interpreted in the Stratonovich sense. Accordingly, Eq. (1) corresponds to the forward Chapman-Kolmogorov equation [1,16]

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}\left\{\left[f(x) + \sigma^2 g(x)\frac{d}{dx}g(x)\right]p(x,t)\right\} + \sigma^2\frac{\partial^2}{\partial x^2}[g(x)^2 p(x,t)] - \lambda p(x,t) + \int_{-\infty}^{+\infty}\lambda b(y)p(x-y,t)dy, \qquad (3)$$

which expresses the evolution of the transition probability density function (PDF) of x, $p(x,t)$. In the following, some special forms of Eq. (3) will be analyzed. In particular, we will distinguish between additive (Sec. II) and multiplicative (Sec. III) Gaussian noise. In the first case, some examples of jump distributions (e.g., two-sided exponential, gamma, and exponential distribution) are studied when the drift is either constant or linear, while the multiplicative noise is studied for linear drift and exponentially distributed jumps. Two possible applications of the results obtained in the previous sections are discussed in detail in Sec. IV.

II. ADDITIVE GAUSSIAN NOISE

In a variety of problems, external fluctuations can be taken into account by simply adding a noise source to the deterministic component, $f(x)$—that is, assuming $g(x)=1$ in Eq. (1) [5,10]. Provided that $f(x)$ attains a finite value when

*Electronic address: edaly@pratt.duke.edu
†Electronic address: amilcare@duke.edu

x tends to $\pm\infty$, while $p(x,t)$ and its derivatives tend to zero, Eq. (3) can be Fourier transformed as

$$\frac{\partial}{\partial t}p^*(s,t) = is\{f(x)p(x,t)\}^* - \sigma^2 s^2 p^*(s,t) + \lambda[b^*(s) - 1]p^*(s,t), \tag{4}$$

where $p^*(s,t)$ is the characteristic function of $p(x,t)$, i.e.,

$$p^*(s,t) = \int_{-\infty}^{+\infty} e^{ixs} p(x,t)dx, \tag{5}$$

and $\{h(x,t)\}^*$ indicates the Fourier transform of the function inside the brackets—i.e., $h^*(s,t)$. The two particular cases of constant and linear drift are now studied in detail, using simple jump distributions $b(z)$ that allow one to obtain exact solutions of Eq. (3).

A. Constant drift

Assuming constant drift, $f(x)=k$, the characteristic function of the PDF becomes $p^*(s,t)=p^*(s,0)p_1^*(s,t)p_2^*(s,t)$, with $p^*(s,0)=\exp(ix_0 s)$, given that $p(x,0)=\delta(x-x_0)$, and

$$p_1^*(s,t) = \exp[(iks - \sigma^2 s^2)t],$$

$$p_2^*(s,t) = \exp\{\lambda[b^*(s) - 1]t\}. \tag{6}$$

Therefore the solution can be represented by the convolution between the transient PDF of the Wiener process with drift, whose characteristic function is $p^*(s,0)p_1^*(s,t)$ [8], and the PDF of the compound Poisson process. A similar result was obtained in [17] in the absence of drift.

The first case we consider is that of jumps extracted from a two-sided exponential distribution, also called a Laplace or Pascalian distribution, $b(z)=\gamma \exp(-\gamma|z|)/2$ $(\gamma>0)$, so that $p_2(s,t)=\exp[-\lambda s^2 t/(\gamma^2+s^2)]$. In this case, the mean and variance vary linearly in time as $\mu_t=x_0+kt$ and $\mathrm{var}_t=2t(\sigma^2+\lambda/\gamma^2)$, respectively. Odd cumulants are zero—i.e., $p(x,t)$ is symmetric—while even cumulants of order higher than 2, as expected, are not affected by the Gaussian noise and read $\kappa_m=m!\lambda t/\gamma^m$. Furthermore, the kurtosis is equal to [19]

$$\beta = 3 + \frac{6\lambda/\gamma^4}{t(\lambda/\gamma^2 + \sigma^2)^2}, \tag{7}$$

which means that for finite time the distribution has tails that are heavier than a Gaussian. It is easy to see that in the limiting case of no Gaussian noise (i.e., $\sigma^2=0$) and both λ and γ^2 tending to $+\infty$ with a finite ratio, the solution tends to a Gaussian distribution [20].

A quite general choice for positive definite jumps $(z>0)$ is the Gamma distribution, i.e., $b(z)=\gamma^a z^{a-1}\exp(-\gamma z)/\Gamma(a)$, where $\Gamma(\cdot)$ is the gamma function [19] and $\gamma>0$ and $a>0$ are two parameters. In this case,

$$p_2^*(s,t) = \exp\left[-\lambda t + \frac{\gamma^a t}{(\gamma - is)^a}\right], \tag{8}$$

and the mean and variance of x depend on time as $\mu_t=x_0+(k+a\lambda/\gamma)t$ and $\mathrm{var}_t=t[2\sigma^2+a\lambda(1+a)/\gamma^2]$, respectively, while the cumulants of order higher than 2 are $\kappa_m=(a+m-1)!\lambda t/[(a-1)!\gamma^m]$. The PDF is therefore asymmetrical (e.g., positive skewness), because of the asymmetry in the jumps, and the kurtosis is

$$\beta = 3 + \frac{6a + 11a^2 + 6a^3 + a^4}{t[(a + a^2)\lambda + 2\gamma^2\sigma^2]^2}\lambda, \tag{9}$$

which is again greater than 3. It is interesting to note that in the particular case when $a=1$—that is, when the gamma distribution reduces to an exponential one, all the even cumulants correspond to those of the previous case, Eq. (7), with jumps distributed as a two-sided exponential PDF.

It is clear that the constant drift only affects the means of the distributions but not their shape. In fact, using the coordinate transformation $\eta=x-kt$ and $\tau=t$, the problem with constant drift can be reduced to a sum of noises without any deterministic component in the new variables η and τ. This latter case was already discussed in part in [17] within a more general context.

B. Linear drift

Another paradigmatic case is that of linear drift, $f(x)=-kx$ $(k>0)$, which, in the absence of the Poisson noise, is the well-known Ornstein-Uhlenbeck process. A general solution of the problem with both the noises can be written again as a product of characteristic functions $p^*(s,0)p_1^*(s,t)p_2^*(s,t)$, with now

$$p_1^*(s,t) = \exp\left[ix_0 s e^{-kt} + \frac{\sigma^2 s^2}{2k}(e^{-2kt} - 1)\right],$$

$$p_2^*(s,t) = \exp\left\{\int_s \frac{\lambda}{ku}[b^*(u) - 1]du\right\} \times \exp\left\{-\int_{s\exp(-kt)} \frac{\lambda}{ku}[b^*(u) - 1]du\right\}, \tag{10}$$

where u is a dummy variable.

Assuming $b(z)$ to be a two-sided exponential distribution, the function $p_2^*(s,t)$ becomes

$$p_2^*(s,t) = \left(\frac{\gamma^2 + s^2 e^{-2kt}}{\gamma^2 + s^2}\right)^{\lambda/2k}, \tag{11}$$

which leads to mean and variance $\mu_t=x_0 e^{-kt}$ and $\mathrm{var}_t=[1-\exp(-2kt)][(\lambda+\gamma^2\sigma^2)/(k\gamma^2)]$, respectively. Odd cumulants are zero (i.e., symmetric PDF), while even cumulants higher than 2 are given by

$$\kappa_m = (1 - e^{-mkt})\frac{(m-1)!\lambda}{k\gamma^m}. \tag{12}$$

In the absence of Gaussian noise (e.g., $\sigma=0$), the steady-state PDF, obtained by inverting $p^*(s,0)p_2^*(s,t)$ [21] for $t\to\infty$, reads

$$p(x) = \frac{2^{\nu}|x|^{-\nu}\gamma^{1-\nu}K_{\nu}(\gamma|x|)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}-\nu\right)}, \quad (13)$$

where $\nu=(1-\lambda/k)/2$, $K_{\nu}(\cdot)$ is the modified Bessel function of the second kind, and $\Gamma(\cdot)$ is the gamma function [19]. The PDF is symmetric, and when $\nu\neq 0$ it decays as a power-exponential distribution [22], since for $|x|\to+\infty$ it tends to zero as $|x|^{\lambda/(2k)}\exp(-\gamma|x|)$. At the origin, it has a finite value when $\nu>0$—i.e., $\lambda<k$—while it goes to $+\infty$ when $k<\lambda$. In particular, when $\nu=0$—i.e., $\lambda=2k$—the PDF has the same distribution as the one of the jumps that is a two-sided exponential PDF, $p(x)=\gamma e^{-\gamma|x|}/2$; this fact was already noted in [23].

Using a one-sided exponential distribution for positive jumps, the case of linear drift leads to

$$p_2^{*} = \left(\frac{\gamma^2+s^2e^{-2kt}}{\gamma^2+s^2}\right)^{\lambda/2k} \times \exp\{i\lambda[\tan^{-1}(s/\gamma) - \tan^{-1}(se^{-kt}/\gamma)]\}, \quad (14)$$

while $p_1^{*}(s,t)$ reads as in Eq. (10). The mean in this case varies as $\mu_t=x_0e^{-kt}+(1-e^{-kt})\lambda/(k\gamma)$, the variance is $\text{var}_t=[1-\exp(-2kt)][(\lambda+\gamma^2\sigma^2)/(k\gamma^2)]$, and the other cumulants are expressed by Eq. (12). This means that the even cumulants of the process with jumps distributed as one-side and two-sided exponential PDF's coincide. In the special case when $\sigma=0$, the process is always positive and provided that $\lambda/k\geqslant 1$ the steady-state PDF is, as well known, a gamma distribution [24,25]

$$p(x) = \frac{\gamma^{\lambda/k}}{\Gamma(\lambda/k)}x^{\lambda/k-1}e^{-\gamma x}. \quad (15)$$

III. MULTIPLICATIVE NOISE

The state-dependent effect of external Gaussian fluctuations is commonly described by the so-called multiplicative noise [2–4]. Differently from the additive noise, the multiplicative one may introduce substantial changes in the properties of the system, such as noise-induced phase transitions [2] and the appearance of power-law tails [18,22,26–28]. The case of noise with state-dependent jumps is also interesting [29], but it will not be discussed here.

Assuming a multiplicative Gaussian noise and random jumps exponentially distributed, Eq. (3) becomes

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}\left\{\left[f(x)+\sigma^2 g(x)\frac{d}{dx}g(x)\right]p(x,t)\right\} + \sigma^2\frac{\partial^2}{\partial x^2}[g^2(x)p(x,t)] - \lambda p(x,t) + \lambda\int_0^x \gamma e^{-\gamma(x-z)}p(z,t)dz, \quad (16)$$

which can be written in the form [1]

$$\frac{\partial}{\partial t}p(x,t) = -\frac{\partial}{\partial x}J(x,t), \quad (17)$$

where the probability current $J(x,t)$ is

$$J(x,t) = \left[f(x)+\sigma^2 g(x)\frac{d}{dx}g(x)\right]p(x,t) - \sigma^2\frac{\partial}{\partial x}[g^2(x)p(x,t)] + \lambda\int_0^x e^{-\gamma(x-z)}p(z,t)dz. \quad (18)$$

Given the mathematical complexity of Eq. (16), transient solutions are not considered here, but the attention is focused on the steady-state conditions, in which case the current is constant. Hereinafter, it will be assumed that either a natural boundary or a reflecting barrier is present at $x=0$, so that the probability current vanishes in steady-state conditions [2].

We analyze in particular the case of linear drift, $f(x)=-kx$, and linear multiplicative Gaussian noise, $g(x)=x$. In this case, the vanishing probability current leads to the ordinary integro-differential equation

$$J(x) = (\sigma^2-k)xp(x) - \sigma^2\frac{d}{dx}[x^2p(x)] + \lambda\int_0^x e^{-\gamma(x-z)}p(z)dz = 0. \quad (19)$$

Laplace transforming the previous relation and using the condition $[xp(x)]_{x=0}=0$ gives

$$\sigma^2 s\frac{d^2}{ds^2}\tilde{p}(s) + (\sigma^2-k)\frac{d}{ds}\tilde{p}(s) - \frac{\lambda}{\gamma+s}\tilde{p}(s) = 0, \quad (20)$$

where $\tilde{p}(s)$ is the Laplace transform of $p(x)$. The solution of Eq. (20) is [30]

$$\tilde{p}(s) = C_1\left(1+\frac{s}{\gamma}\right){}_2F_1(\alpha+1,\beta+1,2;1+s/\gamma), \quad (21)$$

where C_1 is a constant determined by the condition $\tilde{p}(0)=1$, ${}_2F_1(\cdot,\cdot,\cdot;\cdot)$ is the hypergeometric function [19] and

$$\alpha = \frac{-k+\sqrt{k^2+4\lambda\sigma^2}}{2\sigma^2},$$

$$\beta = \frac{-k-\sqrt{k^2+4\lambda\sigma^2}}{2\sigma^2}. \quad (22)$$

Going back to the original variable x [31], the solution reads

$$p(x) = Ce^{-\gamma x}x^{\alpha-1}L_{-\alpha-1}^{-\beta+\alpha}(\gamma x), \quad (23)$$

where C is a constant of normalization and $L_n^m(\cdot)$ is the generalized Laguerre polynomial [19]. A remarkable property of $p(x)$ is that, since it decays at infinity as a power law—e.g., $p(x)\sim x^{-(1+k/\sigma^2)}$ when $x\to+\infty$ [19]—its extremes are scale invariant [22,26–28]. Figure 1 represents in a log-log plot the tails of the PDF's for different values of the noise strength, showing the power-law decay for high values of x with slope that tends to -1 as σ increases. Interestingly, in the absence of Gaussian noise, the PDF loses this property and Eq. (23) reduces to the gamma distribution of Eq. (15). On the con-

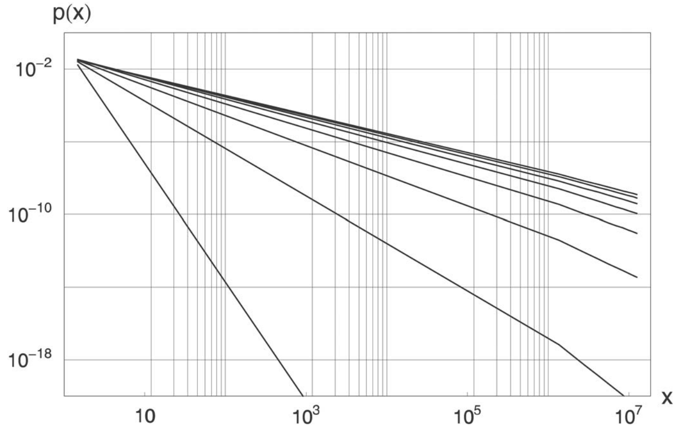

FIG. 1. Log-log plot showing the heavy tails of the PDF in the case of $f(x)=-kx$ and $g(x)=x$ with positive exponentially distributed jumps. The slope of the curves (e.g., $-1-k/\sigma^2$) tends to -1 with increasing σ (in the figure σ varies from 0.1 to 0.8 with steps of 0.1).

trary, when the Poisson noise is turned off, the system is the well-known multiplicative Gaussian process the solution of which is a log-normal distribution with mode exponentially decaying to zero (the process does not reach a steady state).

Figure 2 shows two time series (obtained following the numerical scheme reported in [32]) and the corresponding PDF's, Eq. (23). Figure 2(b), in particular, represents a very intermittent case, in which the drift is zero and both the rate of occurrence and the mean size of the jumps are very low, so that the dynamics is mainly driven by the Gaussian noise. However, it is the presence of the Poisson process that allows the system to reach a steady state by providing a repulsion from the origin. Thus, the Poisson noise, coupled to the multiplicative Gaussian noise, stabilizes the system generating intermittent bursts with power-law tails, with a mechanism that is similar to the ones discussed in [18,22,26]. In the limiting case where the Poisson process has very frequent events (e.g., high values of λ) and small jumps (e.g., high values of γ), the Langevin equation tends to $\dot{x}=\lambda/\gamma-kx+x\xi_t$, which was studied in [27]; the corresponding steady-state PDF [Eq. (23)] tends to

$$p(x)=Cx^{-2-k/\sigma^2}\exp[-\lambda/(\gamma\sigma^2 x)], \tag{24}$$

which can be normalized provided that $k<-\sigma^2$.

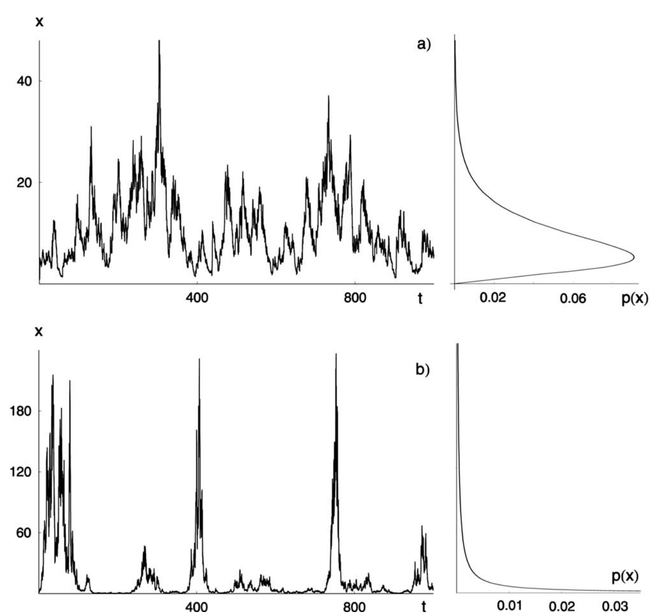

FIG. 2. Examples of time series and corresponding PDF's for linear drift, $f(x)=-kx$, and multiplicative noise, $g(x)=x$. Parameters are (a) $k=0.05$, $\lambda=0.25$, $1/\gamma=1.5$, $\sigma=0.1$ and (b) $k=0$, $\lambda=0.01$, $1/\gamma=0.1$, $\sigma=0.25$.

IV. APPLICATIONS

A. Virtual waiting-time process

The so-called Takàcs process appears in queuing and storage problems [8] and in stochastic input-output systems, as discussed in [5,6]. Assuming, for example, that the dynamics of the stock level $x(t)$ is characterized by a constant decay rate k plus a superposition of continuous small inflows and outflows, modeled as a Wiener process, and large instantaneous inflows, described by a Poisson process with rate λ and with size exponentially distributed with mean $1/\gamma$, the probability current in stationary conditions [e.g., Eq. (18)] is

$$J(x)=-kp(x)-\sigma^2\frac{\partial}{\partial x}p(x)+\lambda\int_0^x e^{-\gamma(x-z)}p(z)dz=0. \tag{25}$$

Solving the equation by Laplace transform leads to the solution [5]

$$p(x)=\frac{k-\lambda/\gamma}{2\sigma^2 a}\left[(k-\gamma\sigma^2+a)\exp\left(\frac{-k-\gamma\sigma^2-a}{2\sigma^2 a}x\right)-(k-\gamma\sigma^2-a)\exp\left(\frac{-k-\gamma\sigma^2+a}{2\sigma^2 a}x\right)\right], \tag{26}$$

where $a=\sqrt{(k-\gamma\sigma^2)^2+4\lambda\sigma^2}$. For the system to be stable and reach a steady state, the condition $\lambda/\gamma<k$ must hold [8]. Examples of two different PDF's for different σ are shown in Fig. 3. As σ decreases, the value of the PDF at zero moves towards higher values, and for $\sigma=0$ the PDF becomes exponential with an atom of probability in $x=0$, which is the classic solution of Takàcs [8].

B. Daily soil moisture dynamics

Simple but realistic models capturing the essential dynamics of the terrestrial water balance are important to analyze the linkage between soil moisture and the climate, soil, and vegetation system. Although in previous studies [25,33–35] the only stochastic component in soil moisture dynamics was assumed to be rainfall intermittency, in what follows we provide a first attempt to include also fluctuations in potential evapotranspiration.

Considering the soil as a constant storage capacity w_0, the daily soil moisture balance at a point is expressed as [15,25]

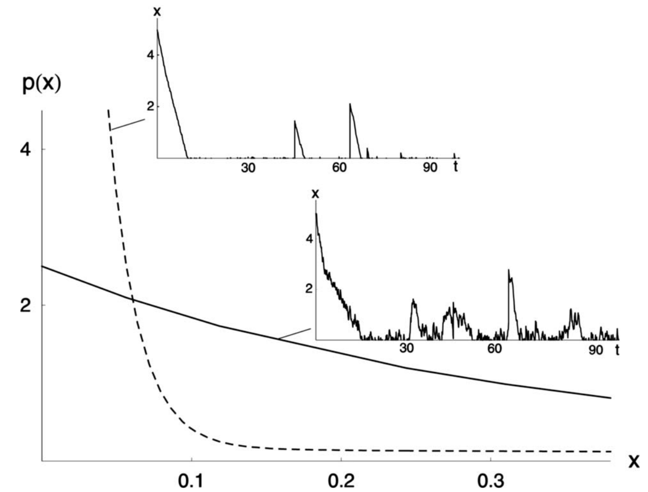

FIG. 3. Different PDF's for the virtual waiting-time problem for noise intensities $\sigma=0.1$ (solid line) and 0.4 (dashed line) and corresponding time series. Common parameters: $\lambda=0.1$, $1/\gamma=1$, and $k=0.5$.

$$w_0\frac{d\dot{x}}{dt} = I(x(t),t) - E(x(t),t), \tag{27}$$

where x is the effective relative soil moisture defined between 0 (e.g., dry soil) and 1 (e.g., soil saturation) [25,33]. Infiltration $I(x(t),t)$ is the amount of rainfall entering into the soil, while $E(x(t),t)$ is the evapotranspiration rate. Although Eq. (27) is modeled in continuous time, the soil water balance—e.g., Eq. (27)—is interpreted at the daily time scale [15,25,33,34].

At this scale, precipitation can be idealized as a compound Poisson process, whose events, occurring with rate λ, carry a random amount of water extracted from an exponential distribution with mean α. An example of daily rainfall values, measured at Duke Forest, NC (USA), is shown in Fig. 4(a). For the sake of simplicity, canopy interception is ignored here (see [33,34] for details). Infiltration is modeled assuming that the soil accommodates all the incoming rainfall if the rainfall depth is smaller than the available storage capacity, $1-x$; otherwise, the rainfall in excess is instantaneously lost as runoff and drainage.

Evapotranspiration $E(x(t),t)$ is assumed to be a function of both soil moisture x and potential evapotranspiration $E_p(t)$—i.e., the maximum rate of transpiration achievable by the extant vegetation under well-watered conditions $(x=1)$. A simple way to model evapotranspiration is to assume $E(x(t),t)$ to decrease linearly from $E_p(t)$ at $x=1$ to zero at $x=0$ [25]. However, as is apparent from Fig. 4(b), potential evapotranspiration fluctuates from day to day depending on climatic conditions (mainly solar radiation, air temperature, and air humidity). A suitable way to model such fluctuations is to assume $E_p(t)$ to be the sum of a mean potential evapotranspiration $\bar{E}_p$ plus a white Gaussian noise $E'(t)$ of intensity σ' (we refer to [15] for details).

With these assumptions, Eq. (27) can be normalized to become

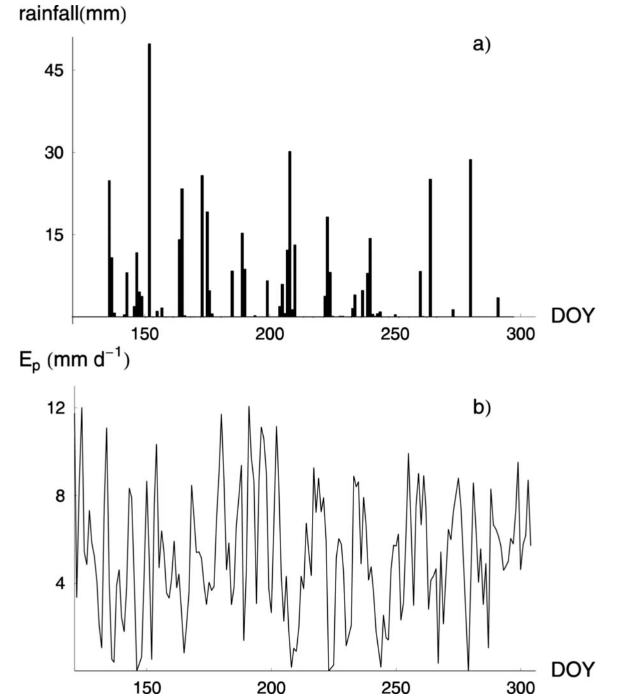

FIG. 4. Time series in the year 2001 of (a) precipitation and (b) potential evapotranspiration from Duke Forest, NC (USA).

$$\frac{d\dot{x}}{dt} = I(x(t),t) - [\eta x(t) + x(t)\xi(t)], \tag{28}$$

where $\eta x(t)$, with $\eta=\bar{E}_p/w_0$, is the deterministic component of evapotranspiration and $x(t)\xi(t)$ is the stochastic forcing of E, where the new noise intensity is $\sigma=\sigma'/w_0$. Following [33], the amount of water carried by each rainfall event is assumed to be extracted from an exponential PDF with mean $1/\gamma=\alpha/w_0$. Although the problem is similar to the case presented in Sec. III, Eq. (28) describes now a process that is bounded at $x=1$ because of saturation. However, as discussed in [33], the presence of the bound at $x=1$ (which acts as a reflecting barrier) only rescales the PDF of x in the range (0, 1). Thus the solution is the same as Eq. (23), where the constant is now determined by the condition $\int_0^1 p(x)dx=1$. We note that the interpretation of the multiplicative noise in Eq. (28) must be according to Stratonovich, since the approximation of external fluctuations in evapotranspiration as a white noise follows from an adiabatic approximation of the weakly colored (differentiable) signal shown in Fig. 4(b) (see [15] for details).

Examples of two series of modeled soil moisture using different values of σ are shown in Fig. 5 along with their PDF's. While in the time series the effect of the fluctuations in the potential evapotranspiration is hardly visible, unless the value of σ is chosen unreasonably high, the PDF's show some interesting differences. Increasing the noise strength moves the mode of the PDF towards lower values, as is common in systems with multiplicative noise [3], while the typical increase of the variance caused by increased noise is in part prevented by the presence of the bound in $x=1$. Overall, the changes induced by evapotranspiration are relatively

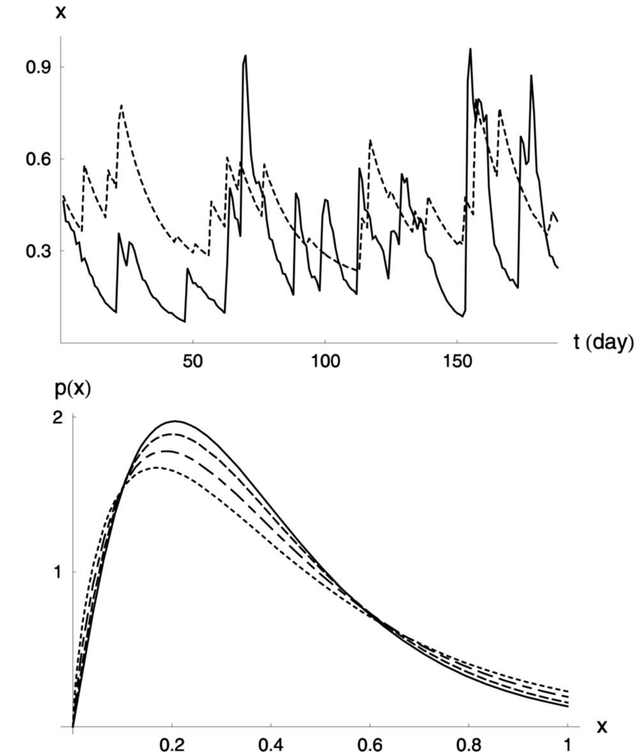

FIG. 5. Top: realizations of soil moisture with different values of noise intensity (σ=0.8, solid line, and σ=0.4, dashed line). Bottom: soil moisture PDF's for σ varying from 0.2 (solid line) to 0.8 (dotted line) with steps of 0.2. Common parameters: λ=0.1 d^{-1}, w_0=9 cm, α=1.5 cm (γ=6), and $\bar{E}_p$=0.4 cm d^{-1}.

small, despite the fact that its potential variability is similar to the one of the rainfall process [15], suggesting that rainfall variability is more effective than fluctuations in evapotranspiration, the action of which is also reduced by their multiplicative nature.

Once the probability distribution is obtained, an equation for the dynamics of the mean water balance (i.e., the mean of x) can be derived. The long-term water balance allows one to distinguish the long-term effect of each single component of the balance expressed by Eq. (28), describing how the rainfall input is partitioned between the different soil water losses [34]. Multiplying Eq. (3) by x and integrating between 0 and 1 leads to

$$\frac{d\langle x\rangle_t}{dt} = \int_0^1 x\frac{\partial}{\partial t}[p(x,t)]dx = -\int_0^1 x\frac{\partial}{\partial x}J(x,t)dx, \qquad (29)$$

which, in stationary conditions, only depends on the probability current

$$-\int_0^1 x\frac{d}{dx}J(x)dx = [-xJ(x)]_0^1 + \int_0^1 J(x)dx = -\int_0^1 \eta x p(x)dx + \int_0^1 \lambda e^{-\gamma x}\left[\int_0^x e^{\gamma z}p(z)dz\right]dx + \int_0^1 \sigma^2 x p(x)dx - \int_0^1 \frac{d}{dx}[\sigma^2 x^2 p(x)]dx = 0. \qquad (30)$$

Solving the integrals with respect to x and reorganizing the terms gives

$$\frac{\lambda}{\gamma} - (\eta - \sigma^2)\langle x\rangle - \frac{\lambda}{\gamma}\int_0^1 e^{-\gamma(1-z)}p(z)dz - \sigma^2[p(x)]_{x=1} = 0. \qquad (31)$$

The first term is the mean rainfall rate, which is the input of water into the system. The second represents the averaged losses due to evapotranspiration, and the last two terms are related to the presence of the bound at $x=1$. In particular, the third term describes the averaged losses caused by leakage and runoff, while the last one, which is in general negligible, is a spurious term introduced by the interaction between the bound and negative tail of the potential evapotranspiration modeled with a Gaussian distribution.

Interestingly, Eq. (31) shows the physical effects of the spurious drift, $\sigma^2\langle x\rangle$, resulting from the necessary Stratonovich interpretation of the fluctuations in potential evapotranspiration to account for their temporal autocorrelation. Here we only notice that the effect is one of a slight reduction of the transpiration losses, when compared to a similar case with same mean but no fluctuations in potential evapotranspiration (e.g., constant $E_p=\bar{E}_p=0.5$ cm d^{-1}) [25], followed by a readjustment of the partitioning between the different soil water losses. A detailed discussion of the hydrologic implications of Eq. (31) is presented elsewhere [15].

V. CONCLUSIONS

Dynamical systems driven by Gaussian and Poisson noises (jump-diffusion processes) have been studied here for both additive and multiplicative Gaussian noise. Formal solutions (characteristic functions and cumulants) of the forward Chapman-Kolmogorov equation have been obtained for general jump distributions in case of constant and linear drift with additive Gaussian noise and jumps distributed as two- and one-sided exponential PDF's.

The case of linear Gaussian multiplicative noise with exponentially distributed jumps has been solved in stationary conditions. Such a solution is characterized by power-law tails, resulting from the interaction of the Poisson noise, which ensures repulsion of the system from the origin, with multiplicative Gaussian fluctuations. The same solution, with slight modifications, also provides a simplified description of the daily soil moisture dynamics in the presence of both rainfall variability and climatic fluctuations.

ACKNOWLEDGMENTS

This research was supported in part by the Office of Science (BER), U.S. Department of Energy, Grant No. DE-FG02-95ER62083, through its Southeast Regional Center (SERC) of the National Institute for Global Environmental Change (NIGEC) under Cooperative Agreement No. DE-FC02-03ER63613.

[1] G. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, 2nd ed. (Springer-Verlag, Berlin, 1990).
[2] W. Horsthemke and R. Lefever, *Noise-Induced Transitions: Theory and Applications in Physics, Chemistry, and Biology* (Springer-Verlag, Berlin, 1984).
[3] A. Schenzle and H. Brand, Phys. Rev. A **20**, 1628 (1979).
[4] C. R. Doering, Phys. Lett. A **122**, 133 (1987).
[5] D. Perry and W. Stadje, Probab. Eng. Mech. **16**, 19, (2002).
[6] O. Kella, D. Perry, and W. Stadje, Probab. Eng. Mech. **17**, 1 (2003).
[7] J. Łuczka, R. Bartussek, and P. Hänggi, Europhys. Lett. **31**, 431 (1995).
[8] D. R. Cox and H. D. Miller, *The Theory of Stochastic Processes* (Methuen, London, 1965).
[9] T. Czernick, J. Kula, J. Łuczka, and P. Hänggi, Phys. Rev. E **55**, 4057 (1997).
[10] J. Łuczka, T. Czernick, and P. Hänggi, Phys. Rev. E **56**, 3968 (1997).
[11] G. L. Gerstein and B. Mandelbrot, Biophys. J. **4**, 41 (1964).
[12] P. Lánský and J. P. Rospars, Biol. Cybern. **72**, 397 (1995).
[13] N. Hohn and A. N. Burkitt, Phys. Rev. E **63**, 031902 (2001).
[14] M. Abundo, Open Syst. Inf. Dyn. **11**, 105 (2004).
[15] E. Daly and A. Porporato (unpublished).
[16] H. C. Tuckwell and F. Y. M. Wan, J. Appl. Probab. **21**, 695 (1984).
[17] Ph. Blanchard and M.-O. Hongler, Phys. Lett. A **180**, 225 (1993).
[18] H. Takayasu, A.-H. Sato, and M. Takayasu, Phys. Rev. Lett. **79**, 966 (1997).
[19] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).
[20] C. Van Den Broeck, J. Stat. Phys. **31**, 467 (1983).
[21] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation* (Birkhäuser Verlag Basel, 1971), Vol. I.
[22] D. Sornette, *Critical Phenomena in Natural Sciences*, 2nd ed. (Springer-Verlag, Berlin, 2004).
[23] R. Zygadło, Phys. Lett. A **329**, 459 (2004).
[24] D. R. Cox and V. Isham, Adv. Appl. Probab. **18**, 558 (1986).
[25] A. Porporato, E. Daly, and I. Rodriguez-Iturbe, Am. Nat. **164**, 625 (2004).
[26] D. Sornette and R. Cont, J. Phys. I France **7**, 431 (1997).
[27] D. Sornette, Physica A **250**, 295 (1998).
[28] S. Solomon and M. Levy, Int. J. Mod. Phys. C **7**, 754 (1996).
[29] A. Porporato and P. D'Odorico, Phys. Rev. Lett. **92**, 110601 (2004).
[30] A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations* (CRC Press, Boca Raton, 1995).
[31] A. P. Prudnikov, Y. A. Brychov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach, New York, 1986), Vol. 5.
[32] J. M. Sancho, M. San Miguel, S. L. Katz, and J. D. Gunton, Phys. Rev. A **26**, 1589, (1982).
[33] I. Rodriguez-Iturbe, A. Porporato, L. Ridolfi, V. Isham, and D. R. Cox, Proc. R. Soc. London, Ser. A **455**, 3789 (1995).
[34] F. Laio, A. Porporato, L. Ridolfi, and I. Rodriguez-Iturbe, Adv. Water Resour. **24**, 707 (2001).
[35] F. Laio, A. Porporato, L. Ridolfi, and I. Rodriguez-Iturbe, Phys. Rev. E **63**, 036105 (2001).