Transport coefficients for an inelastic gas around uniform shear flow: Linear stability analysis

Vicente Garzó*

Departamento de Física, Universidad de Extremadura, E-06071 Badajoz, Spain
(Received 2 June 2005; revised manuscript received 9 November 2005; published 27 February 2006)

The inelastic Boltzmann equation for a granular gas is applied to spatially inhomogeneous states close to uniform shear flow. A normal solution is obtained via a Chapman-Enskog-like expansion around a local shear flow distribution. The heat and momentum fluxes are determined to first order in the deviations of the hydrodynamic field gradients from their values in the reference state. The corresponding transport coefficients are determined from a set of coupled linear integral equations which are approximately solved by using a kinetic model of the Boltzmann equation. The main new ingredient in this expansion is that the reference state $f^{(0)}$ (zeroth-order approximation) retains all the hydrodynamic orders in the shear rate. In addition, since the collisional cooling cannot be compensated locally for viscous heating, the distribution $f^{(0)}$ depends on time through its dependence on temperature. This means that in general, for a given degree of inelasticity, the complete nonlinear dependence of the transport coefficients on the shear rate requires analysis of the *unsteady* hydrodynamic behavior. To simplify the analysis, the steady-state conditions have been considered here in order to perform a linear stability analysis of the hydrodynamic equations with respect to the uniform shear flow state. Conditions for instabilities at long wavelengths are identified and discussed.

I. INTRODUCTION

The understanding of granular systems still remains a topic of interest and controversy. Under rapid flow conditions, they can be modeled as a fluid of hard spheres dissipating part of their kinetic energy during collisions. In the simplest model, the grains are taken to be smooth so that the inelasticity of collisions is characterized through a constant coefficient of normal restitution, $\alpha \lesssim 1$. Energy dissipation has profound consequences on the behavior of these systems since they exhibit a rich phenomenology with many qualitative differences with respect to molecular systems. In particular, the absence of energy conservation yields subtle modifications of the conventional Navier-Stokes equations for states with small gradients of the hydrodynamic fields. The dependence of the corresponding transport coefficients on dissipation may be determined from the Boltzmann kinetic equation conveniently modified to account for inelastic binary collisions [1,2]. The idea is to extend the Chapman-Enskog method [3] to the inelastic case by expanding the velocity distribution function around the local version of the homogeneous cooling state: namely, a homogeneous state whose dependence on time occurs only through the temperature. In the first order of the expansion, explicit expressions for the transport coefficients as functions of the coefficient of restitution have been obtained in the case of a single gas [4] as well as for granular mixtures [5], showing good agreement the analytical results with those obtained from Monte Carlo simulations [6].

Although the Chapman-Enskog method can be in principle applied to get higher orders in the gradients (Burnett and super-Burnett corrections, etc.), it is extremely difficult to evaluate those terms especially for inelastic systems. In

addition, questions about its convergence remain still open [7]. This gives rise to the search for alternative approaches to characterize transport for strongly inhomogeneous situations (i.e., beyond the Navier-Stokes limit). One possibility is to expand in small gradients around a more relevant reference state than the (local) homogeneous cooling state. For example, consider states near a shearing reference steady state such as the so-called uniform (simple) shear flow (USF) [8]. Such an application of the Chapman-Enskog method to a nonequilibrium state requires some care as recently discussed in Ref. [9]. The USF state is probably the simplest flow problem since the only nonzero hydrodynamic gradient is $\partial u_x / \partial y \equiv a = \text{const}$, where **u** is the flow velocity and a is the constant shear rate. Due to its simplicity, this state has been widely used in the past for both elastic [7] and inelastic gases [8] to shed light on the complexities associated with the *non-linear* response of the system to the action of strong shearing. However, the nature of this state for granular systems is different from that of the elastic fluids since the source of energy due to the macroscopic imposed shear field drives the granular system into rapid flow and a steady state is achieved when the amount of energy supplied by shearing work is balanced by the lost one due to the inelastic collisions between the particles. As a consequence, in the steady state the reduced shear rate $a^* \propto a/\sqrt{T}$ (which is the relevant nonequilibrium parameter of the problem) is not an independent quantity but becomes a function of the coefficient of restitution, α. This means that the *quasielastic* limit ($\alpha \to 1$) naturally implies the limit of *small* shear rates ($a^* \ll 1$) and vice versa. The study of the rheological properties of the USF state has received a great deal of attention in recent years in the case of monocomponent [10–20] and multicomponent [21–26] systems.

The aim of this paper is to determine the heat and momentum fluxes of a gas of inelastic hard spheres under simple shear flow in the framework of the Boltzmann equa-

*Electronic address: vicenteg@unex.es

tion. The physical situation is such that the gas is in a state that deviates from the simple shear flow by *small* spatial gradients. The starting point of this study is a recent approximate solution of the Boltzmann equation which is based on Grad's method [22,23,27]. In spite of this approach, the relevant transport properties obtained from this solution compare quite well with Monte Carlo simulations even for strong dissipation [18,22,28], showing again the reliability of Grad's approximation to compute the lowest-velocity moments of the velocity distribution function. Since the system is slightly perturbed from the USF, the Boltzmann equation is solved by applying the Chapman-Enskog method around the (local) shear flow state rather than the (local) homogeneous cooling state. This is the main feature of this expansion since the reference state is not restricted to small values of the shear rate. One important point is that, for general small deviations from the shear flow state, the zeroth-order distribution is not a stationary distribution since the collisional cooling cannot be compensated *locally* for viscous heating. This fact gives rise to conceptual and practical difficulties not present in the previous analysis made for elastic gases to describe transport in thermostatted shear flow states [29]. Due to the difficulties involved in this expansion, here general results will be restricted to particular perturbations for which steady-state conditions apply. In the first order of the expansion, the generalized transport coefficients are given in terms of the solutions of linear integral equations. To get explicit expressions for these coefficients, one needs to know the fourth-degree moments of USF. This requires us to consider higher-order terms in Grad's approximation for the reference distribution function, which is quite an intricate problem. In order to overcome such difficulty, here I have used a convenient kinetic model [31] that preserves the essential properties of the inelastic Boltzmann equation but admits a more practical analysis. The mathematical and physical basis for this model as a good representation of the Boltzmann equation has been discussed in Ref. [31]. In particular, it is worth noting that the results derived from this model coincide with those given from the Boltzmann equation at the level of the rheological properties [18,31]. Furthermore, recent computer simulation results [28] have also shown good agreement between the kinetic model and the Boltzmann equation for the fourth-degree moments, covering this agreement a wide range of values of dissipation (say, for instance, $\alpha \gtrsim 0.5$). This good agreement extends that previously demonstrated for Couette flow in dilute gases [30] and for USF in dense systems [20] and shows the reliability of the kinetic model to capture the main trends of the Boltzmann equation, especially those related to transport properties.

Knowledge of the above generalized transport coefficients allows one to determine the hydrodynamic modes from the associated linearized hydrodynamic equations. This is quite an interesting problem widely analyzed in the literature. As noted by the different molecular dynamics experiments carried out for the USF problem [16,25,32], the development of inhomogeneities and formation of clusters as the flow progresses becomes apparent. Consequently, the USF state is unstable for long enough wavelength spatial perturbations. In order to understand this phenomenon, several stability analy-

ses have been undertaken [33–37]. Most of them are based on Navier-Stokes equations [33–35] and, therefore, they are limited to small velocity gradients, which for the USF problem means small dissipation. Another alternative has been to solve the Boltzmann equation by means of an expansion in a set of basis functions [36,37]. The coefficients of this expansion are then determined by using also an expansion in powers of the parameter $\epsilon \equiv \sqrt{1-\alpha^2}$, which is assumed to be small. All these analytical results have shown that the USF becomes unstable for certain kinds of disturbances. My approach is different from previous works since the conditions for stability are obtained from a linear stability analysis involving the transport coefficients of the *perturbed* USF state instead of the usual Navier-Stokes coefficients. Furthermore, the analysis is not restricted to the low-dissipation limit since the reference state goes beyond this range of values of α. Two different perturbations to the reference state have been considered here: (i) perturbations along the velocity gradient (y direction) only and (ii) perturbations along the vorticity direction (z direction) only. The results show that the USF is linearly stable in the first case while it becomes unstable in the second case. These results agree qualitatively with those previously derived [33,35] in the context of the Navier-Stokes description. On the other hand, at a quantitative level, the comparison carried out here shows significant differences between the Navier-Stokes description and the present results as the collisions become more inelastic. In addition, our results also confirm that the instability is confined to long wavelengths (small wave numbers) and so it can be avoided for small enough systems.

The plan of the paper is as follows. In Sec. II, the Boltzmann kinetic equation is introduced and a brief summary of relevant results concerning the USF problem is given. In Sec. III, the problem we are interested in is described and the set of generalized transport coefficients characterizing the transport around USF is defined. Explicit expressions for these coefficients are provided in Sec. IV by using a kinetic model of the Boltzmann equation. The details of the calculations are displayed along several Appendixes. Section V is devoted to the linear stability analysis around the steady USF state and presents the form of the hydrodynamic modes. The paper is closed in Sec. VI with a discussion of the results obtained here.

II. BOLTZMANN KINETIC EQUATION AND UNIFORM SHEAR FLOW

Let us consider a granular gas composed by smooth spheres of mass m and diameter σ. The inelasticity of collisions among all pairs is accounted for by a *constant* coefficient of restitution $0 \leq \alpha \leq 1$ that only affects the translational degrees of freedom of grains. At low density, a simultaneous interaction of more than two particles is highly unlikely and so can be neglected. Consequently, in a dilute gas the interactions among the particles reduces to a succession of *binary* collisions. In a kinetic theory description all the relevant information on the state of the system is given by the one-particle velocity distribution function $f(\mathbf{r}, \mathbf{v}, t)$. For an inelastic dilute gas, the Boltzmann equation [1,2]

gives the time evolution of $f(\mathbf{r},\mathbf{v},t)$. In the absence of an external force, it has the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)f(\mathbf{r},\mathbf{v},t) = J[\mathbf{v}|f(t),f(t)], \qquad (1)$$

where the Boltzmann collision operator is

$$J[\mathbf{v}_1|f,f] = \sigma^2 \int d\mathbf{v}_2 \int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})$$
$$\times [\alpha^{-2}f(\mathbf{r},\mathbf{v}_1')f(\mathbf{r},\mathbf{v}_2',t) - f(\mathbf{r},\mathbf{v}_1,t)f(\mathbf{r},\mathbf{v}_2,t)]. \qquad (2)$$

Here, $\hat{\boldsymbol{\sigma}}$ is a unit vector along their line of centers, Θ is the Heaviside step function, and $\mathbf{g}=\mathbf{v}_1-\mathbf{v}_2$ is the relative velocity. The primes on the velocities denote the initial values $\{\mathbf{v}_1',\mathbf{v}_2'\}$ that lead to $\{\mathbf{v}_1,\mathbf{v}_2\}$ following a binary collision:

$$\mathbf{v}_1' = \mathbf{v}_1 - \frac{1}{2}(1+\alpha^{-1})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\hat{\boldsymbol{\sigma}}, \quad \mathbf{v}_2' = \mathbf{v}_2 + \frac{1}{2}(1+\alpha^{-1})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\hat{\boldsymbol{\sigma}}. \qquad (3)$$

The first five velocity moments of f define the number density

$$n(\mathbf{r},t) = \int d\mathbf{v}f(\mathbf{r},\mathbf{v},t), \qquad (4)$$

the flow velocity

$$\mathbf{u}(\mathbf{r},t) = \frac{1}{n(\mathbf{r},t)}\int d\mathbf{v}\mathbf{v}f(\mathbf{r},\mathbf{v},t), \qquad (5)$$

and the *granular* temperature

$$T(\mathbf{r},t) = \frac{m}{3n(\mathbf{r},t)}\int d\mathbf{v}V^2(\mathbf{r},t)f(\mathbf{r},\mathbf{v},t), \qquad (6)$$

where $\mathbf{V}(\mathbf{r},t)\equiv\mathbf{v}-\mathbf{u}(\mathbf{r},t)$ is the peculiar velocity. The macroscopic balance equations for density n, momentum $m\mathbf{u}$, and energy $\frac{3}{2}nT$ follow directly from Eq. (1) by multiplying with 1, $m\mathbf{v}$, and $\frac{1}{2}mv^2$ and integrating over $\mathbf{v}$:

$$D_t n + n\boldsymbol{\nabla}\cdot\mathbf{u} = 0, \qquad (7)$$

$$D_t u_i + (mn)^{-1}\nabla_j P_{ij} = 0, \qquad (8)$$

$$D_t T + \frac{2}{3n}(\boldsymbol{\nabla}\cdot\mathbf{q} + P_{ij}\nabla_j u_i) = -\zeta T, \qquad (9)$$

where $D_t=\partial_t+\mathbf{u}\cdot\boldsymbol{\nabla}$. The microscopic expressions for the pressure tensor P, the heat flux $\mathbf{q}$, and the cooling rate ζ are given, respectively, by

$$\mathsf{P}(\mathbf{r},t) = \int d\mathbf{v}m\mathbf{V}\mathbf{V}f(\mathbf{r},\mathbf{v},t), \qquad (10)$$

$$\mathbf{q}(\mathbf{r},t) = \int d\mathbf{v}\frac{1}{2}mV^2\mathbf{V}f(\mathbf{r},\mathbf{v},t), \qquad (11)$$

$$\zeta(\mathbf{r},t) = -\frac{1}{3n(\mathbf{r},t)T(\mathbf{r},t)}\int d\mathbf{v}mV^2J[\mathbf{r},\mathbf{v}|f(t)]. \qquad (12)$$

We assume that the gas is under uniform (or simple) shear flow (USF). This idealized macroscopic state is characterized by a constant density, a uniform temperature, and a simple shear with the local velocity field given by

$$u_i = a_{ij}r_j, \quad a_{ij} = a\delta_{ix}\delta_{jy}, \qquad (13)$$

where a is the *constant* shear rate. This linear velocity profile assumes no boundary layer near the walls and is generated by the Lee-Edwards boundary conditions [38], which are simply periodic boundary conditions in the local Lagrangian frame moving with the flow velocity. For elastic gases, the temperature grows in time due to viscous heating and so a steady state is not possible unless an external (artificial) force is introduced [7]. However, for inelastic gases, the temperature changes in time due to the competition between two (opposite) mechanisms: on the one hand, viscous (shear) heating and, on the other hand, energy dissipation in collisions. A steady state is achieved when both mechanisms cancel each other and the fluid autonomously seeks the temperature at which the above balance occurs. Under these conditions, in the steady state the balance equation (9) becomes

$$aP_{xy} = -\frac{3}{2}\zeta p, \qquad (14)$$

where $p=nT$ is the hydrostatic pressure. Note that for given values of the shear rate a and the coefficient of restitution α, relation (14) gives the temperature T in the steady state as a *unique* function of the density n.

The USF problem is perhaps the nonequilibrium state most widely studied in the past few years both for granular and conventional gases [7,8]. At a microscopic level, it becomes spatially homogeneous when the velocities of the particles are referred to the Lagrangian frame of reference co-moving with the flow velocity $\mathbf{u}$ [39]. Therefore, the one-particle distribution function adopts the *uniform* form $f(\mathbf{r},\mathbf{v})\rightarrow f(\mathbf{V})$ and the Boltzmann equation (1) reads

$$-aV_y\frac{\partial}{\partial V_x}f(\mathbf{V}) = J[\mathbf{V}|f,f]. \qquad (15)$$

This equation is invariant under the transformations

$$V_z \rightarrow -V_z, \quad (V_x,V_y) \rightarrow -(V_x,V_y), \quad (V_x,a) \rightarrow (-V_x,-a). \qquad (16)$$

The elements of the pressure tensor provide information on the relevant transport properties of the USF problem. These elements can be obtained by multiplying the Boltzmann equation (15) by mV_iV_j and integrating over $\mathbf{V}$. The result is

$$a_{i\ell}P_{j\ell} + a_{j\ell}P_{i\ell} = m\int d\mathbf{V}V_iV_jJ[\mathbf{V}|f,f] \equiv \Lambda_{ij}. \qquad (17)$$

The exact expression of the collision integral Λ_{ij} is not known, even in the elastic case. However, a good estimate can be expected by using Grad's approximation:

$$f(\mathbf{V}) \rightarrow f_0(\mathbf{V})\left[1 + \frac{m}{2T}\left(\frac{P_{ij}}{p} - \delta_{ij}\right)V_i V_j\right], \qquad (18)$$

where

$$f_0(\mathbf{V}) = n(m/2\pi T)^{3/2} \exp(-mV^2/2T) \qquad (19)$$

is the local equilibrium distribution function. When Eq. (18) is substituted into the definition of Λ_{ij} and nonlinear terms in $P_{ij}/nT - \delta_{ij}$ are neglected, one gets [23]

$$\Lambda_{ij} = -\nu[\beta(P_{ij} - p\delta_{ij}) + \zeta^* P_{ij}], \qquad (20)$$

where

$$\nu(T) = \frac{16}{5}n\sigma^2 \sqrt{\frac{\pi T}{m}} \qquad (21)$$

is an effective collision frequency,

$$\zeta^* = \frac{\zeta}{\nu} = \frac{5}{12}(1 - \alpha^2) \qquad (22)$$

is the dimensionless cooling rate evaluated in the local equilibrium approximation, and

$$\beta = \frac{1+\alpha}{2}\left(1 - \frac{1-\alpha}{3}\right). \qquad (23)$$

The set of coupled equations for P_{ij} can now be easily solved when one takes into account the approach (20). The expressions for the reduced elements $P_{ij}^* = P_{ij}/p$ are

$$P_{xx}^* = 3 - 2P_{yy}^*, \quad P_{yy}^* = P_{zz}^* = \frac{\beta}{\beta + \zeta^*}, \quad P_{xy}^* = -\frac{\beta}{(\beta + \zeta^*)^2}a^*, \qquad (24)$$

where the (reduced) shear rate $a^* = a/\nu$ is given by

$$a^* = \sqrt{\frac{3}{2}\frac{\zeta^*}{\beta}}(\beta + \zeta^*). \qquad (25)$$

Expression (25) clearly indicates the intrinsic connection between the (reduced) velocity gradient and dissipation in the system. In fact, in the elastic limit ($\alpha = 1$, which implies $a^* = 0$), the equilibrium results of the ordinary gas are recovered—i.e., $P_{ij}^* = \delta_{ij}$. This means that α (or a^*) can be considered as the relevant nonequilibrium parameter of the problem. The analytical results given by Eqs. (24) and (25) agree quite well [22,27] with Monte Carlo simulations of the Boltzmann equation [22,28], even for strong dissipation.

III. SMALL PERTURBATIONS FROM UNIFORM SHEAR FLOW: TRANSPORT COEFFICIENTS

In general, the USF state can be disturbed by small spatial perturbations. The response of the system to these perturbations gives rise to additional contributions to the momentum and heat fluxes, which can be characterized by generalized transport coefficients. This section is devoted to the study of such small perturbations.

In order to analyze this problem we have to start from the Boltzmann equation with a general time and space depen-

dence. Let $\mathbf{u}_0 = \mathsf{a} \cdot \mathbf{r}$ be the flow velocity of the *undisturbed* USF state. Here, the only nonzero element of the tensor a is $a_{ij} = a\delta_{ix}\delta_{jy}$. In the *disturbed* state, however, the true velocity $\mathbf{u}$ is in general different from $\mathbf{u}_0$ since $\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}$, $\delta\mathbf{u}$ being a small perturbation to $\mathbf{u}_0$. As a consequence, the true peculiar velocity is now $\mathbf{c} \equiv \mathbf{v} - \mathbf{u} = \mathbf{V} - \delta\mathbf{u}$, where $\mathbf{V} = \mathbf{v} - \mathbf{u}_0$. In the Lagrangian frame moving with $\mathbf{u}_0$, the Boltzmann equation can be written as

$$\frac{\partial}{\partial t}f - aV_y\frac{\partial}{\partial V_x}f + (\mathbf{V} + \mathbf{u}_0) \cdot \boldsymbol{\nabla}f = J[\mathbf{V}|f,f], \qquad (26)$$

where here the derivative $\boldsymbol{\nabla}f$ is taken at constant $\mathbf{V}$. The corresponding macroscopic balance equations associated with this disturbed USF state follow from the general equations (7)–(9) when one takes into account that $\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}$. The result is

$$\partial_t n + \mathbf{u}_0 \cdot \boldsymbol{\nabla}n = -\boldsymbol{\nabla} \cdot (n\delta\mathbf{u}), \qquad (27)$$

$$\partial_t \delta\mathbf{u} + \mathsf{a} \cdot \delta\mathbf{u} + (\mathbf{u}_0 + \delta\mathbf{u}) \cdot \boldsymbol{\nabla}\delta\mathbf{u} = -(mn)^{-1}\boldsymbol{\nabla} \cdot \mathsf{P}, \qquad (28)$$

$$\frac{3}{2}n\partial_t T + \frac{3}{2}n(\mathbf{u}_0 + \delta\mathbf{u}) \cdot \boldsymbol{\nabla}T + aP_{xy} + \boldsymbol{\nabla} \cdot \mathbf{q} + \mathsf{P}:\boldsymbol{\nabla}\delta\mathbf{u} = -\frac{3}{2}p\zeta, \qquad (29)$$

where the pressure tensor P, the heat flux $\mathbf{q}$, and the cooling rate ζ are defined by Eqs. (10), (11), and (12), respectively, with the replacement $\mathbf{V} \rightarrow \mathbf{c}$.

We assume now that the deviations from the USF state are small, which means that the spatial gradients of the hydrodynamic fields,

$$A(\mathbf{r},t) \equiv \{n(\mathbf{r},t), T(\mathbf{r},t), \delta\mathbf{u}(\mathbf{r},t)\}, \qquad (30)$$

are small. Under these conditions, a solution to the Boltzmann equation (26) can be obtained by means of a generalization of the conventional Chapman-Enskog method [3] where the velocity distribution function is expanded about a *local* shear flow reference state in terms of the small spatial gradients of the hydrodynamic fields relative to those of USF. This type of Chapman-Enskog-like expansion has been considered in the case of elastic gases to get the set of shear-rate-dependent transport coefficients [7,29] in a thermostatted shear flow problem, and it has also been recently considered [9] in the context of inelastic gases.

To construct the Chapman-Enskog expansion let us look for a *normal* solution of the form

$$f(\mathbf{r},\mathbf{V},t) \equiv f(A(\mathbf{r},t),\mathbf{V}). \qquad (31)$$

This special solution expresses the fact that the space dependence of the reference shear flow is completely absorbed in the relative velocity $\mathbf{V}$ and all other space and time dependence occurs entirely through a functional dependence on the fields $A(\mathbf{r},t)$. The functional dependence can be made local by an expansion of the distribution function in powers of the hydrodynamic gradients:

$$f(\mathbf{r},\mathbf{V},t) = f^{(0)}(A(\mathbf{r},t),\mathbf{V}) + f^{(1)}(A(\mathbf{r},t),\mathbf{V}) + \cdots, \qquad (32)$$

where the reference zeroth-order distribution function corresponds to the USF distribution function but taking into ac-

count the local dependence of the density and temperature and the change $\mathbf{V} \rightarrow \mathbf{V} - \delta\mathbf{u}(\mathbf{r}, t)$ [see Eqs. (D3) and (D4) for the explicit form of $f^{(0)}$ in the steady state given by a kinetic model of the Boltzmann equation]. The successive approximations $f^{(k)}$ are of order k in the gradients of n, T, and $\delta\mathbf{u}$ but retain all the orders in the shear rate a. This is the main feature of this expansion. In this paper, only the first-order approximation will be considered. More details on this Chapman-Enskog-like type of expansion can be found in Ref. [9].

The expansion (32) yields the corresponding expansion for the fluxes and the cooling rate when one substitutes Eq. (32) into their definitions (10)–(12):

$$\mathsf{P} = \mathsf{P}^{(0)} + \mathsf{P}^{(1)} + \cdots, \quad \mathbf{q} = \mathbf{q}^{(0)} + \mathbf{q}^{(1)} + \cdots,$$
$$\zeta = \zeta^{(0)} + \zeta^{(1)} + \cdots. \tag{33}$$

Finally, as in the usual Chapman-Enskog method, the time derivative is also expanded as

$$\partial_t = \partial_t^{(0)} + \partial_t^{(1)} + \partial_t^{(2)} + \cdots, \tag{34}$$

where the action of each operator $\partial_t^{(k)}$ is obtained from the hydrodynamic equations (27)–(29). These results provide the basis for generating the Chapman-Enskog solution to the inelastic Boltzmann equation (26).

A. Zeroth-order approximation

Substituting the expansions (32) and (34) into Eq. (26), the kinetic equation for $f^{(0)}$ is given by

$$\partial_t^{(0)} f^{(0)} - a V_y \frac{\partial}{\partial V_x} f^{(0)} = J[\mathbf{V}|f^{(0)}, f^{(0)}]. \tag{35}$$

To lowest order in the expansion the conservation laws give

$$\partial_t^{(0)} n = 0, \quad \partial_t^{(0)} T = -\frac{2}{3n} a P_{xy}^{(0)} - T \zeta^{(0)}, \tag{36}$$

$$\partial_t^{(0)} \delta u_i + a_{ij} \delta u_j = 0. \tag{37}$$

As said before, for given values of a and α, the steady-state condition (14) establishes a mapping between the density and temperature so that every density corresponds to one and only one temperature. Since the density $n(\mathbf{r}, t)$ and temperature $T(\mathbf{r}, t)$ are specified separately in the *local* USF state, the viscous heating only partially compensates for the collisional cooling and so $\partial_t^{(0)} T \neq 0$. Consequently, the zeroth-order distribution $f^{(0)}$ depends on time through its dependence on the temperature. Because of the steady-state condition, Eq. (14) does not apply in general locally, the reduced shear rate a^* $= a/\nu(n, T)$ depends on space and time so that a^* and α must be considered as independent parameters for general infinitesimal perturbations around the USF state. Since $f^{(0)}$ is a normal solution, then

$$\partial_t^{(0)} f^{(0)} = \frac{\partial f^{(0)}}{\partial n} \partial_t^{(0)} n + \frac{\partial f^{(0)}}{\partial T} \partial_t^{(0)} T + \frac{\partial f^{(0)}}{\partial \delta u_i} \partial_t^{(0)} \delta u_i$$

$$= -\left(\frac{2}{3n} a P_{xy}^{(0)} + T \zeta^{(0)} \right) \frac{\partial}{\partial T} f^{(0)} - a_{ij} \delta u_j \frac{\partial}{\partial \delta u_i} f^{(0)}$$

$$= -\left(\frac{2}{3n} a P_{xy}^{(0)} + T \zeta^{(0)} \right) \frac{\partial}{\partial T} f^{(0)} + a_{ij} \delta u_j \frac{\partial}{\partial c_i} f^{(0)}, \tag{38}$$

where in the last step we have taken into account that $f^{(0)}$ depends on $\delta\mathbf{u}$ only through the peculiar velocity $\mathbf{c}$. Substituting Eq. (38) into Eq. (35) yields the following kinetic equation for $f^{(0)}$:

$$-\left(\frac{2}{3n} a P_{xy}^{(0)} + T \zeta^{(0)} \right) \frac{\partial}{\partial T} f^{(0)} - a c_y \frac{\partial}{\partial c_x} f^{(0)} = J[\mathbf{V}|f^{(0)}, f^{(0)}]. \tag{39}$$

The zeroth-order solution leads to $\mathbf{q}^{(0)} = 0$. On the other hand, to solve Eq. (39) one needs to know the temperature dependence of the zeroth momentum flux $P_{xy}^{(0)}$. A closed set of equations for $\mathsf{P}^{(0)}$ is obtained when one considers Grad's approximation (18):

$$-\left(\frac{2}{3n} a P_{xy}^{(0)} + T \zeta^{(0)} \right) \frac{\partial}{\partial T} P_{ij}^{(0)} + a_{i\ell} P_{j\ell}^{(0)} + a_{j\ell} P_{i\ell}^{(0)}$$

$$= -\nu [\beta (P_{ij}^{(0)} - p \delta_{ij}) + \zeta^* P_{ij}^{(0)}], \tag{40}$$

where

$$\zeta^* = \frac{\zeta^{(0)}}{\nu} = \frac{5}{12}(1 - \alpha^2). \tag{41}$$

The steady-state solution of Eq. (40) is given by Eqs. (24) and (25). However, in general Eqs. (40) must be solved numerically to get the dependence of the zeroth-order pressure tensor $P_{ij}^{(0)}(T)$ on temperature. A detailed study of the unsteady hydrodynamic solution of Eq. (40) has been carried out in Ref. [27]. In what follows, $P_{ij}^{(0)}(T)$ will be considered as a known function of T.

B. First-order approximation

The analysis to first order in the gradients is worked out in Appendix A. Only the final results are presented in this section. The distribution function $f^{(1)}$ is of the form

$$f^{(1)} = \mathbf{X}_n \cdot \boldsymbol{\nabla} n + \mathbf{X}_T \cdot \boldsymbol{\nabla} T + \mathsf{X}_u : \boldsymbol{\nabla} \delta\mathbf{u}, \tag{42}$$

where the vectors $\mathbf{X}_n$ and $\mathbf{X}_T$ and the tensor X_u are functions of the true peculiar velocity $\mathbf{c}$. They are the solutions of the following linear integral equations:

$$-\left[\left(\frac{2}{3n} a P_{xy}^{(0)} + T \zeta^{(0)} \right) \partial_T + a c_y \frac{\partial}{\partial c_x} - \mathcal{L} \right] X_{n,i}$$

$$+ \frac{T}{n} \left[\frac{2a}{3p} (1 - n \partial_n) P_{xy}^{(0)} - \zeta^{(0)} \right] X_{T,i} = Y_{n,i}, \tag{43}$$

$$-\left[\left(\frac{2}{3n}aP_{xy}^{(0)}+T\zeta^{(0)}\right)\partial_T+\frac{2a}{3p}T(\partial_TP_{xy}^{(0)})+\frac{3}{2}\zeta^{(0)}\right.$$

$$\left.+ac_y\frac{\partial}{\partial c_x}-\mathcal{L}\right]X_{T,i}=Y_{T,i}, \qquad (44)$$

$$-\left[\left(\frac{2}{3n}aP_{xy}^{(0)}+T\zeta^{(0)}\right)\partial_T+ac_y\frac{\partial}{\partial c_x}-\mathcal{L}\right]X_{u,k\ell}-a\delta_{ky}X_{u,x\ell}$$

$$-\zeta_{u,k\ell}T\partial_Tf^{(0)}=Y_{u,k\ell}, \qquad (45)$$

where $\mathbf{Y}_n(\mathbf{c})$, $\mathbf{Y}_T(\mathbf{c})$, and $\mathbf{Y}_u(\mathbf{c})$ are defined by Eqs. (A9), (A10), and (A11), respectively, and $\zeta_{u,k\ell}$ is defined by Eq. (A14). While the Y functions are given in terms of the reference state distribution $f^{(0)}$, $\zeta_{u,k\ell}$ is a functional of the unknown $X_{u,k\ell}$. In addition, $\mathcal{L}$ is the linearized Boltzmann collision operator around the reference state:

$$\mathcal{L}X\equiv-(J[f^{(0)},X]+J[X,f^{(0)}]). \qquad (46)$$

A good estimate of $\zeta_{u,k\ell}$ can be obtained by expanding $X_{u,k\ell}$ in a complete set of polynomials (for instance, Sonine polynomials) and then truncating the series after the first few terms. In practice, the leading term in these expansions provides a very accurate result over a wide range of dissipation. This contribution is obtained in Appendix B and is given by Eq. (B9).

With the distribution function $f^{(1)}$ determined by Eq. (42), the first-order corrections to the fluxes are

$$P_{ij}^{(1)}=-\eta_{ijk\ell}\frac{\partial\delta u_k}{\partial r_\ell}, \qquad (47)$$

$$q_i^{(1)}=-\kappa_{ij}\frac{\partial T}{\partial r_j}-\mu_{ij}\frac{\partial n}{\partial r_j}, \qquad (48)$$

where

$$\eta_{ijk\ell}=-\int d\mathbf{c}\,mc_ic_jX_{u,k\ell}(\mathbf{c}), \qquad (49)$$

$$\kappa_{ij}=-\int d\mathbf{c}\frac{m}{2}c^2c_iX_{T,j}(\mathbf{c}), \qquad (50)$$

$$\mu_{ij}=-\int d\mathbf{c}\frac{m}{2}c^2c_iX_{n,j}(\mathbf{c}). \qquad (51)$$

Upon writing Eqs. (47)–(51) use has been made of the symmetry properties of $X_{n,i}$, $X_{T,i}$, and $X_{u,ij}$. In general, the set of *generalized* transport coefficients $\eta_{ijk\ell}$, κ_{ij}, and μ_{ij} is composed of nonlinear functions of the coefficient of restitution, α, and the reduced shear rate a^*. The anisotropy induced in the system by the shear flow gives rise to new transport coefficients, reflecting broken symmetry. The momentum flux is expressed in terms of a viscosity tensor $\eta_{ijk\ell}(\alpha)$ of rank 4 which is symmetric and traceless in ij due to the properties of the pressure tensor $P_{ij}^{(1)}$. The heat flux is expressed in terms of a thermal conductivity tensor $\kappa_{ij}(\alpha)$ and a new tensor $\mu_{ij}(\alpha)$.

C. Steady-state conditions

As shown in the above subsections, the evaluation of the complete nonlinear dependence of the generalized transport coefficients on the shear rate and dissipation requires the analysis of the hydrodynamic behavior of the *unsteady* reference state. This involves the corresponding numerical integrations of the differential equations obeying the velocity moments of the zeroth-order solution. This is quite an intricate and long problem. However, given that here we are mainly interested in performing a linear stability analysis of the hydrodynamic equations with respect to the steady state, we want to evaluate the transport coefficients in this special case. As a consequence, $\partial_t^{(0)}T=0$ and so the condition

$$a^*P_{xy}^*=-\frac{3}{2}\zeta^* \qquad (52)$$

applies. In Eq. (52), it is understood that a^* and $P_{xy}^*=P_{xy}^{(0)}/p$ are evaluated in the steady state; namely, they are given by Eqs. (24) and (25), respectively. A consequence of Eq. (52) is that the first term on the left-hand side of the integral equations (43)–(45) vanishes. In addition, the dependence of the pressure tensor $P_{ij}^{(0)}$ on density and temperature occurs explicitly through $p=nT$ and through its dependence on a^*. In this case, the derivatives $\partial_nP_{ij}^{(0)}$ and $\partial_TP_{ij}^{(0)}$ can be written more explicitly as

$$n\partial_nP_{ij}^{(0)}=n\partial_npP_{ij}^*(a^*)=p\left(1-a^*\frac{\partial}{\partial a^*}\right)P_{ij}^*(a^*), \qquad (53)$$

$$T\partial_TP_{ij}^{(0)}=T\partial_TpP_{ij}^*(a^*)=p\left(1-\frac{1}{2}a^*\frac{\partial}{\partial a^*}\right)P_{ij}^*(a^*). \qquad (54)$$

The dependence of P_{ij}^* on a^* near the steady state is determined in Appendix C so that all the terms appearing in the integral equations are explicitly known in the steady state. Under the above conditions, Eqs. (43)–(45) become

$$\left(-ac_y\frac{\partial}{\partial c_x}+\mathcal{L}\right)X_{n,i}+\frac{2a}{3}\frac{T}{n}(P_{xy}^*+a^*\partial_a^*P_{xy}^*)X_{T,i}=Y_{n,i},$$

$$\qquad (55)$$

$$\left(-ac_y\frac{\partial}{\partial c_x}-\frac{1}{3}a(P_{xy}^*-a^*\partial_{a^*}P_{xy}^*)+\mathcal{L}\right)X_{T,i}=Y_{T,i}, \qquad (56)$$

$$\left(-ac_y\frac{\partial}{\partial c_x}+\mathcal{L}\right)X_{u,k\ell}-a\delta_{ky}X_{u,x\ell}-\zeta_{u,k\ell}T\partial_Tf^{(0)}=Y_{u,k\ell},$$

$$\qquad (57)$$

where it is understood again that in Eqs. (55)–(57) all the quantities are evaluated in the steady state. Henceforth, I will restrict my calculations to this particular case.

Given that in the steady state the coefficient of restitution and the reduced shear rate are coupled, the usual Navier-Stokes transport coefficients for ordinary gases are recovered for elastic collisions ($a^*=0$). Thus, when $\alpha\to1$ the coefficients become

$$\eta_{ijk\ell} \rightarrow \eta_0 \left(\delta_{ik}\delta_{j\ell} + \delta_{jk}\delta_{i\ell} - \frac{2}{3}\delta_{ij}\delta_{k\ell} \right), \quad \kappa_{ij} \rightarrow \kappa_0 \delta_{ij},$$

$$\mu_{ij} \rightarrow 0, \tag{58}$$

where $\eta_0 = p/\nu$ and $\kappa_0 = 15\eta_0/4m$ are the shear viscosity and thermal conductivity coefficients given by the (elastic) Boltzmann equation.

IV. RESULTS FROM A SIMPLE KINETIC MODEL

The explicit form of the generalized transport coefficients μ_{ij}, κ_{ij}, and $\eta_{ijk\ell}$ requires one to solve the integral equations (55)–(57). Apart from the mathematical difficulties embodied in the Boltzmann collision operator $\mathcal{L}$, the fourth-degree velocity moments of the distribution $f^{(0)}$ are also needed to determine μ_{ij} and κ_{ij} and they are not provided in principle by the Grad approximation. Nevertheless, an accurate estimate of these moments from the Boltzmann equation is a formidable task since it would require at least to include the fourth-degree moments in Grad's solution. In this case, to overcome such difficulties it is useful to consider a model kinetic equation of the Boltzmann equation. As for elastic collisions, the idea is to replace the true Boltzmann collision operator with a simpler, more tractable operator that retains the most relevant physical properties of the Boltzmann operator. Here, I consider a kinetic model [31] based on the well-known Bhatnagar-Gross-Krook (BGK) [7] model for ordinary gases where the operator $J[f,f]$ is [40]

$$J[f,f] \rightarrow -\nu_{\text{BGK}}(f - f_0) + \frac{\zeta}{2}\frac{\partial}{\partial \mathbf{c}} \cdot (\mathbf{c}f). \tag{59}$$

Here, $\nu_{\text{BGK}} \propto nT^{1/2}$ is an effective collision frequency, f_0 is the local equilibrium distribution (19), and ζ is the cooling rate defined by Eq. (12). As said before, an estimate of ζ to first order in the gradients has been derived in Appendix B. The collision frequency ν_{BGK} can be considered as an adjustable parameter to optimize the agreement with the Boltzmann equation. In this paper, the α dependence of ν_{BGK} is chosen to reproduce the rheological properties of USF derived from Grad's approximation (18). This leads to take ν_{BGK} as

$$\nu_{\text{BGK}} = \nu\beta = \frac{(1 + \alpha)(2 + \alpha)}{6}\nu, \tag{60}$$

where ν and β are given by Eqs. (21) and (23), respectively. It must be remarked that the expression of the Navier-Stokes shear viscosity coefficient derived from the model with the choice (60) also agrees with the one obtained from the true Boltzmann equation [4]. A slightly different choice for ν_{BGK}—namely, $\nu_{\text{BGK}} = \nu(1+\alpha)/2$—is considered in Ref. [28].

By taking moments with respect to 1, $\mathbf{c}$, and c^2, the model kinetic equation (59) yields the same form of the macroscopic balance equations for mass, momentum, and energy, Eqs. (7), (8), and (9), as those given from the Boltzmann equation. When $\alpha=1$, then $\nu_{\text{BGK}}=\nu$, $\zeta=0$, and so the kinetic model (59) reduces to the BGK equation whose utility to address complex states not accessible via the Boltzmann
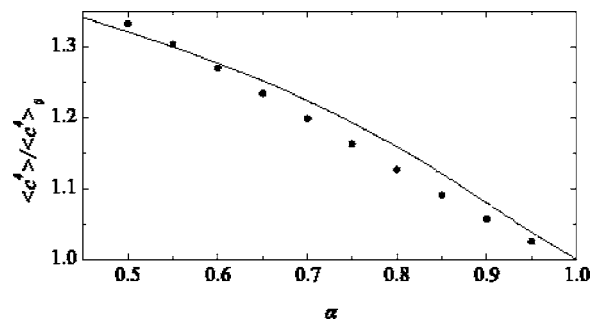

FIG. 1. Fourth-degree velocity moment $\langle c^4 \rangle$ relative to its local equilibrium value as a function of the coefficient of restitution. The solid line is the prediction of the kinetic model while the symbols are simulation results [28].

equation is well established for elastic gases [7]. In the case of granular gases, as said before, it is easy to show that the kinetic model leads to the same results for the pressure tensor in the USF problem as those given from Grad's solution to the Boltzmann equation, Eqs. (24) and (25). This result, along with those of Refs. [20,30], confirms the reliability of the kinetic model for granular media as well. A summary of the USF results derived from the kinetic model is provided in Appendix D. In particular, beyond rheological properties, recent computer simulations [28] have confirmed the accuracy of the kinetic model to capture the dependence of the fourth-degree velocity moments (whose expressions are needed to get the coefficients μ_{ij} and κ_{ij}) on dissipation in the USF state. To illustrate it, in Fig. 1 we plot the fourth-degree moment

$$\langle c^4 \rangle = \int d\mathbf{c}\, c^4 f(\mathbf{c}) \tag{61}$$

relative to its local equilibrium value $\langle c^4 \rangle_0 = 15nT^2/m^2$. The symbols refer to the numerical results obtained from the DSMC method [28]. It is quite apparent that the analytical results agree well with simulation data (the discrepancies between both results are smaller than 3%), showing again that the reliability of the kinetic model goes beyond the quasielastic limit.

Let us consider the perturbed USF problem in the context of the kinetic model. By using the model (59), the integral equations (55)–(57) still apply with the only replacements being

$$\mathcal{L}X \rightarrow \nu\beta X - \frac{\zeta^{(0)}}{2}\frac{\partial}{\partial \mathbf{c}} \cdot (\mathbf{c}X) \tag{62}$$

in the case of $X_{n,i}$ and $X_{T,i}$ and

$$\mathcal{L}X_{ij} \rightarrow \nu\beta X_{ij} - \frac{\zeta^{(0)}}{2}\frac{\partial}{\partial \mathbf{c}} \cdot (\mathbf{c}X_{ij}) - \frac{\zeta_{u,ij}}{2}\frac{\partial}{\partial \mathbf{c}} \cdot (\mathbf{c}f^{(0)}) \tag{63}$$

in the case of $X_{u,ij}$. In the above equations, $\zeta^{(0)}$ is the zeroth-order approximation to ζ which is given by Eq. (41). With the changes, Eqs. (62) and (63), all the generalized transport coefficients can be easily evaluated from Eqs. (55)–(57). Details of these calculations are also given in Appendix B; a
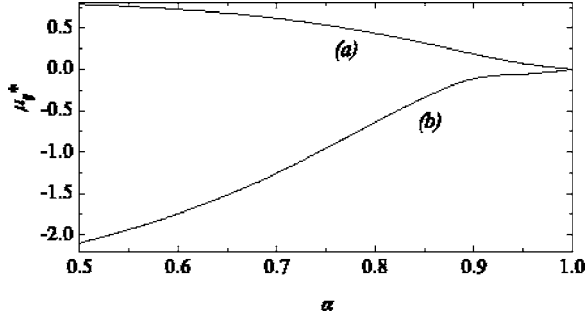
FIG. 2. Plot of the reduced coefficients (a) μ_{yy}^* and (b) μ_{xy}^* as a function of the coefficient of restitution α.

FIG. 3. Plot of the reduced coefficients (a) κ_{yy}^* and (b) κ_{xy}^* as a function of the coefficient of restitution α.

more complete listing can be obtained on request from the author.

The dependence of the generalized transport coefficients on the coefficient of restitution α is illustrated in Figs. 2–4 for the (reduced) coefficients μ_{xy}^*, μ_{yy}^*, κ_{yy}^*, η_{xxyy}^*, η_{yyyy}^*, η_{zzyy}^*, and η_{xyyy}^*. Here, $\mu_{ij}^* = n\mu_{ij}/T\kappa_0$, $\kappa_{ij}^* = \kappa_{ij}/\kappa_0$, and $\eta_{ijk\ell}^* = \eta_{ijk\ell}/\eta_0$, where $\eta_0 = p/\nu$ and $\kappa_0 = 5\eta_0/2m$ are the elastic values of the shear viscosity and thermal conductivity coefficients given by the BGK kinetic model. In general, we observe that the influence of dissipation on the transport coefficients is quite significant.

With all the transport coefficients known, the new constitutive equations (47) and (48) are completed and the corresponding set of closed hydrodynamic equations (27)–(29) can be derived. They are given by

$$\partial_t n + \mathbf{u}_0 \cdot \nabla n + \nabla \cdot (n\,\delta\mathbf{u}) = 0, \tag{64}$$

$$\partial_t \delta u_i + a_{ij}\delta u_j + (\mathbf{u}_0 + \delta\mathbf{u}) \cdot \nabla \delta u_i$$
$$+ \frac{1}{mn}\frac{\partial}{\partial r_j}\left(P_{ij}^{(0)} - \eta_{ijk\ell}\frac{\partial \delta u_k}{\partial r_\ell}\right) = 0, \tag{65}$$

$$\frac{3}{2}n\partial_t T + \frac{3}{2}n(\mathbf{u}_0 + \delta\mathbf{u}) \cdot \nabla T - a\,\eta_{xyij}\frac{\partial \delta u_i}{\partial r_j}$$
$$- \frac{\partial}{\partial r_i}\left(\mu_{ij}\frac{\partial n}{\partial r_j} + \kappa_{ij}\frac{\partial T}{\partial r_j}\right) + \left(P_{ij}^{(0)} - \eta_{ijk\ell}\frac{\partial \delta u_k}{\partial r_\ell}\right)$$
$$\times \frac{\partial \delta u_i}{\partial r_j} + aP_{xy}^{(0)} = -\frac{3}{2}nT\zeta - \frac{3}{2}nT\zeta_{u,ij}\frac{\partial \delta u_i}{\partial r_j}. \tag{66}$$

Note also that consistency would require one to consider the term $aP_{xy}^{(2)}$ which is of second order in gradients, and so it should be retained. Given that this would require one to determine the second-order contributions to the fluxes, this term will be neglected in our study. A similar problem occurs in calculations of the transport coefficients for the homogeneous cooling state of a granular gas since the true Navier-Stokes equations require one to consider the second-order contributions $\zeta^{(2)}$ to the cooling rate [4,41]. Given that $\zeta^{(2)}$ is very small, it is usual to neglect its contribution to the Navier-Stokes equations. As in the homogenous case, the same type of approximation is accepted here and so one neglects the term $aP_{xy}^{(2)}$ in Eq. (66). Since $P_{xy}^{(2)}$ is of second order in gradients, one expects that its contribution to the pressure
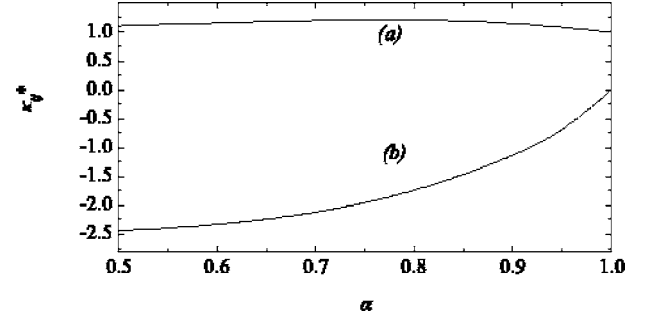
tensor is negligible as compared with $P_{xy}^{(0)}$ and $P_{xy}^{(1)}$. An important feature of the linearized hydrodynamic equations (64)–(66) is that they are not restricted to small values of the (reduced) shear rate or, equivalently, to small inelasticity. This allows us to go beyond the usual Navier-Stokes hydrodynamics. The hydrodynamic equations (64)–(66) are the starting point of the linear stability analysis of the USF of the next section.

V. LINEAR STABILITY ANALYSIS OF THE STEADY SHEAR FLOW STATE

As said in the Introduction, computer simulations [32] have clearly shown that the USF state is unstable with respect to long enough wavelength perturbations. These results have also been confirmed by different analytical results [33–36], most of them based on the Navier-Stokes description that applies to first order in the shear rate. However, given that USF is inherently non-Newtonian [27], the full nonlinear dependence of the transport coefficients on the shear rate is required to perform a consistent linear stability analysis of the nonlinear hydrodynamic equations (64)–(66) with respect to the USF state for small initial excitations. This analysis allows one to determine the hydrodynamic modes for states near USF as well as the conditions for instabilities at long wavelengths. A growth of these modes signals the onset of instability, which is ultimately controlled by the dominance of nonlinear terms. Note also that while all the works have been mainly devoted to dense systems, much less attention has been paid to dilute gases.
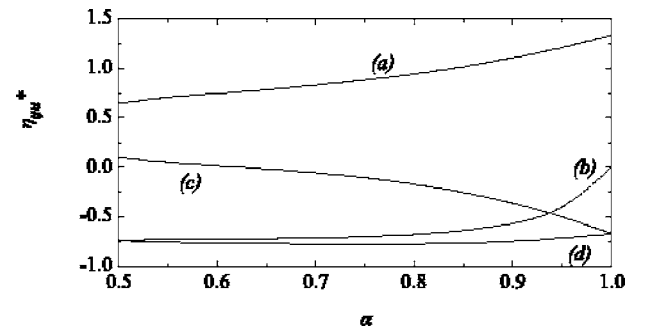

FIG. 4. Plot of the reduced coefficients (a) η_{yyyy}^*, (b) η_{xyyy}^*, (c) η_{zzyy}^*, and (d) η_{xxyy}^* as a function of the coefficient of restitution, α.

Let us assume that the deviations $\delta x_\mu(\mathbf{r},t) = x_\mu(\mathbf{r},t) - x_{0\mu}(\mathbf{r})$ are small, where $\delta x_\mu(\mathbf{r},t)$ denotes the deviation of $\{n, \mathbf{u}, T\}$ from their values in the USF state $\{n_0, \mathbf{u}_0, T_0\}$. The quantities in the USF verify

$$\boldsymbol{\nabla} n_0 = \boldsymbol{\nabla} T_0 = 0, \quad \mathbf{u}_0 = \mathsf{a} \cdot \mathbf{r}, \quad \partial_t T_0 = 0. \quad (67)$$

Now, let us linearize Eqs. (64)–(66) with respect to

$$\{\delta x_\mu(\mathbf{r},t)\} \equiv \{\delta n(\mathbf{r},t), \delta T(\mathbf{r},t), \delta \mathbf{u}(\mathbf{r},t)\}. \quad (68)$$

The resulting set of five linearized hydrodynamic equations follows from Eqs. (64)–(66):

$$\partial_t \delta n + ay \frac{\partial}{\partial x} \delta n + n_0 \cdot \delta \mathbf{u} = 0, \quad (69)$$

$$\frac{3}{2} n_0 \partial_t \delta T + ay \frac{\partial}{\partial x} \delta T + a \delta_{ix} \delta u_y + a[(\partial_n P_{xy}^{(0)}) \delta n + (\partial_T P_{xy}^{(0)}) \delta T]$$
$$+ (P_{k\ell}^{(0)} - a \eta_{xyk\ell}) \frac{\partial \delta u_k}{\partial r_\ell} - \mu_{ij} \frac{\partial^2 \delta n}{\partial r_i \partial r_j} - \kappa_{ij} \frac{\partial^2 \delta T}{\partial r_i \partial r_j}$$
$$= -\frac{3}{2} \zeta_0 n_0 T_0 \left(2 \frac{\delta n}{n_0} + \frac{3}{2} \frac{\delta T}{T_0}\right) - \frac{3}{2} n_0 T_0 \zeta_{u,k\ell} \frac{\partial \delta u_k}{\partial r_\ell}, \quad (70)$$

$$\partial_t \delta u_k + ay \frac{\partial}{\partial x} \delta u_k + a \delta_{kx} \delta u_y + \frac{1}{mn_0}$$
$$\left[(\partial_n P_{k\ell}^{(0)}) \frac{\partial \delta n}{\partial r_\ell} + (\partial_T P_{k\ell}^{(0)}) \frac{\partial \delta T}{\partial r_\ell} - \eta_{k\ell ij} \frac{\partial^2 \delta u_i}{\partial r_\ell \partial r_j}\right] = 0. \quad (71)$$

Here, it is understood that the pressure tensor $P_{ij}^{(0)}$ and its derivatives with respect to n and T, the cooling rate ζ_0, and the transport coefficients $\eta_{ijk\ell}$, μ_{ij}, and κ_{ij} are evaluated in the steady USF state.

To analyze the linearized hydrodynamic equations (69)–(71) it is convenient to transform to the local Lagrangian frame, $r_i' = r_i - ta_{ij}r_j$. The Lees-Edwards boundary conditions then become simple periodic boundary conditions in the variable $\mathbf{r}'$ [29]. A Fourier representation is defined as

$$\delta \tilde{x}_\mu(\mathbf{k},t) = \int d\mathbf{r}' e^{i\mathbf{k}\cdot\mathbf{r}'} \delta x_\mu(\mathbf{r},t) = \int d\mathbf{r} e^{i\mathbf{k}(t)\cdot\mathbf{r}} \delta x_\mu(\mathbf{r},t), \quad (72)$$

where in the second equality $k_i(t) = k_j(\delta_{ij} - ta_{ji})$. Periodicity conditions require that $k_i = 2n_i\pi/L_i$, where n_i are integers and L_i are the linear dimensions of the system. In this Fourier representation, the resulting set of five linear equations defines the hydrodynamic modes—i.e., linear response excitations to small perturbations. If at least one of the modes grows in time, the reference USF state is linearly unstable. Given the mathematical difficulties involved in the general problem, for the sake of simplicity, here I consider two kinds of perturbations: (i) perturbations along the velocity gradient direction only ($k_x = k_z = 0$, $k_y \neq 0$) and (ii) perturbations in the vorticity direction only ($k_x = k_y = 0$, $k_z \neq 0$). In both cases, the linearized hydrodynamic equations have time-independent coefficients.

A. Perturbations in the velocity gradient direction ($k_x = k_z = 0$, $k_y \neq 0$)

Let us consider first perturbations along the y direction only. In this case, Eqs. (69)–(71) in this Fourier representation can be written in the matrix form

$$\partial_\tau \delta \tilde{x}_\mu^* + F_{\mu\nu} \delta \tilde{x}_\nu^* = 0, \quad (73)$$

where the dimensionless quantities $\tau = \nu_0 t$ and $\delta \tilde{x}_\mu^* \equiv \{\rho_k, \theta_k, \mathbf{w}_k\}$, with

$$\rho_k = \frac{\delta \tilde{n}}{n_0}, \quad \theta_k = \frac{\delta \tilde{T}}{T_0}, \quad \mathbf{w}_k = \frac{\delta \tilde{\mathbf{u}}}{\sqrt{T_0/m}}, \quad (74)$$

have been introduced. The matrix $F_{\mu\nu}$ is

$$F_{\mu\nu} = 2C\delta_{\mu 2}\delta_{\nu 1} + C\delta_{\mu 2}\delta_{\nu 2} + a^*\delta_{\mu 3}\delta_{\nu 4} - ik^* G_{\mu\nu} + k^{*2}H_{\mu\nu}, \quad (75)$$

where $a^* = a/\nu_0$, ν_0 is the collision frequency (21) of the reference state, and $k^* = \ell_0 k$, $\ell_0 = \sqrt{T_0/m}/\nu_0$ being of the order of the mean free path. In addition, we have introduced the coefficient

$$C(\alpha) = -\frac{1}{3}a^*(1 + a^*\partial_a^*)P_{xy}^* \quad (76)$$

and the square matrices

$$\mathsf{G} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3}(P_{xy}^* - a^*\eta_{xyxy}^*) + \zeta_{xy}^* & \frac{2}{3}(P_{yy}^* - a^*\eta_{xyyy}^*) + \zeta_{yy}^* & 0 \\ (1 - a^*\partial_{a^*})P_{xy}^* & \left(1 - \frac{1}{2}a^*\partial_{a^*}\right)P_{xy}^* & 0 & 0 & 0 \\ (1 - a^*\partial_{a^*})P_{yy}^* & \left(1 - \frac{1}{2}a^*\partial_{a^*}\right)P_{yy}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (77)$$
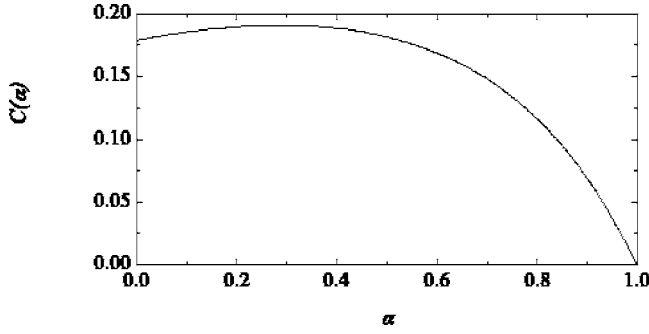
FIG. 5. Dependence of $C(\alpha)$ on the coefficient of restitution, α.

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \dfrac{5}{3}\mu_{yy}^{*} & \dfrac{5}{3}\kappa_{yy}^{*} & 0 & 0 & 0 \\ 0 & 0 & \eta_{xyxy}^{*} & \eta_{xyyy}^{*} & 0 \\ 0 & 0 & \eta_{yyxy}^{*} & \eta_{yyyy}^{*} & 0 \\ 0 & 0 & 0 & 0 & \eta_{zyzy}^{*} \end{pmatrix} \quad (78)$$

have been also introduced. Here, $P_{ij}^{*}=P_{ij}^{(0)}/n_0 T_0$ and

$$\zeta_{ij}^{*} = -\frac{1}{48}(1-\alpha^2)(P_{k\ell}^{*}-\delta_{k\ell})\eta_{k\ell ij}^{*}. \quad (79)$$

The eigenvalues $\lambda_\mu(k,\alpha)$ of the matrix $\mathsf{F}(k,\alpha)$ determine the time evolution of $\delta\tilde{x}_\mu^{*}(k,t)$. In the case that the real parts of the eigenvalues $\lambda_\mu(k,\alpha)$ are positive, then the USF state will be linearly stable. Before considering the general case, it is convenient to consider some special limits. Thus, in the elastic limit ($\alpha=1$), the hydrodynamic modes of the Navier-Stokes equations (for the particular case considered here and in the context of the BGK model) are recovered [42]: namely, two sound modes, a heat mode, and a twofold-degenerate shear mode. To second order in k^{*} they are given by

$$\lambda_\mu(k,\alpha=1) \rightarrow \left\{ i\sqrt{\frac{5}{3}}k^{*}+k^{*2}, -i\sqrt{\frac{5}{3}}k^{*}+k^{*2}, k^{*2}, k^{*2}, k^{*2} \right\}, \quad (80)$$

and consequently, excitations around equilibrium are damped. It is also quite illustrative to get the modes by setting $k=0$—namely, consider small, homogeneous deviations from the steady shear flow state. In this case, it is easy to see that ρ_k and $w_{y,k}$ are constant and

$$w_{x,k}(\tau) = w_{x,k}(0) - a\tau w_{y,k}(0), \quad (81)$$

$$\theta_k(\tau) = \theta_k(0)e^{-C\tau} - 2\rho_k(0). \quad (82)$$

The mode associated with $w_{x,k}$ is unstable to an initial perturbation in $w_{y,k}$, leading to an unbounded linear change in time. However, stability is still possible at finite k if this behavior is modulated by exponential damping factors. With respect to the temperature field, initial disturbances decay at $\tau \rightarrow \infty$ if the coefficient $C(\alpha)>0$. Figure 5 shows that the
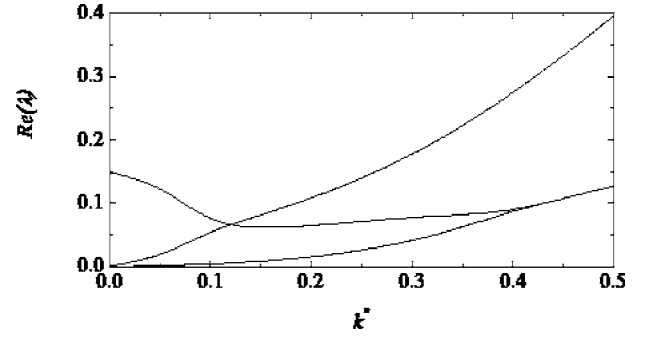

FIG. 6. Dispersion relations for a granular gas with $\alpha=0.8$ in the case of perturbations along the velocity gradient direction. Only the real parts of the eigenvalues are plotted.

coefficient C is positive for any value of α, and so this mode is (marginally) stable with a finite decay constant.

The analysis for $k \neq 0$ requires to get the eigenvalues $\lambda_\mu(k^{*},\alpha)$ with the full nonlinear dependence of k^{*}. However, the structure of $\mathsf{F}(k,\alpha)$ shows that the perturbation $\delta\tilde{x}_5^{*} \propto \delta\tilde{u}_z$ is decoupled from the other four modes and hence can be obtained more easily. This is due to the choice of gradients along the y direction only. The eigenvalue associated with this mode is positive and is simply given by

$$\lambda_5(k,\alpha) = \eta_{zyzy}^{*}k^{*2}, \quad \eta_{zyzy}^{*} = \frac{\beta}{(\beta+\zeta^{*})^2}, \quad (83)$$

where ζ^{*} is defined by Eq. (22). The remaining modes correspond to ρ_k, θ_k, and the components of the velocity field, $w_{x,k}$ and $w_{y,k}$. They are the solutions of a quartic equation with coefficients that depend on k^{*} and α. The results show that $\mathrm{Re}\lambda_\mu(k^{*},\alpha)>0$ for all values of the coefficient of restitution, α, and consequently, the flow remains stable to this kind of perturbation. As an illustration, the dispersion relations for a gas with $\alpha=0.8$ are plotted in Fig. 6. It is apparent that all real parts of the eigenvalues λ_μ are positive in the range of values of wave number k^{*} considered. Our conclusion agrees with previous stability analyses [33,35] based on the Navier-Stokes constitutive equations where a minimum value of the solid fraction (around 0.156) below which the USF is stable was found. Given that our system is a dilute gas (zero density), the present results confirm previous findings when one uses the improved transport coefficients.

B. Perturbations in the vorticity direction ($k_x=k_y=0$, $k_z \neq 0$)

The variation of the hydrodynamic modes with wave number $k=k_z$ in the vorticity direction is considered next. This situation has not been widely studied in the literature since most of the studies have been focused on two-dimensional (2D) flows due to the relative computational efficiency with which they can be analyzed. Here, for the sake of simplicity, I consider perturbations for which $\delta u_x = \delta u_y=0$, and so the eigenvalues $\lambda_\mu(k^{*},\alpha)$ obey a cubic equation. The analysis is similar to the one carried out in the previous section, and so details will be omitted. For a given value of α, it can be seen that this dispersion relation has one
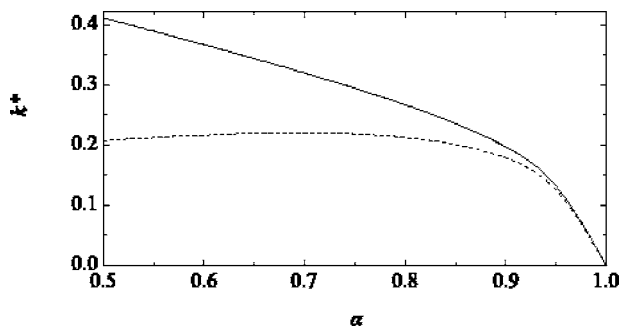
FIG. 7. Stability lines $k_s^*(\alpha)$ corresponding to the perturbation along the vorticity direction. The solid line corresponds to the results derived here while the dashed line refers to the results obtained from the Navier-Stokes approximation. The region above the curve corresponds to the stable domain, while the region below the curve corresponds to the unstable domain.

real root and a complex conjugate pair of damping modes. The instability arises from the real root since this mode $\lambda_\mu(k^*, \alpha) > 0$ if k^* is larger than a certain threshold value $k_s^*(\alpha)$. This value can be obtained by solving $\lambda_\mu(k^*, \alpha) = 0$. As a consequence, the USF state is linearly stable against excitations with a wave number $k^* > k_s^*(\alpha)$. The stability line $k_s^*(\alpha)$ is plotted in Fig. 7 as a function of the coefficient of restitution. Above this line the modes are stable, while below this line they are unstable. For comparison, the corresponding stability line obtained from the approximations made in previous works [33,35] is also plotted. This line can be formally obtained from the results derived in this paper when one replaces the expressions of the coefficients $\eta_{ijk\ell}$, κ_{ij}, and μ_{ij} by their corresponding Navier-Stokes expressions [4]. It is apparent that the Navier-Stokes approximation captures the qualitative dependence of k_s^* on α, although as expected quantitative discrepancies between both descriptions appear as the dissipation increases. Thus, for instance, for $\alpha = 0.8$ the discrepancies between both approaches are about 22% while for $\alpha = 0.5$ the discrepancies are about 49%. The prediction of a long-wavelength instability for the USF state has been observed in early molecular dynamics simulations [32] and qualitatively agrees with the previous analytical results based on the Navier-Stokes equations [33–36]. At a quantitative level, the lack of numerical results from the Boltzmann equation prevents us from carrying out a more detailed comparison to confirm the results derived from this kinetic model. We hope that the results offered here will stimulate the performance of such computer simulations.

VI. SUMMARY AND DISCUSSION

The objective of this paper has been to study the transport properties of a granular gas of inelastic hard spheres for the special nonequilibrium states near the uniform (simple) shear flow. Although the derivation of the Navier-Stokes equations (with explicit expressions for the transport coefficients appearing in them) from a microscopic description has been widely worked out in the past [4,5], the analysis of transport in a strongly shearing granular gas has received little attention due perhaps to its complexity and technical difficulties.

Very recently, a generalized Chapman-Enskog method has been proposed to analyze transport around nonequilibrium states in granular gases [9]. In the case of the USF state, due to the anisotropy induced in the system by the presence of shear flow, tensorial quantities are required to describe the momentum and heat fluxes instead of the usual Navier-Stokes transport coefficients [4,5]. In this paper we have been interested in a physical situation where *weak* spatial gradients of density, velocity, and temperature coexist with a *strong* shear rate. Under these conditions, the corresponding generalized transport coefficients characterizing heat and momentum transport are *nonlinear* functions of both the (reduced) shear rate a^* and the coefficient of restitution, α. The determination of such transport coefficients has been the primary aim of this paper.

Due to the difficulties embodied in this problem, a low-density gas described by the inelastic Boltzmann equation has been considered. Although the exact solution to the Boltzmann equation in the (steady) USF is not known, a good estimate of the relevant transport properties can be obtained by means of Grad's method [22,23,27]. The reliability of this approximation has been recently assessed by comparison with Monte Carlo simulations of the Boltzmann equation [22,28]. Assuming that the USF state is slightly perturbed, the Boltzmann equation has been solved by a Chapman-Enskog-like expansion where the shear flow state is used as the reference state rather than the local equilibrium or the (local) homogeneous cooling state. Due to the spatial dependence of the zeroth-order distribution $f^{(0)}$ (reference state), this distribution is not in general stationary and only in very special conditions has a simple relation with the (steady) USF distribution [9]. Here, since one of the main goals has been to address a stability analysis of the USF state, for practical purposes my results have been specialized to the steady state; namely, when the hydrodynamic variables satisfy the balance condition (52). In this situation, the (reduced) shear rate a^* is coupled with the coefficient of restitution, α [see Eq. (25)], so that the latter is the relevant parameter of the problem. In the first order of the expansion the momentum and heat fluxes are given by Eqs. (47) and (48), respectively, where the components of the set of generalized transport coefficients $\eta_{ijk\ell}$, μ_{ij}, and κ_{ij} are given in terms of solutions of the linear integral equations (55)–(57). As expected, there are many new transport coefficients in comparison to the case of states near equilibrium or cooling state. These coefficients provide all the information on the physical mechanisms involved in the transport of momentum and energy under shear flow.

Practical applications require one to solve the integral equations (55)–(57), which is in general quite a complex problem. In addition, the fourth-degree velocity moments of USF [whose evaluation would require one to consider higher-order terms in Grad's solution (18) of the Boltzmann equation] are needed to determine the coefficients κ_{ij} and μ_{ij}. To overcome such mathematical difficulties, here a kinetic model of the Boltzmann equation [31] has been used. This kinetic model can be considered as an extension of the well-known BGK equation to inelastic gases. Although the kinetic model is only a crude representation of the Boltzmann equation, it does preserve the most important features for trans-

port, such as the homogeneous cooling state and the macroscopic conservation laws. The model has a free parameter ν_{BGK} to be adjusted to fit a given property of the Boltzmann equation. Here, ν_{BGK} is given by Eq. (60) to get the same results for rheological properties in the USF problem as those derived from the Boltzmann equation by means of Grad's method. In addition, with the choice (60), the model reproduces quite well the α dependence of the Navier-Stokes transport coefficients of an inelastic gas [4], even for not small values of dissipation. On the other hand, given that the model does not intend to mimic the behavior of the true distribution function beyond the thermal velocity region, discrepancies between the kinetic model and the Boltzmann equation are expected beyond the second-degree velocity moments (which quantify the elements of the pressure tensor). Nevertheless, a recent comparison with Monte Carlo simulations of the Boltzmann equation [28] have shown the accuracy of the kinetic model predictions for the fourth-degree moments. As illustrated in Fig. 1, the semiquantitative agreement between theory and simulation is not restricted to the quasielastic limit ($\alpha \approx 0.99$) since it covers values of large dissipation ($\alpha \gtrsim 0.5$). The use of this kinetic model allows one to get the explicit dependence of the generalized transport coefficients on the coefficient of restitution. This dependence has been illustrated in some cases showing that in general the deviation of the transport coefficients from their corresponding elastic values is quite significant.

With these new expressions for the fluxes, a closed set of generalized hydrodynamic equations for states close to USF has been derived. A stability analysis of these linearized hydrodynamic equations with respect to the USF state has been also carried out to identify the conditions for stability in terms of dissipation. Two different kinds of perturbations to the USF state have been analyzed: (i) perturbations along the velocity gradient only ($k_y \neq 0$) and (ii) perturbations along the vorticity direction only ($k_z \neq 0$). In the first case, previous results [33,35] have shown that the USF is stable for a dilute gas while the USF becomes unstable in the second case for all α [43]. These results agree with these findings, and the USF is unstable for any finite value of dissipation at sufficiently long wavelengths when disturbances are generated in the orthogonal direction to the shear flow plane. On the other hand, as expected, quantitative discrepancies between our results and those given [33,35] from the Navier-Stokes approximation become significant as the dissipation increases. These differences are illustrated in Fig. 7 for the stability line. Although the instability of the USF has been extensively studied for many authors by using a Navier-Stokes description [33–35] as well as solutions of the Boltzmann equation in the quasielastic limit [36,37], I am not aware of any previous solution of the hydrodynamic equations where the generalized transport coefficients describing transport around USF were taken into account. The analytical results found in this paper allow a quantitative comparison with numerical solutions to the Boltzmann equation for finite dissipation. As happens for the USF problem for elastic [29,44,49] and inelastic [22,28] gases, one expects that the results reported here compare well with such simulations, confirming again the reliability of the kinetic theory results to characterize the onset and first stages of evolution of the clustering instability. We hope to carry out these simulations in the near future.

It is apparent that the results presented here are relevant to make a comparison with numerical simulations. Granular gases are notorious for the difficulties they pose to the experimenter. However, a variety of experimental methods have been employed in this field in the last few years which have allowed one to make some comparisons with theories. Among the recent additions to the experimental tool kit in the field of granular gases are NMR methods [45,46], the positron emission particles tracking method [47], and ultrafast video recording systems [48]. Although some experimental results are known for the simple shear flow problem [8], from my knowledge there are no data on the transport coefficients near USF. Since in many cases the experimentalists are now able to follow the motions of most of the grains, I hope that the results derived here encourage experimentalists to carry out studies where the present theory can be tested against suitable experimental data. Previous comparisons between kinetic theory and NMR experiments [46] have shown the reliability of theory to describe transport in granular systems.

On the other hand, the stability analysis performed here has only considered spatial variations along the y and z directions. More complex dynamics is expected in the general case of an arbitrary direction for the spatial perturbation. This will be worked out elsewhere along with a comparison with direct Monte Carlo computer simulations of the Boltzmann equation. Another possible direction of study is the extension of the present approach to other physically interesting reference states, such as nonlinear Couette flow. This is a more realistic shearing problem than the USF state since combined heat and momentum transport appears in the system. Given that an exact solution to the kinetic model used here is known for the Couette flow problem [30], the reference distribution for the Chapman-Enskog-like expansion is available. Finally, it would be also interesting to extend the analysis made in this paper for a single gas to the intriguing and important subject of granular mixtures. Given the difficulties associated with multicomponent systems, self-diffusion could be perhaps a good starting point to provide some insight into the general problem. Work along these lines will be carried out in the near future.

APPENDIX A: CHAPMAN-ENSKOG EXPANSION

Inserting the expansions (32) and (34) into Eq. (26), one gets the kinetic equation for $f^{(1)}$,

$$\left(\partial_t^{(0)} - aV_y \frac{\partial}{\partial V_x} + \mathcal{L} \right) f^{(1)} = -\left[\partial_t^{(1)} + (\mathbf{V} + \mathbf{u}_0) \cdot \boldsymbol{\nabla} \right] f^{(0)},$$

$$(A1)$$

where $\mathcal{L}$ is the linearized Boltzmann collision operator:

$$\mathcal{L}X \equiv -(J[f^{(0)}, X] + J[X, f^{(0)}]). \qquad (A2)$$

The velocity dependence on the right side of Eq. (A1) can be obtained from the macroscopic balance equations to first order in the gradients. They are given by

$$\partial_t^{(1)} n + \mathbf{u}_0 \cdot \boldsymbol{\nabla} n = -\boldsymbol{\nabla} \cdot (n\,\delta \mathbf{u}), \qquad (A3)$$

$$\partial_t^{(1)} \delta \mathbf{u} + (\mathbf{u}_0 + \delta \mathbf{u}) \cdot \boldsymbol{\nabla} \delta \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} \cdot \mathsf{P}^{(0)}, \qquad (A4)$$

$$\frac{3}{2} n \partial_t^{(1)} T + \frac{3}{2} n (\mathbf{u}_0 + \delta \mathbf{u}) \cdot \boldsymbol{\nabla} T + a P_{xy}^{(1)} + \mathsf{P}^{(0)} : \boldsymbol{\nabla} \delta \mathbf{u} = -\frac{3}{2} p \zeta^{(1)},$$

$$(A5)$$

where $\rho = mn$ is the mass density,

$$P_{ij}^{(1)} = \int d\mathbf{c}\, m c_i c_j f^{(1)}(\mathbf{c}), \qquad (A6)$$

and

$$\zeta^{(1)} = \frac{1}{3p} \int d\mathbf{c}\, m c^2 \mathcal{L} f^{(1)}. \qquad (A7)$$

Use of Eqs. (A3)–(A5) in Eq. (A1) yields

$$\left(\partial_t^{(0)} - aV_y \frac{\partial}{\partial V_x} + \mathcal{L} \right) f^{(1)} - \zeta^{(1)} T \frac{\partial f^{(0)}}{\partial T} = \mathbf{Y}_n \cdot \boldsymbol{\nabla} n + \mathbf{Y}_T \cdot \boldsymbol{\nabla} T$$

$$+ \mathsf{Y}_u : \boldsymbol{\nabla} \delta \mathbf{u}, \qquad (A8)$$

where

$$Y_{n,i} = -\frac{\partial f^{(0)}}{\partial n} c_i + \frac{1}{\rho} \frac{\partial f^{(0)}}{\partial \delta u_j} \frac{\partial P_{ij}^{(0)}}{\partial n}, \qquad (A9)$$

$$Y_{T,i} = -\frac{\partial f^{(0)}}{\partial T} c_i + \frac{1}{\rho} \frac{\partial f^{(0)}}{\partial \delta u_j} \frac{\partial P_{ij}^{(0)}}{\partial T}, \qquad (A10)$$

$$Y_{u,ij} = n \frac{\partial f^{(0)}}{\partial n} \delta_{ij} - \frac{\partial f^{(0)}}{\partial \delta u_i} c_j + \frac{2}{3n} \frac{\partial f^{(0)}}{\partial T} (P_{ij}^{(0)} - a\eta_{xyij}).$$

$$(A11)$$

According to Eqs. (A9) and (A10), $Y_{u,ij}$ has the same symmetry properties (16) as the distribution function $f^{(0)}$ while $Y_{n,i}$ and $Y_{T,i}$ are odd functions in the velocity $\mathbf{c}$.

The solution to Eq. (A8) has the form

$$f^{(1)} = X_{n,i}(\mathbf{c}) \boldsymbol{\nabla}_i n + X_{T,i}(\mathbf{c}) \boldsymbol{\nabla}_i T + X_{u,ji}(\mathbf{c}) \boldsymbol{\nabla}_i \delta u_j. \quad (A12)$$

Note that in Eq. (A11) the coefficients $\eta_{ijk\ell}$ are defined through Eq. (49). Substitution of the solution (A12) into

the relation (A7) allows one to write the cooling rate in the form

$$\zeta^{(1)} = \zeta_{n,i} \boldsymbol{\nabla}_i n + \zeta_{T,i} \boldsymbol{\nabla}_i T + \zeta_{u,ji} \boldsymbol{\nabla}_i \delta u_j, \qquad (A13)$$

where

$$\begin{pmatrix} \zeta_{n,i} \\ \zeta_{T,i} \\ \zeta_{u,ij} \end{pmatrix} = \frac{1}{3p} \int d\mathbf{c}\, m c^2 \mathcal{L} \begin{pmatrix} X_{n,i} \\ X_{T,i} \\ X_{u,ij} \end{pmatrix}. \qquad (A14)$$

However, given that $X_{n,i}$ and $X_{T,i}$ are odd functions in $\mathbf{c}$ [see, for instance, Eqs. (A19) and (A20) below], the terms proportional to $\boldsymbol{\nabla} n$ and $\boldsymbol{\nabla} T$ vanish by symmetry—i.e.,

$$\zeta_{n,i} = \zeta_{T,i} = 0. \qquad (A15)$$

Thus, the only nonzero contribution to $\zeta^{(1)}$ comes from the term proportional to the tensor $\boldsymbol{\nabla}_i \delta u_j$:

$$\zeta^{(1)} = \zeta_{u,ji} \boldsymbol{\nabla}_i \delta u_j. \qquad (A16)$$

An estimate of the tensor $\zeta_{u,ij}$ has been made in Appendix B by considering the leading terms in a Sonine polynomial expansion of the distribution $f^{(1)}$. Its expression is given by Eq. (B9). As expected, $\zeta_{u,ij}$ vanishes in the elastic limit ($\alpha = 1$).

The coefficients $X_{n,i}$, $X_{T,i}$, and $X_{u,ij}$ are functions of the peculiar velocity $\mathbf{c}$ and the hydrodynamic fields. In addition, there are contributions from the time derivative $\partial_t^{(0)}$ acting on the temperature and velocity gradients given by

$$\partial_t^{(0)} \boldsymbol{\nabla}_i T = \boldsymbol{\nabla}_i \partial_t^{(0)} T = \left(\frac{2a}{3n^2} (1 - n\partial_n) P_{xy}^{(0)} - \frac{\zeta^{(0)} T}{n} \right) \boldsymbol{\nabla}_i n$$

$$- \left(\frac{2a}{3n} \partial_T P_{xy}^{(0)} + \frac{3}{2} \zeta^{(0)} \right) \boldsymbol{\nabla}_i T, \qquad (A17)$$

$$\partial_t^{(0)} \boldsymbol{\nabla}_i \delta u_j = \boldsymbol{\nabla}_i \partial_t^{(0)} \delta u_j = -a_{jk} \boldsymbol{\nabla}_i \delta u_k. \qquad (A18)$$

Substituting (A16) into (A8) and identifying coefficients of independent gradients gives the set of equations

$$-\left[\left(\frac{2}{3n} a P_{xy}^{(0)} + T\zeta^{(0)} \right) \partial_T + ac_y \frac{\partial}{\partial c_x} - \mathcal{L} \right] X_{n,i}$$

$$+ \frac{T}{n} \left[\frac{2a}{3p} (1 - n\partial_n) P_{xy}^{(0)} - \zeta^{(0)} \right] X_{T,i} = Y_{n,i}, \quad (A19)$$

$$-\left[\left(\frac{2}{3n} a P_{xy}^{(0)} + T\zeta^{(0)} \right) \partial_T + \frac{2a}{3p} T(\partial_T P_{xy}^{(0)}) + \frac{3}{2} \zeta^{(0)} \right.$$

$$\left. + ac_y \frac{\partial}{\partial c_x} - \mathcal{L} \right] X_{T,i} = Y_{T,i}, \qquad (A20)$$

$$-\left[\left(\frac{2}{3n}aP_{xy}^{(0)}+T\zeta^{(0)}\right)\partial_T+ac_y\frac{\partial}{\partial c_x}-\mathcal{L}\right]X_{u,k\ell}-a\delta_{ky}X_{u,x\ell}$$

$$-\zeta_{u,k\ell}T\partial_Tf^{(0)}=Y_{u,k\ell}. \tag{A21}$$

Upon writing Eqs. (A19)–(A21), use has been made of the property

$$\partial_t^{(0)}X=\frac{\partial X}{\partial T}\partial_t^{(0)}T+\frac{\partial X}{\partial\delta u_i}\partial_t^{(0)}\delta u_i=-\left(\frac{2}{3n}aP_{xy}^{(0)}+T\zeta^{(0)}\right)\frac{\partial X}{\partial T}$$

$$+a_{ij}\delta u_j\frac{\partial X}{\partial c_i}, \tag{A22}$$

where in the last step we have taken into account that X depends on $\delta\mathbf{u}$ through $\mathbf{c}=\mathbf{V}-\delta\mathbf{u}$.

APPENDIX B: EVALUATION OF THE COOLING RATE

In this appendix the contribution $\zeta_{u,ij}$ to the cooling rate $\zeta^{(1)}$ is evaluated by expanding $X_{u,ij}$ as series in Sonine polynomials and taking the lowest-order truncation. The tensor $\zeta_{u,ij}$ is given by

$$\zeta_{u,ij}=\frac{1}{3p}\int d\mathbf{c}_1mc_1^2\mathcal{L}X_{u,ij}$$

$$=-\frac{1}{3p}\int d\mathbf{c}_1mc_1^2\{J[\mathbf{c}_1|f^{(0)},X_{u,ij}]+J[\mathbf{c}_1|X_{u,ij},f^{(0)}]\}. \tag{B1}$$

A useful identity for an arbitrary function $h(\mathbf{c}_1)$ is

$$\int d\mathbf{c}_1h(\mathbf{c}_1)J[\mathbf{c}_1|f,g]=\sigma^2\int d\mathbf{c}_1\int d\mathbf{c}_2f(\mathbf{c}_1)g(\mathbf{c}_2)$$

$$\times\int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})[h(\mathbf{c}_1'')-h(\mathbf{c}_1)], \tag{B2}$$

where $\mathbf{g}=\mathbf{c}_1-\mathbf{c}_2$ and

$$\mathbf{c}_1''=\mathbf{c}_1-\frac{1}{2}(1+\alpha)(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})\hat{\boldsymbol{\sigma}}. \tag{B3}$$

Using Eq. (B2), Eq. (B1) can be written as

$$\zeta_{u,ij}=\frac{m}{6p}\sigma^2(1-\alpha^2)\int d\mathbf{c}_1\int d\mathbf{c}_2f^{(0)}$$

$$\times(\mathbf{c}_1)X_{u,ij}(\mathbf{c}_2)\int d\hat{\boldsymbol{\sigma}}\Theta(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})(\hat{\boldsymbol{\sigma}}\cdot\mathbf{g})^3. \tag{B4}$$

The integration over $\hat{\boldsymbol{\sigma}}$ in Eq. (B4) yields

$$\zeta_{u,ij}=\frac{m}{12p}\pi\sigma^2(1-\alpha^2)\int d\mathbf{c}_1\int d\mathbf{c}_2f^{(0)}(\mathbf{c}_1)X_{u,ij}(\mathbf{c}_2). \tag{B5}$$

This equation is still exact. To perform the integrals over $\mathbf{c}_1$ and $\mathbf{c}_2$ one takes the Grad approximation (18) to $f^{(0)}$ and expands $X_{u,ij}$ in Sonine polynomials. In this case and accord-

ing to the anisotropy of the USF problem, one takes the approximation

$$X_{u,k\ell}(\mathbf{c})\rightarrow-\frac{1}{2nT^2}D_{ij}\eta_{ijk\ell}f_0(c), \tag{B6}$$

where

$$f_0(c)=n\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left(-\frac{mc^2}{2T}\right) \tag{B7}$$

is the Maxwellian distribution and

$$D_{ij}(\mathbf{c})=m\left(c_ic_j-\frac{1}{3}c^2\delta_{ij}\right). \tag{B8}$$

Next, change the variables to the (dimensionless) relative velocity $\mathbf{g}^*=(\mathbf{c}_1-\mathbf{c}_2)/v_0$ and center of mass $\mathbf{G}^*=(\mathbf{c}_1+\mathbf{c}_2)/2v_0$, where $v_0=\sqrt{2T/m}$ is the thermal velocity. A lengthy calculation leads to

$$\zeta_{u,ij}=-\frac{1}{6}\frac{v_0\sigma^2}{\pi^2T}(1-\alpha^2)\int d\mathbf{g}^*\int d\mathbf{G}^*g^{*3}e^{-2G^{*2}}e^{-g^{*2}/2}$$

$$\times\left[G_k^*G_\ell^*G_m^*G_n^*-\frac{1}{18}g^{*2}G^{*2}(\delta_{km}\delta_{\ell n}+\delta_{kn}\delta_{\ell m})\right.$$

$$\left.+\frac{1}{16}g_k^*g_\ell^*g_m^*g_n^*\right]\left(\frac{P_{mn}}{nT}-\delta_{mn}\right)\eta_{k\ell ij}$$

$$=-\frac{1}{15}\sigma^2\sqrt{\frac{\pi}{mT}}(1-\alpha^2)\left(\frac{P_{k\ell}}{nT}-\delta_{k\ell}\right)\eta_{k\ell ij}. \tag{B9}$$

Of course, when $\alpha=1$, then $\zeta_{u,ij}=0$.

APPENDIX C: BEHAVIOR OF THE ZEROTH-ORDER VELOCITY MOMENTS NEAR THE STEADY STATE

This appendix addresses the behavior of the velocity moments of the zeroth-order distribution $f^{(0)}$ near the steady state. Let us start with the elements of the pressure tensor $P_{ij}^{(0)}$. In the context of the Boltzmann equation and by using Grad's approximation (18), they verify the equation

$$-\left(\frac{2}{3n}aP_{xy}^{(0)}+T\zeta^{(0)}\right)\frac{\partial}{\partial T}P_{ij}^{(0)}+a_{i\ell}P_{j\ell}^{(0)}+a_{j\ell}P_{i\ell}^{(0)}$$

$$=-\nu[\beta(P_{ij}^{(0)}-p\delta_{ij})+\zeta^*P_{ij}^{(0)}]. \tag{C1}$$

Since we are interested in the *hydrodynamic* solution, the temperature derivative term can be written as

$$T\partial_TP_{ij}^{(0)}=T\partial_TpP_{ij}^*=p\left(1-\frac{1}{2}a^*\frac{\partial}{\partial a^*}\right)P_{ij}^*, \tag{C2}$$

where $P_{ij}^*=P_{ij}^{(0)}/p$. Upon deriving Eq. (C2), use has been made of the fact that the dimensionless pressure tensor P_{ij}^* depends on T only through its dependence on the reduced shear rate $a^*=a/\nu(n,T)$. In dimensionless form, the set of equations (C1) becomes

$$-\left(\frac{2}{3}a^*P_{xy}^* + \zeta^*\right)\left(1 - \frac{1}{2}a^*\frac{\partial}{\partial a^*}\right)P_{ij}^* + a_{i\ell}^*P_{j\ell}^* + a_{j\ell}^*P_{i\ell}^*$$

$$= -[\beta(P_{ij}^* - \delta_{ij}) + \zeta^*P_{ij}^*],$$ (C3)

where

$$\zeta^* = \frac{\zeta^{(0)}}{\nu} = \frac{5}{12}(1 - \alpha^2).$$ (C4)

Let us consider the elements P_{xy}^* and $P_{yy}^* = P_{zz}^*$. From Eq. (C1), one gets

$$-\left(\frac{2}{3}a^*P_{xy}^* + \zeta^*\right)\left(1 - \frac{1}{2}a^*\frac{\partial}{\partial a^*}\right)P_{xy}^* + a^*P_{yy}^* = -(\beta + \zeta^*)P_{xy}^*,$$ (C5)

$$-\left(\frac{2}{3}a^*P_{xy}^* + \zeta^*\right)\left(1 - \frac{1}{2}a^*\frac{\partial}{\partial a^*}\right)P_{yy}^* = -(\beta + \zeta^*)P_{yy}^* + \beta.$$ (C6)

This set of equations has a singular point corresponding to the steady-state solution—i.e., when $a^*(T) = a_s^*$ where $a_s^*(\alpha)$ is the steady-state value of a^* given by Eq. (25). Since we are interested in the solution of Eqs. (C5) and (C6) near the steady state, we assume that in this region P_{xy}^* and P_{yy}^* behave as

$$P_{xy}^* = P_{xy,s}^* + \left(\frac{\partial P_{xy}^*}{\partial a^*}\right)_s(a^* - a_s^*) + \cdots,$$ (C7)

$$P_{yy}^* = P_{yy,s}^* + \left(\frac{\partial P_{yy}^*}{\partial a^*}\right)_s(a^* - a_s^*) + \cdots,$$ (C8)

where the subscript s means that the quantities are evaluated in the steady state. Substitution of Eqs. (C7) and (C8) into Eqs. (C5) and (C6) allows one to determine the corresponding derivatives. The result is

$$\left(\frac{\partial P_{yy}^*}{\partial a^*}\right)_s = 4P_{yy,s}^*\frac{a_s^*C + P_{xy,s}^*}{2a_s^{*2}C + 6\beta + 3\zeta^*},$$ (C9)

where $C \equiv (\partial P_{xy}^*/\partial a^*)_s$ is the real root of the cubic equation

$$2a_s^{*4}C^3 + 12a_s^{*2}(\zeta^* + \beta)C^2 + \frac{9}{2}(7\zeta^{*2} + 14\zeta^*\beta + 4\beta^2)C$$

$$+ 9\beta(\zeta^* + \beta)^{-2}(2\beta^2 - 2\zeta^{*2} - \beta\zeta^*).$$ (C10)

Equations (C9) and (C10) can also be obtained a different way. Let us write the set of equations (C5) and (C6) as

$$\frac{\partial P_{xy}^*}{\partial a^*} = \frac{-2P_{yy}^* - \frac{2}{a^*}P_{xy}^*\left(\beta - \frac{2}{3}P_{xy}^*a^*\right)}{\zeta^* + \frac{2}{3}a^*P_{xy}^*},$$ (C11)

$$\frac{\partial P_{yy}^*}{\partial a^*} = \frac{2\beta - 2P_{yy}^*\left(\beta - \frac{2}{3}P_{xy}^*a^*\right)}{a^*\left(\zeta^* + \frac{2}{3}a^*P_{xy}^*\right)}.$$ (C12)

In the steady-state limit ($a^* \to a_s^*$), the numerators and denominators of Eqs. (C11) and (C12) vanish. Evaluating the corresponding limit by means of l'Hopital's rule, one reobtains the above results (C9) and (C10). This procedure can be used to get the behavior of the remaining velocity moments near the steady state.

The behavior of the fourth-degree velocity moments of the distribution $f^{(0)}$ near the steady state is also needed to determine the transport coefficients μ_{ij} and κ_{ij} associated with the heat flux in the first-order solution. To evaluate this behavior we use the Boltzmann kinetic model (59). Let us introduce the velocity moments of the zeroth-order distribution:

$$M_{k_1,k_2,k_3}^{(0)} = \int d\mathbf{c}\, c_x^{k_1}c_y^{k_2}c_z^{k_3}f^{(0)}(\mathbf{c}).$$ (C13)

These moments verify the equation

$$-\left(\frac{2}{3n}aP_{xy}^{(0)} + T\zeta^{(0)}\right)\partial_T M_{k_1,k_2,k_3}^{(0)} + ak_1 M_{k_1-1,k_2+1,k_3}^{(0)}$$

$$= -\nu\beta(M_{k_1,k_2,k_3}^{(0)} - N_{k_1,k_2,k_3}) - k\frac{\zeta^{(0)}}{2}M_{k_1,k_2,k_3}^{(0)},$$ (C14)

where $k = k_1 + k_2 + k_3$ and N_{k_1,k_2,k_3} are the velocity moments of the Gaussian distribution f_0. As before, the derivative $\partial_T M_{k_1,k_2,k_3}^{(0)}$ can be written as

$$T\partial_T M_{k_1,k_2,k_3}^{(0)} = T\partial_T n\left(\frac{2T}{m}\right)^{k/2}M_{k_1,k_2,k_3}^*(a^*)$$

$$= n\left(\frac{2T}{m}\right)^{k/2}\frac{1}{2}(k - a^*\partial_{a^*})M_{k_1,k_2,k_3}^*(a^*).$$ (C15)

In dimensionless form, Eq. (C14) becomes

$$-\left(\frac{2}{3}a^*P_{xy}^* + \zeta^*\right)\frac{1}{2}(k - a^*\partial_{a^*})M_{k_1,k_2,k_3}^* + k_1a^*M_{k_1-1,k_2+1,k_3}^*$$

$$+ \left(\beta + \frac{k}{2}\zeta^*\right)M_{k_1,k_2,k_3}^* - \beta N_{k_1,k_2,k_3}^* = 0,$$ (C16)

where N_{k_1,k_2,k_3}^* are the reduced moments of the Gaussian distribution given by

$$N_{k_1,k_2,k_3}^* = \pi^{-3/2}\Gamma\left(\frac{k_1+1}{2}\right)\left(\frac{k_2+1}{2}\right)\left(\frac{k_3+1}{2}\right)$$ (C17)

if k_1, k_2, and k_3 are even, being zero otherwise. Equation (C16) gives the expressions of the reduced moments $M_{k_1,k_2,k_3,s}^*$ in the steady state. To get $\partial_{a^*}M_{k_1,k_2,k_3}^*$ in the steady state, we differentiate with respect to a^* both sides of Eq. (C16) and then take the limit $a^* \to a_s^*$. In general, it is easy to see that the problem becomes linear so that it can be easily

solved. To illustrate the procedure, let us consider for simplicity the moment M^*_{040}, which obeys the equation

$$-\left(\frac{2}{3}a^*P^*_{xy}+\zeta^*\right)\left(2-\frac{1}{2}a^*\partial_{a^*}\right)M^*_{040}+(\beta+2\zeta^*)M^*_{040}$$

$$-\frac{3}{4}\beta=0. \tag{C18}$$

From this equation, one gets the identity

$$-\left(\frac{2}{3}a^*P^*_{xy}+\zeta^*\right)\partial_{a^*}\left[\left(2-\frac{1}{2}a^*\partial_{a^*}\right)M^*_{040}\right]$$

$$-\frac{2}{3}[P^*_{xy}+a^*(\partial_{a^*}P^*_{xy})]\left(2-\frac{1}{2}a^*\partial_{a^*}\right)M^*_{040}$$

$$+(\beta+2\zeta^*)\partial_{a^*}M^*_{040}=0. \tag{C19}$$

In the steady-state limit, Eq. (52) applies and the first term on the left-hand side vanishes. In this case, one easily gets

$$\left(\frac{\partial}{\partial a^*}M^*_{040}\right)_s=\frac{4\chi_s}{a^*_s\chi_s+2\beta+4\zeta^*}M^*_{040,s}, \tag{C20}$$

where

$$\chi_s=\frac{2}{3}\left[P^*_{xy,s}+a^*_s\left(\frac{\partial P^*_{xy}}{\partial a^*}\right)_s\right] \tag{C21}$$

is a known function and

$$M^*_{040,s}=\frac{3}{4}\frac{\beta}{\beta+2\zeta^*}. \tag{C22}$$

Proceeding in a similar way, all the derivatives of the form $\partial_{a^*}M^*$ can be analytically computed in the steady state.

APPENDIX D: KINETIC MODEL RESULTS IN THE STEADY STATE

In this appendix, I display the results obtained from the model kinetic equation chosen here for the determination of the generalized transport coefficients. In the model, the Boltzmann collision operator is replaced by the term [31]

$$J[f,f]\rightarrow-\beta\nu(f-f_0)+\frac{\zeta}{2}\frac{\partial}{\partial\mathbf{c}}\cdot(\mathbf{c}f), \tag{D1}$$

where ν and β are given by Eqs. (21) and (23), respectively, f_0 is the local equilibrium distribution (19), and ζ is the cooling rate (12).

1. Steady-state solution for the (unperturbed) USF

Let us consider first the steady-state solution to the (unperturbed) USF problem. In this case, $\delta\mathbf{u}=0$ and so $\mathbf{c}=\mathbf{V}$. The one-particle distribution function $f(\mathbf{V})$ obeys the kinetic equation

$$-aV_y\frac{\partial}{\partial V_x}f(\mathbf{V})=-\beta\nu(f-f_0)+\frac{\zeta^{(0)}}{2}\frac{\partial}{\partial\mathbf{V}}\cdot(\mathbf{V}f), \tag{D2}$$

where here ζ has been approximated by its local equilibrium approximation $\zeta^{(0)}$ given by Eq. (22). The main advantage of using a kinetic model instead of the Boltzmann equation is that the model lends itself to an exact solution [7,28]. It can be written as

$$f(\mathbf{V})=n\left(\frac{m}{2T}\right)^{3/2}f^*(\boldsymbol{\xi}),\quad\boldsymbol{\xi}=\sqrt{\frac{m}{2T}}\mathbf{V}, \tag{D3}$$

where the reduced velocity distribution function f^* is a function of the coefficient of restitution, α, and the reduced peculiar velocity $\boldsymbol{\xi}$:

$$f^*(\boldsymbol{\xi})=\pi^{-3/2}\int_0^\infty ds\,e^{-(1-3\bar{\zeta}/2)s}\exp[-e^{\bar{\zeta}s}(\boldsymbol{\xi}+s\bar{\mathbf{a}}\cdot\boldsymbol{\xi})^2]. \tag{D4}$$

Here, $\bar{\mathbf{a}}=\mathbf{a}/(\nu\beta)$ and $\bar{\zeta}=\zeta^{(0)}/(\nu\beta)$. It has been recently shown that the distribution function (D4) presents an excellent agreement with Monte Carlo simulations in the region of thermal velocities, even for strong dissipation [28].

The explicit knowledge of the velocity distribution function allows one to compute *all* the velocity moments. We introduce the moments

$$M_{k_1,k_2,k_3}=\int d\mathbf{v}\,V_x^{k_1}V_y^{k_2}V_z^{k_3}f(\mathbf{V}). \tag{D5}$$

According to the symmetry of the USF distribution (D4), the only nonvanishing moments correspond to even values of k_1+k_2 and k_3. In this case, after some algebra, one gets [28]

$$M_{k_1,k_2,k_3}=n\left(\frac{2T}{m}\right)^{k/2}M^*_{k_1,k_2,k_3}, \tag{D6}$$

where the reduced moments $M^*_{k_1,k_2,k_3}$ are given by

$$M^*_{k_1,k_2,k_3}=\pi^{-3/2}\sum_{\substack{q=0\\q+k_1=\text{even}}}^{k_1}(-\bar{a})^q\left(1+\frac{\bar{\zeta}}{2}k\right)^{-(1+q)}\frac{k_1!}{(k_1-q)!}$$

$$\times\Gamma\left(\frac{k_1-q+1}{2}\right)\Gamma\left(\frac{k_2+q+1}{2}\right)\Gamma\left(\frac{k_3+1}{2}\right), \tag{D7}$$

with $\bar{a}=a/(\nu\beta)=a^*/\beta$. It is easy to see that the expressions for the second-degree velocity moments (rheological properties) coincide with those given from the Boltzmann equation by using Grad's approximation, Eqs. (24) and (25).

2. Transport coefficients

Let us now evaluate the generalized transport coefficients $\eta_{ijk\ell}$, κ_{ij}, and μ_{ij} in the steady state. They can be obtained from Eqs. (55)–(57) with the replacement given by Eqs. (62) and (63). With these changes, Eqs. (55)–(57) become

$$\left(-ac_y\frac{\partial}{\partial c_x}+\nu\beta-\frac{\zeta^{(0)}}{2}\frac{\partial}{\partial \mathbf{c}}\cdot\mathbf{c}\right)X_{n,i}+\frac{2a}{3}\frac{T}{n}(P_{xy}^*+a^*\partial_{a^*}P_{xy}^*)X_{T,i}$$

$$=Y_{n,i},\tag{D8}$$

$$\left(-ac_y\frac{\partial}{\partial c_x}-\frac{1}{3}a(P_{xy}^*-a^*\partial_{a^*}P_{xy}^*)+\mathcal{L}\right)X_{T,i}=Y_{T,i},\tag{D9}$$

$$\left(-ac_y\frac{\partial}{\partial c_x}+\nu\beta-\frac{\zeta^{(0)}}{2}\frac{\partial}{\partial\mathbf{c}}\cdot\mathbf{c}\right)X_{u,j\ell}$$

$$-\frac{1}{2}\zeta_{u,j\ell}\left[\frac{\partial}{\partial\mathbf{c}}\cdot(\mathbf{c}f^{(0)})+2T\frac{\partial}{\partial T}f^{(0)}\right]$$

$$-a\delta_{jy}X_{u,x\ell}=Y_{u,j\ell}.\tag{D10}$$

In order to get the transport coefficients κ_{ij}, μ_{ij}, and $\eta_{ijk\ell}$, it is convenient to introduce the velocity moments

$$A_{k_1,k_2,k_3}^{(i)}=\int d\mathbf{c}\,c_x^{k_1}c_y^{k_2}c_z^{k_3}X_{n,i},\tag{D11}$$

$$B_{k_1,k_2,k_3}^{(i)}=\int d\mathbf{c}\,c_x^{k_1}c_y^{k_2}c_z^{k_3}X_{T,i},\tag{D12}$$

$$C_{k_1,k_2,k_3}^{(ij)}=\int d\mathbf{c}\,c_x^{k_1}c_y^{k_2}c_z^{k_3}X_{u,ij}.\tag{D13}$$

Knowledge of these moments allows one to get all the transport coefficients of the *perturbed* USF problem. Now, we multiply Eqs. (D8)–(D10) by $c_x^{k_1}c_y^{k_2}c_z^{k_3}$ and integrate over velocity. The result is

$$ak_1A_{k_1-1,k_2+1,k_3}^{(i)}+\left(\nu\beta+\frac{1}{2}k\zeta^{(0)}\right)A_{k_1,k_2,k_3}^{(i)}+\omega_nB_{k_1,k_2,k_3}^{(i)}$$

$$=\int d\mathbf{c}\,c_x^{k_1}c_y^{k_2}c_z^{k_3}Y_{n,i},\tag{D14}$$

$$ak_1B_{k_1-1,k_2+1,k_3}^{(i)}+\left(\nu\beta+\frac{1}{2}k\zeta^{(0)}+\omega_T\right)B_{k_1,k_2,k_3}^{(i)}$$

$$=\int d\mathbf{c}\,c_x^{k_1}c_y^{k_2}c_z^{k_3}Y_{T,i},\tag{D15}$$

$$ak_1C_{k_1-1,k_2+1,k_3}^{(j\ell)}+\left(\nu\beta+\frac{1}{2}k\zeta^{(0)}\right)C_{k_1,k_2,k_3}^{(j\ell)}$$

$$+\frac{1}{2}\zeta_{u,j\ell}(k-2T\partial_T)M_{k_1,k_2,k_3}^{(0)}-a\delta_{jy}C_{k_1,k_2,k_3}^{(x\ell)}$$

$$=\int d\mathbf{c}\,c_x^{k_1}c_y^{k_2}c_z^{k_3}Y_{u,j\ell}.\tag{D16}$$

Here, $M_{k_1,k_2,k_3}^{(0)}$ are the moments of the zeroth-order distribution $f^{(0)}$ and we have introduced the quantities

$$\omega_n=\frac{2a}{3}\frac{T}{n}(P_{xy}^*+a^*\partial_{a^*}P_{xy}^*),\quad \omega_T=-\frac{1}{3}a(P_{xy}^*-a^*\partial_{a^*}P_{xy}^*).\tag{D17}$$

The right-hand-side terms of Eqs. (D14)–(D16) can be easily evaluated with the result

$$A_{k_1,k_2,k_3}^{(\ell)}\equiv\int d\mathbf{c}\,c_x^{k_1}c_y^{k_2}c_z^{k_3}Y_{n,\ell}=-\frac{\partial}{\partial n}M_{k_1+\delta_{\ell x},k_2+\delta_{\ell y},k_3+\delta_{\ell z}}$$

$$+\frac{1}{\rho}\frac{\partial P_{\ell j}^{(0)}}{\partial n}(\delta_{jx}k_1M_{k_1-1,k_2,k_3}+\delta_{jy}k_2M_{k_1,k_2-1,k_3}$$

$$+\delta_{jz}k_3M_{k_1,k_2,k_3-1})$$

$$=-\left(\frac{2T}{m}\right)^{(k+1)/2}\left[(1-a^*\partial_{a^*})M_{k_1+\delta_{\ell x},k_2+\delta_{\ell y},k_3+\delta_{\ell z}}^*\right.$$

$$-\frac{1}{2}(1-a^*\partial_{a^*})P_{\ell j}^*(\delta_{jx}k_1M_{k_1-1,k_2,k_3}^*$$

$$\left.+\delta_{jy}k_2M_{k_1,k_2-1,k_3}^*+\delta_{jz}k_3M_{k_1,k_2,k_3-1}^*)\right],\tag{D18}$$

$$B_{k_1,k_2,k_3}^{(\ell)}\equiv\int d\mathbf{c}\,c_x^{k_1}c_y^{k_2}c_z^{k_3}Y_{T,\ell}=-\frac{\partial}{\partial T}M_{k_1+\delta_{\ell x},k_2+\delta_{\ell y},k_3+\delta_{\ell z}}$$

$$+\frac{1}{\rho}\frac{\partial P_{\ell j}^{(0)}}{\partial T}(\delta_{jx}k_1M_{k_1-1,k_2,k_3}+\delta_{jy}k_2M_{k_1,k_2-1,k_3}$$

$$+\delta_{jz}k_3M_{k_1,k_2,k_3-1})=-n\left(\frac{2T}{m}\right)^{\frac{k+1}{2}}$$

$$\times\left[\frac{1}{2T}(k+1-a^*\partial_{a^*})M_{k_1+\delta_{\ell x},k_2+\delta_{\ell y},k_3+\delta_{\ell z}}^*\right.$$

$$-\frac{1}{2T}\left(1-\frac{1}{2}a^*\partial_{a^*}\right)P_{\ell j}^*(\delta_{jx}k_1M_{k_1-1,k_2,k_3}^*$$

$$\left.+\delta_{jy}k_2M_{k_1,k_2-1,k_3}^*+\delta_{jz}k_3M_{k_1,k_2,k_3-1}^*)\right],\tag{D19}$$

$$C_{k_1,k_2,k_3}^{(j\ell)}\equiv\int d\mathbf{c}\,c_x^{k_1}c_y^{k_2}c_z^{k_3}Y_{u,j\ell}=-\delta_{j\ell}\left(1-n\frac{\partial}{\partial n}\right)M_{k_1,k_2,k_3}$$

$$+\frac{2}{3n}(P_{j\ell}^{(0)}-a\,\eta_{xyj\ell})\frac{\partial}{\partial T}M_{k_1,k_2,k_3}$$

$$-M_{k_1,k_2,k_3}(\delta_{jx}\delta_{\ell x}k_1+\delta_{jy}\delta_{\ell y}k_2+\delta_{jz}\delta_{\ell z}k_3)$$

$$-k_1\delta_{jx}(\delta_{\ell y}M_{k_1-1,k_2+1,k_3}+\delta_{\ell z}M_{k_1-1,k_2,k_3+1})$$

$$-k_2\delta_{jy}(\delta_{\ell x}M_{k_1+1,k_2-1,k_3}+\delta_{\ell z}M_{k_1,k_2-1,k_3+1})$$

$$-k_3\delta_{jz}(\delta_{\ell x}M_{k_1+1,k_2,k_3-1}+\delta_{\ell y}M_{k_1,k_2+1,k_3-1})$$

$$=-n\left(\frac{2T}{m}\right)^{k/2}\left[\delta_{j\ell}a^*\partial_{a^*}M_{k_1,k_2,k_3}^*\right.$$

$$-\frac{1}{3nT}(P_{j\ell}^{(0)}-a\,\eta_{xyj\ell})(k-a^*\partial_{a^*})M_{k_1,k_2,k_3}^*$$

$$+ M^*_{k_1,k_2,k_3}(\delta_{jx}\delta_{\ell x}k_1 + \delta_{jy}\delta_{\ell y}k_2 + \delta_{jz}\delta_{\ell z}k_3)$$

$$+ k_1\delta_{jx}(\delta_{\ell y}M^*_{k_1-1,k_2+1,k_3} + \delta_{\ell z}M^*_{k_1-1,k_2,k_3+1})$$

$$+ k_2\delta_{jy}(\delta_{\ell x}M^*_{k_1+1,k_2-1,k_3} + \delta_{\ell z}M^*_{k_1,k_2-1,k_3+1})$$

$$+ k_3\delta_{jz}(\delta_{\ell x}M^*_{k_1+1,k_2,k_3-1} + \delta_{\ell y}M^*_{k_1,k_2+1,k_3-1})\Bigg].$$

$$(D20)$$

Here, $M^*_{k_1,k_2,k_3}$ are the reduced moments of the distribution $f^{(0)}$ defined by Eq. (D6). In the steady state, $M^*_{k_1,k_2,k_3}$ is given by Eq. (D7) while the derivatives $\partial_{a^*}M^*_{k_1,k_2,k_3}$ can be obtained by following the procedure described in Appendix C.

The solution to Eqs. (D14)–(D16) can be written as

$$A^{(i)}_{k_1,k_2,k_3} = (\nu\beta)^{-1}\sum_{q=0}^{k_1}(-\bar{a})^q$$

$$\times\left(1 + \frac{k\bar{\zeta}}{2}\right)^{-(1+q)}\frac{k_1!}{(k_1-q)!}$$

$$\times[\mathcal{A}^{(i)}_{k_1-q,k_2+q,k_3} - \omega_n B^{(i)}_{k_1-q,k_2+q,k_3}], \quad (D21)$$

$$B^{(i)}_{k_1,k_2,k_3} = (\nu\beta)^{-1}\sum_{q=0}^{k_1}(-\bar{a})^q$$

$$\times\left(1 + \bar{\omega}_T + \frac{k\bar{\zeta}}{2}\right)^{-(1+q)}\frac{k_1!}{(k_1-q)!}$$

$$\times\mathcal{B}^{(i)}_{k_1-q,k_2+q,k_3}, \quad (D22)$$

$$C^{(j\ell)}_{k_1,k_2,k_3} = (\nu\beta)^{-1}\sum_{q=0}^{k_1}(-\bar{a})^q\left(1 + \frac{k\bar{\zeta}}{2}\right)^{-(1+q)}\frac{k_1!}{(k_1-q)!}$$

$$\times\Bigg[\mathcal{C}^{(j\ell)}_{k_1-q,k_2+q,k_3} + a\delta_{jy}C^{(x\ell)}_{k_1-q,k_2+q,k_3}$$

$$- \frac{1}{2}n\left(\frac{2T}{m}\right)^{k/2}\zeta_{u,j\ell}a^*\partial_{a^*}M^*_{k_1-q,k_2+q,k_3}\Bigg], \quad (D23)$$

where $\bar{\omega}_T = \omega_T/(\nu\beta)$. From Eqs. (D21)–(D23) one can get the expressions for the transport coefficients κ_{ij}, μ_{ij}, and $\eta_{ijk\ell}$ in terms of β, $\bar{\zeta}$, and $\bar{a}$.

[1] A. Goldshtein and M. Shapiro, J. Fluid Mech. **282**, 41 (1995).

[2] J. J. Brey, J. W. Dufty, and A. Santos, J. Stat. Phys. **87**, 1051 (1997).

[3] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, England, 1970).

[4] J. J. Brey, J. W. Dufty, C. S. Kim, and A. Santos, Phys. Rev. E **58**, 4638 (1998); J. J. Brey and D. Cubero, in *Granular Gases*, Lecture Notes in Physics, edited by T. Pöschel and S. Luding (Springer-Verlag, Berlin, 2001), p. 59; V. Garzó and J. M. Montanero, Physica A **313**, 336 (2002).

[5] V. Garzó and J. W. Dufty, Phys. Fluids **14**, 1476 (2002); V. Garzó and J. M. Montanero, Phys. Rev. E **69**, 021301 (2004).

[6] J. J. Brey, M. J. Ruiz-Montero, and D. Cubero, Europhys. Lett. **48**, 359 (1999); V. Garzó and J. M. Montanero, Physica A **313**, 336 (2002); J. M. Montanero and V. Garzó, Phys. Rev. E **67**, 021308 (2003).

[7] V. Garzó and A. Santos, *Kinetic Theory of Gases in Shear Flows. Nonlinear Transport* (Kluwer Academic, Dordrecht, 2003).

[8] See, for instance, C. S. Campbell, Annu. Rev. Fluid Mech. **22**, 57 (1990); I. Goldhirsch, *ibid.* **35**, 267 (2003).

[9] J. Lutsko, e-print cond-mat/0510749, Phys. Rev. E. (to be published).

[10] C. K. K. Lun, S. B. Savage, D. J. Jeffrey, and N. Chepurniy, J. Fluid Mech. **140**, 223 (1984).

[11] J. T. Jenkins and M. W. Richman, J. Fluid Mech. **192**, 313 (1988).

[12] C. S. Campbell, J. Fluid Mech. **203**, 449 (1989).

[13] M. A. Hopkins and H. H. Shen, J. Fluid Mech. **244**, 477 (1992).

[14] P. J. Schmid and H. K. Kytömaa, J. Fluid Mech. **264**, 255 (1994).

[15] C. K. K. Lun and A. A. Bent, J. Fluid Mech. **258**, 335 (1994).

[16] I. Goldhirsch and M. L. Tan, Phys. Fluids **8**, 1752 (1996).

[17] N. Sela, I. Goldhirsch, and S. H. Noskowicz, Phys. Fluids **8**, 2337 (1997).

[18] J. J. Brey, M. J. Ruiz-Montero, and F. Moreno, Phys. Rev. E **55**, 2846 (1997).

[19] C.-S. Chou and M. W. Richman, Physica A **259**, 430 (1998); C.-S. Chou, *ibid.* **287**, 127 (2000); **290**, 341 (2001).

[20] J. M. Montanero, V. Garzó, A. Santos, and J. J. Brey, J. Fluid Mech. **389**, 391 (1999).

[21] R. Clelland and C. M. Hrenya, Phys. Rev. E **65**, 031301 (2002).

[22] J. M. Montanero and V. Garzó, Physica A **310**, 17 (2002); Mol. Simul. **29**, 357 (2003); V. Garzó and J. M. Montanero, Granular Matter **5**, 165 (2003).

[23] V. Garzó, Phys. Rev. E **66**, 021308 (2002).

[24] J. F. Lutsko, Phys. Rev. E **70**, 061101 (2004).

[25] M. Alam and S. Luding, J. Fluid Mech. **476**, 69 (2003).

[26] J. M. Montanero, V. Garzó, M. Alam, and S. Luding, e-print cond-mat/0411548, Granular Matter (to be published).

[27] A. Santos, V. Garzó, and J. W. Dufty, Phys. Rev. E **69**, 061303 (2004).

[28] A. Santos and A. Astillero, Phys. Rev. E **72**, 031308 (2005); A. Astillero and A. Santos, *ibid.* **72**, 031309 (2005).

[29] M. Lee and J. W. Dufty, Phys. Rev. E **56**, 1733 (1997).

[30] M. Tij, E. Tahiri, J. M. Montanero, V. Garzó, A. Santos, and J. W. Dufty, J. Stat. Phys. **103**, 1035 (2001).

[31] J. J. Brey, J. W. Dufty, and A. Santos, J. Stat. Phys. **97**, 281 (1999).

[32] O. R. Walton and R. L. Braun, J. Rheol. **30**, 949 (1986); Acta Mech. **73**, 86 (1986); C. S. Campbell and C. E. Brennen, J.

Fluid Mech. **151**, 167 (1985); M. H. Hopkins and M. Y. Louge, Phys. Fluids A **3**, 47 (1991); M. Alam and S. Luding, Phys. Fluids **15**, 2298 (2003).

[33] S. B. Savage, J. Fluid Mech. **241**, 109 (1992).

[34] M. Babic, J. Fluid Mech. **254**, 127 (1993).

[35] M. Alam and P. R. Nott, J. Fluid Mech. **343**, 267 (1997); **377**, 99 (1998); J. D. Goddard and M. Alam, Part. Sci. Technol. **17**, 69 (1999).

[36] V. Kumaran, Physica A **275**, 283 (2000); **284**, 246 (2000).

[37] V. Kumaran, Phys. Fluids **13**, 2258 (2001).

[38] A. W. Lees and S. F. Edwards, J. Phys. C **5**, 1921 (1972).

[39] J. W. Dufty, A. Santos, J. J. Brey, and R. F. Rodríguez, Phys. Rev. A **33**, 459 (1986).

[40] The model kinetic equation of Ref. [31] uses the exact local homogeneous cooling state of the Boltzmann equation for the distribution f_0. An excellent approximation to f_0 in the region of thermal velocities is given by the local equilibrium distribution function (19). Given that here we are mainly interested in computing transport properties, the local equilibrium form (19) is adopted for f_0.

[41] V. Garzó, Phys. Rev. E **72**, 021106 (2005).

[42] P. Resibois and M. de Leener, *Classical Kinetic Theory of Fluids* (Wiley, New York, 1977).

[43] J. F. Lutsko, e-print cond-mat/0403551.

[44] M. Lee, J. W. Dufty, J. M. Montanero, A. Santos, and J. F. Lutsko, Phys. Rev. Lett. **76**, 2702 (1996).

[45] E. Fukushima, Annu. Rev. Fluid Mech. **31**, 95 (1999).

[46] X. Yang, C. Huan, D. Candela, R. W. Mair, and R. L. Walsworth, Phys. Rev. Lett. **88**, 044301 (2002); C. Huan, X. Yang, D. Candela, R. W. Mair, and R. L. Walsworth, Phys. Rev. E **69**, 041302 (2004).

[47] R. D. Wildman, J. M. Huntley, and D. J. Parker, Phys. Rev. Lett. **86**, 3304 (2001).

[48] M. Medved, D. Dawson, H. M. Jaeger, and S. R. Nagel, Chaos **9**, 691 (1999).

[49] J. M. Montanero, A. Santos, M. Lee, J. W. Dufty, and J. F. Lutsko, Phys. Rev. E **57**, 546 (1998).