Experimental and theoretical study of vibrational resonance in a bistable system with asymmetry

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We present a theoretical and experimental study of the phenomenon of vibrational resonance in a bistable vertical cavity surface emitting laser with asymmetrical double-well quasipotential. Several relations, found analytically, are compared with the experimental and numerical results. Additionally, we investigate the effect of additive noise.

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Strong high-frequency (HF) periodic excitations of a bistable system may lead to a change of its stability and result in the amplification of a low-frequency (LF) subthreshold signal near the critical point. Such a type of excitation has been proposed in the context of stochastic resonance (SR), where the added noise has been replaced by the HF modulation [1]. The phenomenon, showing up as a nonmonotonic dependence of the LF response on the HF amplitude, has been named vibrational resonance (VR) [1]. An evidence of VR was demonstrated in analog electric circuits modeling noise-induced structures [2], excitable systems [3], and an overdamped bistable oscillator [4], and in a vertical cavity surface emitting laser (VCSEL) operating in the regime of polarization bistability for both nearly symmetric and strongly asymmetric quasipotentials [5]. Recently, the phenomenon of VR has received much attention both theoretically and experimentally because of its possible practical importance for the amplification and the improvement of the signal-to-noise ratio for the detection of LF signals in bistable noisy systems [6–9]. In spite of a number of papers devoted to theoretical and experimental investigations of VR, the effect of asymmetry was not considered except in Ref. [5] where it was experimentally shown that asymmetry leads to qualitative and quantitative changes in the dynamics and response of a bistable oscillator with respect to symmetric case.

Here we present the result of detailed theoretical and experimental investigations of VR in VCSEL operating in the regime of the polarization bistability for the case of strongly asymmetric double-well quasipotential. The aim of the paper is to show that the main experimentally observed features of VR in this case can be qualitatively understood from an analysis of a threshold system with a static nonlinearity. From such a consideration we found analytically the dependence of the gain factor as a function of the amplitude of the HF modulation as well as relations which are in good qualitative agreement with the experimental results. The measurements were carried out in a bistable VCSEL quasistatically excited by two periodic forces with very different frequencies. Besides, we present experimental and numerical results on the effect of additive noise, demonstrating the noise-induced gain degradation of VR. In a sense, the dynamics and response of an asymmetric overdamped bistable oscillator in VR are very similar to the manifestation of VR in an excitable system of the FitzHugh-Nagumo model [3], therefore the analytical results presented here can be applied and tested in such a system as well.

Theoretically, the dynamics of the polarization switchings induced by a deterministic modulation in the VCSEL can be described in the framework of a Langevin equation. We consider here the case of the asymmetric two-well potential. The system is excited by two periodic signals with amplitudes A_L and A_H, and frequencies Ω_L and Ω_H, such that $\Omega_H \gg \Omega_L$. In this case, the dynamics can be described by the equation

$$\partial x/\partial t = -V'(x) + A_L \sin \Omega_L t + A_H \sin \Omega_H t, \tag{1}$$

where $V'(x)$ is the derivative with respect to x of an asymmetric bistable potential function $V(x) = -(\alpha/2)x^2 + (\beta/4)x^4 - \gamma x$, where α, β, and γ are positive numbers. In the symmetric case ($\gamma=0$), the local minima $x_0^{\pm}$ and the static threshold μ_0 (corresponding to the amplitude of the applied modulation necessary to induce the potential barrier crossing) are determined by the expressions

$$x_0^{\pm} = \pm \sqrt{\alpha/\beta}, \tag{2}$$

$$\mu_0 = \sqrt{4\alpha^3/27\beta}. \tag{3}$$

The addition of an asymmetry ($\gamma \neq 0$) leads to a change of the location of minima which are determined by the expressions

$$x_1 = 2\sqrt{1/3} \cos\left(\frac{\pi}{3} - \frac{1}{3} \arccos \frac{\gamma}{\mu_0}\right), \tag{4}$$

$$x_2 = -2\sqrt{1/3} \cos\left(\frac{1}{3} \arccos \frac{\gamma}{\mu_0}\right), \tag{5}$$

where x_1 and x_2 are normalized by $\sqrt{\alpha/\beta}$. In this case the switching threshold is given by

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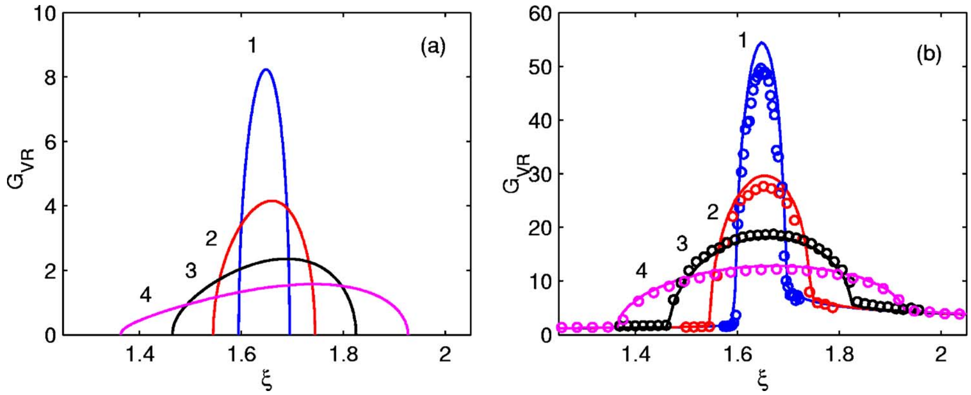

FIG. 1. (Color online) The gain factor G_{VR} versus ξ for different values of the LF signal amplitude $\varepsilon=0.05$ (1), 0.1 (2), 0.16 (3), 0.28 (4). (a) Analytics and (b) numerics (solid lines) and experiment (open circles).

$$\mu=\mu_0 \pm \gamma, \tag{6}$$

where the sign depends on the well from which the switching occurs. The response of the system to a periodic modulation depends on the level of the asymmetry γ and is different in the right and left wells. In the linear approximation, one can find that the response

$$R_0=\frac{A_L}{\sqrt{\Omega_L^2+\alpha^2(1-3x_i^2)^2}}, \tag{7}$$

where x_i $(i=1,2)$ is determined by expressions (4) and (5), respectively. In the quasistatical regime $(\Omega_L,\Omega_H \ll \alpha)$ the critical value of the HF amplitude needed for the switching between two states in the presence of the LF signal is given by the expression

$$\xi_c=1+\Delta-\varepsilon, \tag{8}$$

where $\xi=A_H/\mu_H$ and $\varepsilon=A_L/\mu_L$ are the normalized amplitudes of the LF and HF modulations, μ_L and μ_H are the switching thresholds at the frequencies Ω_L and Ω_H, respectively, and $\Delta=\gamma/\mu_0$. In the quasistatical regime $\mu_H=\mu_L=\mu_0$.

Now we consider the response of the system to the LF modulation as a function of ξ. We simplify our further analysis by considering a threshold system with a static nonlinearity. In this case, we determine the output $F(t)$ of the system by the rule

$$F(t)=\begin{cases}0, & \text{for } \varepsilon f_L(t)+(\xi-1)f_H(t)<\Delta,\\ \Delta, & \text{for } \varepsilon f_L(t)+(\xi-1)f_H(t)>\Delta,\end{cases} \tag{9}$$

where $f_L(t)$ and $f_H(t)$ are LF and HF periodic functions. In order to simplify our theoretical analysis we use the square wave shape of the HF signal, since as the numerical simulation has shown the shape of the LF response practically does not depend on the shape of the HF modulation. In the following we use $f_H(t)=\mathrm{sgn}[\sin(\Omega_H t)]$, where $\mathrm{sgn}(a)=1$ if $a>0$ and $\mathrm{sgn}(a)=-1$ if $a<0$. The input LF signal is $f_L(t)=\sin(\Omega_L t)$. In this case, the condition of threshold crossing is

$$\varepsilon \sin(\Omega_L t)+(\xi-1)\mathrm{sgn}[\sin(\Omega_H t)]=\Delta. \tag{10}$$

We calculate the response at the frequency Ω_L as

$$R_L=\frac{2}{T_L}\int_0^{T_L} F(t)\exp\left(-i\frac{2\pi}{T_L}t\right)dt, \tag{11}$$

where $T_L=2\pi/\Omega_L$. After some calculations, we obtain

$$G_{VR}=\frac{2(\xi-1)}{\pi\varepsilon}\sqrt{1-\frac{(1+\Delta-\xi)^2}{\varepsilon^2}}, \tag{12}$$

where we used the definition $G_{VR}=|R_L|\varepsilon^{-1}$. From expression (12) one can find that the optimal value ξ_{opt}, corresponding to the maximum gain, is shifted to higher values of ξ with increasing ε and depends on ε and Δ as

$$\xi_{opt}=1+\Delta\left(\tfrac{3}{4}+\tfrac{1}{4}\sqrt{1+8\varepsilon^2/\Delta^2}\right). \tag{13}$$

In order to check our predictions, we numerically integrated Eq. (1) with parameters $\alpha=\beta=4$ and $\gamma=1$ for the asymmetrical potential. For a quantitative characterization of VR we define the gain factor

$$G_{VR}=R_L(\Omega_L)/R_0(\Omega_L), \tag{14}$$

where $R_L(\Omega_L)$ and $R_0(\Omega_L)$ are the responses at the low frequency in the presence and the absence of the HF modulation, respectively, which were evaluated from the spectra of the Fourier transformed time series. In the numerical study of the effect of noise on VR, a source of white, Gaussian noise $\zeta(t)$ with $\langle\zeta(t)\zeta(t')\rangle=2D\delta(t-t')$ and mean $\langle\zeta(t)\rangle=0$ was added to the equation (1).

The experimental setup is essentially the same as was used earlier for the investigation of VR [5]. We studied the

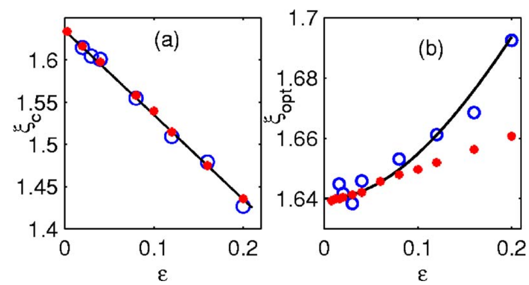

FIG. 2. (Color online) (a) Critical value of the switching threshold ξ_c and (b) optimal value ξ_{opt} of the HF amplitude versus ε. Open circles—experiment, asterisks—simulation, solid lines were plotted using expressions (a) (8) and (b) (13), respectively $(\Delta=0.64)$.

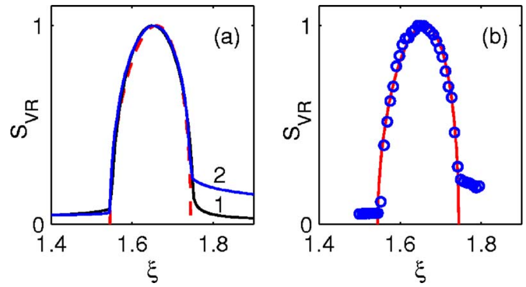

FIG. 3. (Color online) $S_{VR}=G_{VR}/G_{VR}^{\max}$ versus ξ for $\varepsilon=0.1$. (a) Solid lines—simulation with squarewave (1) and sinusoidal (2) shapes of the HF modulation, a dashed line plotted using (12) ($\Omega_L/2\pi=10^{-4}$, $\Omega_H/2\pi=10^{-2}$, $\Delta=0.64$). (b) Open circles—experiment, a solid line—expression (12).

laser response after polarization selection when the mixture of two periodic signals was applied to the injection current. The sinusoidal LF and HF signals have frequencies $\Omega_L=1$ kHz and $\Omega_H=100$ kHz, and amplitudes A_L and A_H, respectively. Both frequencies Ω_L and Ω_H were chosen much lower than the cutoff frequency of the laser polarization switching bandwidth, in order to ensure a quasistatical regime of the excitation of VR. The normalizations defined for the different parameters allow for a direct comparison with experimental results. We tune the injection current of the laser to the value which corresponds to a strongly asymmetric quasipotential (see [10]). In our experiment the value of the threshold was set to $\Delta\cong 0.64$ (in units of the switching threshold of the symmetrical quasipotential). The laser response was detected by a fast photodetector and recorded by a digital oscilloscope coupled with a computer to store and process the data.

In Fig. 1 the gain factor as a function of ξ for different values of ε obtained from analytics [Fig. 1(a)], numerics, and experiment [shown in Fig. 1(b) together] are presented. First of all, one can note a qualitative similarity in the behavior of the gain factor on ξ and ε in both figures. One can see that an increase of ε results in an increase of the width of the response curve, a shift of the critical value of the switching threshold to a lower value of ξ, and the weak shift of the optimal value of ξ_{opt} (corresponding to $G_{VR}^{\max}$) to higher values of ξ. One can note also a good quantitative agreement between numerical and experimental results shown in Fig. 1(b) for large enough values of ε (curves 2, 3, and 4).

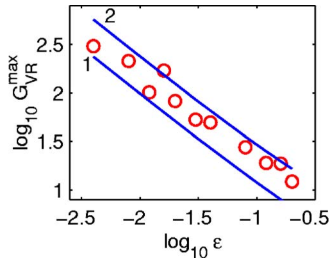

FIG. 4. (Color online) $G_{VR}^{\max}$ versus ε. Open circles—experiment. Solid lines correspond to numerical results with $\Delta=0.64$ (see text).

In Figs. 2–4 the analytical predictions are compared with the data obtained from the processing of numerical and experimental results. In Fig. 2(a) is shown the experimentally measured critical value of the switching threshold ξ_c as a function of ε (shown by open circles) along with an analytical prediction (8) (shown by a solid line) and the results of the numerical simulation (shown by asterisks). One can note the agreement between them for the range of the amplitude of the LF signal ε used in the experiment. The experimental evidence of the shift of ξ_{opt} (13) is demonstrated in Fig. 2(b) where the experimental data are shown by circles while a solid line corresponds to the analytical expression (13) and the asterisks are the results of the numerical simulation. We can note an agreement between analytical, numerical, and experimental results for $\varepsilon<0.1$, though for higher values of ε (>0.1) the numerical simulation gives values lower than the analytical and experimental results. In order to explain the point we performed further numerical simulations concluding that this shift depends on the shape of both periodical signals as well as on the frequency of HF modulation with respect to cutoff frequency in the system.

In Fig. 3 the analytical shape of G_{VR} (12) is compared with the numerical [Fig. 3(a)] and experimental [Fig. 3(b)] results. In this case, for comparison purposes we have normalized the gain factor G's by corresponding $G_{VR}^{\max}$'s. In Fig. 3(a) the analytical curve (shown by a dashed line) fits excellently the numerical shapes obtained with square wave (curve 1) and sinusoidal (curve 2) HF signals. This means that our analytical approach describes the LF response ad-

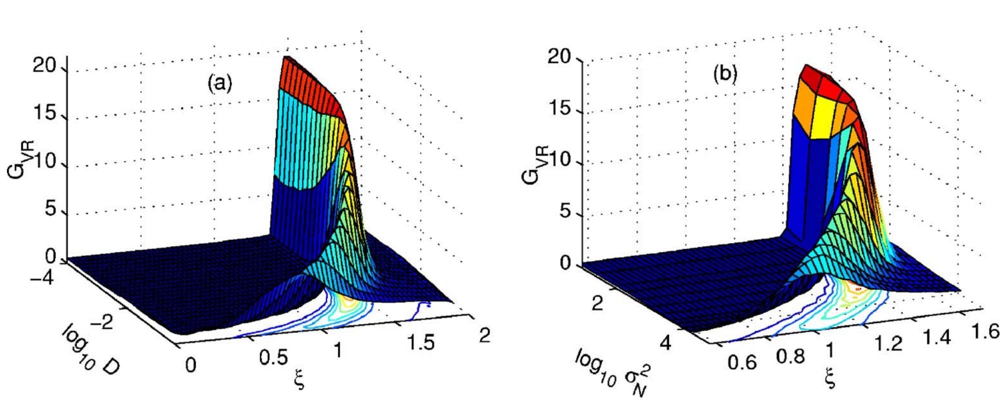

FIG. 5. (Color online) (a) Numerics. The gain factor G_{VR} as a function of ξ and D ($\Omega_L=2\pi\times 10^{-4}$, $\Omega_H=2\pi\times 10^{-2}$, $\varepsilon=0.08$, $\Delta\approx 0.5$). (b) Experiment. The gain factor G_{VR} as a function of ξ and the variance σ_N^2 of added noise ($\varepsilon=0.08,\Delta\approx 0.5$).

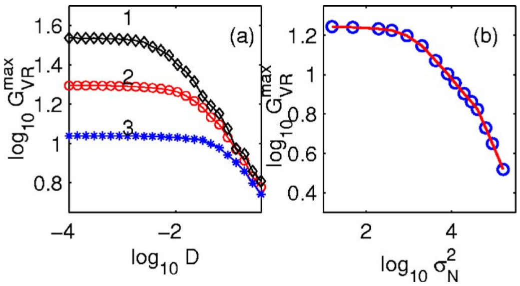

FIG. 6. (Color online) (a) Numerics. G_{VR}^{max} versus D for different values of ε=0.04 (1), 0.08 (2), 0.16 (3) (Δ=0.5). (b) Experiment. G_{VR}^{max} versus σ_N^2 for ε=0.08 and $\Delta \approx 0.5$. Each point was obtained by averaging over 20 signals.

equately for both HF signals. There is a difference only in the value of G_{VR}^{max} which is larger with the use of the square wave shape by about 1.33 for the parameters used in the simulation. One can note also a very good agreement between analytical and experimental shapes observed in Fig. 3(b).

In Fig. 4 we plotted the experimental maximum gain G_{VR}^{max} as a function of ε (open circles). The solid lines [(1) and (2) in Fig. 4] correspond to the numerical results with $\Delta = \gamma / \mu_0 \approx 0.64$. The two possible behaviors of G_{VR}^{max} are a consequence of our definition of the gain (14), since the response $R_0(\Omega_L)$ of the asymmetric bistable system to a periodic modulation depends on the initial conditions as was shown in the linear approximation (7). It should be noted that for low values of ε the maximum gain G_{VR}^{max} is strongly influenced by internal noise in VCSEL, leading to the degradation of the gain factor, therefore the experimental points for weak LF signals have a deviation from line (2) in Fig. 4. Nevertheless, one can note a good agreement between experimental and numerical results. A more detailed numerical simulation has revealed that G_{VR}^{max} depends also on the shape of both periodic signals.

Finally, we present the numerical and experimental results for the effect of additive noise. It should be noted that in the symmetrical case the effect of noise leads to a strong degradation of the gain factor and of the signal-to-noise ratio for weak LF periodic signals. In this case indicators follow scaling laws, depending on the level of noise [8,9]. A very similar picture is observed for the asymmetrical case as shown in Figs. 5(a) and 5(b). Increasing the level of noise, the gain factor decreases. At the same time, the optimal values of ξ corresponding to the maximum of the curve are shifted to lower values of the HF amplitude. A quantitative characterization of the gain degradation obtained numerically for this case is shown in Fig. 6(a). Similar to the symmetrical case [8], we can distinguish two regions in the dependence of G_{VR}^{max} on D. In the first one, G_{VR}^{max} practically does not depend on D, whereas in the second one, G_{VR}^{max} decreases with increasing D. In the second region, a fitting yields $G_{VR}^{max} \sim D^{-\eta}$ where η depends on the amplitude of LF signal. For instance, in the range of $D \in [10^{-2} - 5 \times 10^{-1}]$, $\eta \approx 0.4$, 0.33, and 0.27 for the values of ε=0.04, 0.08, and 0.16, respectively. We observe a very similar behavior in the experiment [Fig. 6(b)]. A fitting of the experimental data gives $G_{VR}^{max} \sim D^{-\eta}$, where $\eta \approx 0.27$ in the range of $\sigma_N^2 \in [10^3 - 5 \times 10^4]$, which is in agreement with the numerical results.

To conclude, we have shown that vibrational resonance in a bistable system with asymmetry in the quasistatical regime of the excitation can be well described in the framework of a threshold crossing system with a static nonlinearity. All experimentally observed quantities characterizing the response are in reasonable agreement with the analytical and numerical results.

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[1] P. S. Landa and P. V. E. McClintock, J. Phys. A **33**, L433 (2000).

[2] A. A. Zaikin, L. Lopez, J. P. Baltanas, J. Kurths, and M. A. F. Sanjuan, Phys. Rev. E **66**, 011106 (2002).

[3] E. Ullner, A. Zaikin, J. Garcia-Ojalvo, R. Bascones, and J. Kurths, Phys. Lett. A **312**, 348 (2003).

[4] J. P. Baltanas, L. Lopez, I. I. Blechman, P. S. Landa, A. Zaikin, J. Kurths, and M. A. F. Sanjuan, Phys. Rev. E **67**, 066119 (2003).

[5] V. N. Chizhevsky, E. Smeu, and G. Giacomelli, Phys. Rev. Lett. **91**, 220602 (2003).

[6] I. I. Blechman and P. S. Landa, Int. J. Non-Linear Mech. **39**, 421 (2004).

[7] J. Casado-Pascual and J. P. Baltanas, Phys. Rev. E **69**, 046108 (2004).

[8] V. N. Chizhevsky and G. Giacomelli, Phys. Rev. E **70**, 062101 (2004).

[9] V. N. Chizhevsky and G. Giacomelli, Phys. Rev. A **71**, 011801(R) (2005).

[10] S. Barbay, G. Giacomelli, and F. Marin, Phys. Rev. E **61**, 157 (2000).