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Chapman-Enskog expansion about nonequilibrium states with application to the sheared granular fluid

James F. Lutsko
Center for Nonlinear Phenomena and Complex Systems, Université Libre de Bruxelles, Code Postal 231, Boulevard du Triomphe, 1050 Brussels, Belgium

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The Chapman-Enskog method of solution of kinetic equations, such as the Boltzmann equation, is based on an expansion in gradients of the deviations of the hydrodynamic fields from a uniform reference state (e.g., local equilibrium). This paper presents an extension of the method so as to allow for expansions about *arbitrary*, far-from-equilibrium reference states. The primary result is a set of hydrodynamic equations for studying variations from the arbitrary reference state which, unlike the usual Navier-Stokes hydrodynamics, does not restrict the reference state in any way. The method is illustrated by application to a sheared granular gas which cannot be studied using the usual Navier-Stokes hydrodynamics.

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I. INTRODUCTION

The determination of the one-body distribution function, which gives the probability of finding a particle at some given position, with a given velocity at a given time, is one of the central problems in nonequilibrium statistical mechanics. Its time-evolution is in many cases well-described by approximate kinetic equations such as the Boltzmann equation [1] for low-density gases and the revised Enskog equation [2,3] for denser hard-sphere gases and solids. Only rarely are exact solutions of these equations possible. Probably the most important technique for generating approximate solutions to one-body kinetic equations is the Chapman-Enskog method which, as originally formulated, consists of a gradient expansion about a local-equilibrium state [4,1]. The goal in this approach is to construct a particular type of solution, called a "normal solution," in which all space and time dependence of the one-body distribution occurs implicitly via its dependence on the macroscopic hydrodynamic fields. The latter are, for a simple fluid, the density, velocity, and temperature fields corresponding to the conserved variables of particle number, momentum, and energy, respectively. (In a multicomponent system, the partial densities are also included.) The Chapman-Enskog method proceeds to develop the solution perturbatively in the gradients of the hydrodynamic fields: the distribution is developed as a functional of the fields and their gradients and at the same time the equations of motion of the fields, the hydrodynamic equations, are also developed. The zeroth-order distribution is the local-equilibrium distribution; at first order, this is corrected by terms involving linear gradients of the hydrodynamic fields which in turn are governed by the Euler equations (with an explicit prescription for the calculation of the pressure from the kinetic theory). At second order, the hydrodynamic fields are governed by the Navier-Stokes equations, at third order, by the Burnett equations, etc. The calculations involved in extending the solution to each successive higher order are increasingly difficult and since the Navier-Stokes equations are usually considered an adequate description of fluid dynamics, results above third order (Burnett order) for the Boltzmann equation and above second (Navier-Stokes) order for the Enskog equation are sparse. The extension of the Chapman-Enskog method beyond the Navier-Stokes level is, however, not physically irrelevant since only by doing so is it possible to understand non-Newtonian viscoelastic effects such as shear thinning and normal stresses which occur even in simple fluids under extreme conditions [5,6].

Recently, interest in non-Newtonian effects has increased because of their importance in fluidized granular materials. Granular systems are composed of particles—grains—which lose energy when they collide. As such, there is no equilibrium state: an undriven homogeneous collection of grains will cool continuously. This has many interesting consequences such as the spontaneous formation of clusters in the homogeneous gas and various segregation phenomena in mixtures [7–10]. The applicability of standard kinetic theory to fluidized granular materials has long been questioned (see, e.g., Ref. [11]). While it is certainly true that the assumption of binary collisions that underlies the Boltzmann and Enskog equations does not apply to some important circumstances such as when jamming and clusters form, there is support both from experiment and from computer simulation studies for the usefulness of the kinetic-theory description under conditions in which the system remains fluidized so that binary collisons dominate the dynamics. (A recent discussion of the evidence supporting the use of kinetic theory in fluidized granular systems can be found, e.g., in Ref. [12].) The collisional cooling also gives rise to a unique class of nonequilibrium steady states due to the fact that the cooling can be balanced by the viscous heating that occurs in inhomogeneous flows. One of the most widely studied examples of such a system is a granular fluid undergoing planar Couette flow where the velocity field takes the form $\mathbf{v}(\mathbf{r})=ay\hat{\mathbf{x}}$, where a is the shear rate. The common presence of non-Newtonian effects, such as normal stresses, in these systems has long been recognized as signaling the need to go beyond the Navier-Stokes description [13]. As emphasized by Santos *et al.* [14], the balance between the velocity gradients, which determine the rate of viscous heating, and the cooling, aris-

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ing from a material property, means that such fluids are inherently non-Newtonian in the sense that the sheared state cannot be viewed as a perturbation of the unsheared, homogeneous fluid and so the usual Navier-Stokes equations cannot be used to study either the rheology or the stability of the sheared granular fluid. One of the goals of the present work is to show that a more general hydrodynamic description can be derived for this, and other flow states, which is able to accurately describe such far-from-equilibrium states. The formalism developed here is general and not restricted to granular fluids although they do provide the most obvious application. Indeed, an application of this form of hydrodynamics has recently been presented by Garzó [15] who studied the stability of a granular fluid under strong shear.

The extension of the Chapman-Enskog method to derive the hydrodynamics for fluctuations about an arbitrary nonequilibrium state might at first appear trivial but in fact it involves a careful application of the ideas underlying the method. To illustrate, let $f(\mathbf{r},\mathbf{v},t)$ be the probability to find a particle at position $\mathbf{r}$ with velocity $\mathbf{v}$ at time t. For a D-dimensional system in equilibrium, this is just the (space and time-independent) Gaussian distribution

$$f(\mathbf{r},\mathbf{v},t)=\phi_0(\mathbf{v};n,T,U)=n\left(\frac{m}{2\pi k_B T}\right)^{D/2}\exp(-(\mathbf{v}-\mathbf{U})^2/k_BT), \tag{1}$$

where n is the number density, k_B is Boltzmann's constant, T is the temperature, m is the mass of the particles, and $\mathbf{U}$ is the center-of-mass velocity. The zeroth-order approximation in the Chapman-Enskog method is the localized distribution $f^{(0)}(\mathbf{r},\mathbf{v},t)=\phi_0(\mathbf{v};n(\mathbf{r},t),T(\mathbf{r},t),\mathbf{U}(\mathbf{r},t))$ or, in other words, the local equilibrium distribution. In contrast, a homogeneous nonequilibrium steady state might be characterized by some time-independent distribution

$$f(\mathbf{r},\mathbf{v},t)=\Phi_{ss}(\mathbf{v};n,T,\mathbf{U}) \tag{2}$$

but the zeroth-order approximation in the Chapman-Enskog method will *not* in general be the localized steady-state distribution, $f^{(0)}(\mathbf{r},\mathbf{v},t)\neq\Phi_{ss}(\mathbf{v};n(\mathbf{r},t),T(\mathbf{r},t),\mathbf{U}(\mathbf{r},t))$. The reason is that a steady state is the result of a balance—in the example given above, it is a balance between viscous heating and collisional cooling. Thus any change in density must be compensated by, say, a change in temperature or the system is no longer in a steady state. This therefore gives a relation between density and temperature in the steady state, say $n=n_{ss}(T)$, so that one has $\Phi_{ss}(\mathbf{v};n,T,\mathbf{U})=\Phi_{ss}(\mathbf{v};n_{ss}(T),T,\mathbf{U})$. Clearly, it makes no sense to simply "localize" the hydrodynamic variables as the starting point of the Chapman-Enskog method since, in a steady state, the hydrodynamic variables are not independent. Limited attempts have been made in the past to perform the type of generalization suggested here. In particular, Lee and Dufty considered this problem for the specific case of an ordinary fluid under shear with an artificial thermostat present so as to make possible a steady state [16,17]. However, the issues discussed in this paper were circumvented through the use of a very particular type of thermostat so that, while of theoretical interest, that calculation cannot serve as a template for the more general problem.

In Sec. II, the abstract formulation of the Chapman-Enskog expansion for fluctuations about a nonequilibrium state is discussed. It not only requires care in understanding the zeroth order approximation, but a generalization in the concept of a normal solution. In Sec. III, the method is illustrated by application to the simple kinetic theory for a sheared granular gas proposed by Brey, Dufty, and Santos [18]. While a kinetic model can only be considered as a crude approximation to a more complete kinetic description, such as the Boltzmann equation, there is evidence supporting the semiquantitative accuracy of such models in describing the low-density fluids under shear flow. Comparisons of numerical solutions of the Boltzmann equation to the predictions of kinetic models such as the one used here show surprisingly good agreement in the description of such phenomena as normal stresses, shear thinning, and the pressure tensor and heat-flux vector in the steady state are all reasonably well-reproduced [19–21]. It is therefore of practical relevence, as well as of some intrinisic interest, to extend the analysis of this simple model. On the other hand, the simplicity of the model will allow for a clearer illustration of the principles of the extension of the Chapman-Enskog method to nonequilibrium reference states than would be possible with a more complex example such as the Boltzmann equation. The result is that exact expressions in the form of ordinary differential equations can be given for the full complement of transport coefficients. The section concludes with a brief summary of the resulting hydrodynamics and of the linearized form of the hydrodynamic equations which leads to considerable simplification including explicit expressions for the transport coefficients in the steady state. The paper ends in Sec. IV with a summary of the results, a comparison of the present results to the results of the standard Chapman-Enskog analysis, and a discussion of further applications.

II. THE CHAPMAN-ENSKOG EXPANSION ABOUT AN ARBITRARY STATE

A. Kinetic theory

Consider a single-component fluid composed of particles of mass m in D dimensions. In general, the one-body distribution will obey a kinetic equation of the form

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)f(\mathbf{r},\mathbf{v},t)=J[\mathbf{r},\mathbf{v},t|f] \tag{3}$$

where the collision operator $J[\mathbf{r},\mathbf{v},t|f]$ is a function of position and velocity and a *functional* of the distribution function. No particular details of the form of the collision operator will be important here but all results are formulated with the examples of BGK-type relaxation models, the Boltzmann equation and the Enskog equation in mind. The first five velocity moments of f define the number density

$$n(\mathbf{r},t)=\int d\mathbf{v}f(\mathbf{r},\mathbf{v},t), \tag{4}$$

the flow velocity

$$\mathbf{u}(\mathbf{r},t) = \frac{1}{n(\mathbf{r},t)} \int d\mathbf{v}\mathbf{v}f(\mathbf{r},\mathbf{v},t), \tag{5}$$

and the kinetic temperature

$$T(\mathbf{r},t) = \frac{m}{Dn(\mathbf{r},t)k_B} \int d\mathbf{v}C^2(\mathbf{r},t)f(\mathbf{r},\mathbf{v},t), \tag{6}$$

where $\mathbf{C}(\mathbf{r},t) \equiv \mathbf{v} - \mathbf{u}(\mathbf{r},t)$ is the peculiar velocity. The macroscopic balance equations for density n, momentum $m\mathbf{u}$, and energy $(D/2)nk_BT$ follow directly from Eq. (3) by multiplying with 1, $m\mathbf{v}$, and $\frac{1}{2}mC^2$ and integrating over $\mathbf{v}$:

$$D_t n + n\nabla \cdot \mathbf{u} = 0,$$

$$D_t u_i + (mn)^{-1}\nabla_j P_{ij} = 0,$$

$$D_t T + \frac{2}{Dnk_B}(\nabla \cdot \mathbf{q} + P_{ij}\nabla_j u_i) = -\zeta T, \tag{7}$$

where $D_t = \partial_t + \mathbf{u}\cdot\nabla$ is the convective derivative. The microscopic expressions for the pressure tensor $\mathsf{P} = \mathsf{P}[f]$, the heat flux $\mathbf{q} = \mathbf{q}[f]$ depend on the exact form of the collision operator (see Refs. [1,22] for a general discussion) but as indicated, they are in general functionals of the distribution, while the cooling rate ζ is given by

$$\zeta(\mathbf{r},t) = -\frac{1}{Dn(\mathbf{r},t)k_BT(\mathbf{r},t)} \int d\mathbf{v}mC^2 J[\mathbf{r},\mathbf{v},t|f]. \tag{8}$$

B. Formulation of the gradient expansion

The goal of the Chapman-Enskog method is to construct a so-called *normal* solution to the kinetic equation, Eq. (3). In the standard formulation of the method [1], this is defined as a distribution $f(\mathbf{r},\mathbf{v},t)$ for which all of the space and time dependence occurs through the hydrodynamic variables, denoted collectively as $\psi \equiv \{n,\mathbf{u},T\}$, and their derivatives so that

$$f(\mathbf{r},\mathbf{v},t) = f(\mathbf{v};\psi(\mathbf{r},t),\nabla\psi(\mathbf{r},t),\nabla\nabla\psi(\mathbf{r},t),\ldots). \tag{9}$$

The distribution is therefore a *functional* of the fields $\psi(\mathbf{r},t)$ or, equivalently in this case, a *function* of the fields and their gradients to all orders. In the following, this particular type of functional dependence will be denoted more compactly with the notation $f(\mathbf{v};[\nabla^{(n)}\psi(\mathbf{r},t)])$ where the index, n, indicates the maximum derivative that is used. When all derivatives are possible, as in Eq. (9) the notation $f(\mathbf{r},\mathbf{v},t) = f(\mathbf{v};[\nabla^{(\infty)}\psi(\mathbf{r},t)])$ will be used. The kinetic equation, Eq. (3), the balance equations, Eqs. (7), and the definitions of the various fluxes and sources then provide a closed set of equations from which to determine the distribution. Note that since the fluxes and sources are functionals of the distribution, their space and time dependence also occurs implicitly via their dependence on the hydrodynamic fields and their derivatives.

Given such a solution has been found for a particular set of boundary conditions yielding the hydrodynamic state $\psi_0(\mathbf{r},t)$ with distribution $f_0(\mathbf{v};[\nabla^{(\infty)}\psi_0(\mathbf{r},t)])$, the aim is to describe deviations about this state, denoted $\delta\psi$, so that the total hydrodynamic fields are $\psi = \psi_0 + \delta\psi$. In the Chapman-Enskog method, it is assumed that the deviations are smooth in the sense that

$$\delta\psi \gg l\nabla\,\delta\psi \gg l^2\nabla\nabla\delta\psi\cdots, \tag{10}$$

where l is the mean free path, so that one can work perturbatively in terms of the gradients of the perturbations to the hydrodynamic fields. To develop this perturbation theory systematically, it is convenient to introduce a fictitious small parameter, ϵ, and to write the gradient operator as $\nabla = \nabla^{(0)} + \epsilon\nabla^{(1)}$ where the two operators on the right are defined by $\nabla_0\psi = \nabla\psi_0$ and $\nabla_1\psi = \nabla\delta\psi$. This then generates an expansion of the distribution that looks like

$$\begin{aligned} f(\mathbf{v};[\nabla^{(\infty)}\psi(\mathbf{r},t)]) &= f^{(0)}(\mathbf{v};\nabla_0^{(\infty)}\psi(\mathbf{r},t)) \\ &\quad + \epsilon f^{(1)}(\mathbf{v};\nabla_1\delta\psi,\nabla_0^{(\infty)}\psi(\mathbf{r},t)) \\ &\quad + \epsilon^2 f^{(2)}(\mathbf{v};\nabla_1\nabla_1\delta\psi,(\nabla_1\delta\psi)^2,\nabla_0^{(\infty)}\psi(\mathbf{r},t)) + \cdots, \end{aligned} \tag{11}$$

where $f^{(1)}$ will be linear in $\nabla_1\delta\psi$, $f^{(2)}$ will be linear in $\nabla_1\nabla_1\delta\psi$ and $(\nabla_1\delta\psi)^2$, etc. This notation is meant to be taken literally: the quantity $\nabla_0^{(\infty)}\psi(\mathbf{r},t) = \{\psi(\mathbf{r},t),\nabla_0\psi(\mathbf{r},t),\ldots\} = \{\psi(\mathbf{r},t),\nabla\psi_0(\mathbf{r},t),\ldots\}$ so that at each order in perturbation theory, the distribution is a function of the exact field $\psi(\mathbf{r},t)$ as well as all gradients of the reference field. This involves a departure from the usual formulation of the Chapman-Enskog definition of a normal state. In the standard form, the distribution is assumed to be a functional of the *exact* fields $\psi(\mathbf{r},t)$ whereas here it is proposed that the distribution is a functional of the exact field $\psi(\mathbf{r},t)$ *and* of the reference state $\psi_0(\mathbf{r},t)$. Of course, it is obvious that in order to study deviations about a reference state within the Chapman-Enskog framework, the distribution will have to be a functional of that reference state. Nevertheless, this violates, or generalizes, the usual definition of a normal solution since there are now two sources of space and time dependence in the distribution: the exact hydrodynamics fields and the reference hydrodynamic state. For deviations from an equilibrium state, this point is moot since $\nabla\psi_0(\mathbf{r},t) = 0$, etc.

The perturbative expansion of the distribution will generate a similar expansion of the fluxes and sources through their functional dependence on the distribution, see, e.g., Eq. (8), so that one writes

$$P_{ij} = P_{ij}^{(0)} + \epsilon P_{ij}^{(1)} + \cdots \tag{12}$$

and so forth. Since the balance equations link space and time derivatives, it is necessary to introduce a multiscale expansion of the time derivatives in both the kinetic equation and the balance equations as

$$\frac{\partial}{\partial t}f = \partial_t^{(0)}f + \epsilon\partial_t^{(1)}f + \cdots. \tag{13}$$

The precise meaning of the symbols $\partial_t^{(0)}$, $\partial_t^{(1)}$ is that the balance equations define $\partial_t^{(i)}$ in terms of the spatial gradients of the hydrodynamic fields and these definitions, together with the normal form of the distribution, define the action of these symbols on the distribution. Finally, to maintain generality, note that sometimes (specifically in the Enskog theory) the collision operator itself is nonlocal and must be expanded as well in gradients in $\delta\psi$ so that we write

$$J[\mathbf{r},\mathbf{v},t|f] = J_0[\mathbf{r},\mathbf{v},t|f] + \epsilon J_1[\mathbf{r},\mathbf{v},t|f] + \cdots \tag{14}$$

and it is understood that $J_0[\mathbf{r},\mathbf{v},t|f]$ by definition involves no gradients with respect to the perturbations $\delta\psi(\mathbf{r},t)$ but will, in general, contain gradients of *all* orders in the reference fields $\psi_0(\mathbf{r},t)$. (Note that the existence of a normal solution is plausible if the spatial and temporal dependence of the collision operator is also normal which is, in fact, generally the case. However, for simplicity, no effort is made here to indicate this explicitly.) A final property of the perturbative expansion concerns the relation between the various distributions and the hydrodynamic variables. The zeroth order distribution is required to reproduce the exact hydrodynamic variables via

$$\begin{pmatrix} n(\mathbf{r},t) \\ n(\mathbf{r},t)\mathbf{u}(\mathbf{r},t) \\ Dn(\mathbf{r},t)k_BT \end{pmatrix} = \int \begin{pmatrix} 1 \\ \mathbf{v} \\ mC^2 \end{pmatrix} f^{(0)}(\mathbf{v};\nabla_0^{(\infty)}\psi(\mathbf{r},t))d\mathbf{v} \tag{15}$$

while the higher order terms are orthogonal to the first three velocity moments

$$\int \begin{pmatrix} 1 \\ \mathbf{v} \\ mC^2 \end{pmatrix} f^{(n)}(\mathbf{v};\nabla_0^{(\infty)}\psi(\mathbf{r},t))d\mathbf{v} = 0, n > 0, \tag{16}$$

so that the total distribution $f=f^{(0)}+f^{(1)}+\cdots$ satisfies Eqs. (4)–(6).

C. The reference state

Recall that the goal is to describe deviations from the reference state $\psi_0(\mathbf{r},t)$ which corresponds to the distribution $f_0(\mathbf{r},\mathbf{v},t;[\psi_0])$ and in fact the distribution and fields are related by the definitions given in Eqs. (4)–(6). The reference distribution is itself assumed to be normal so that the dependence on $\mathbf{r}$ and t occurs implicitly through the fields. In terms of the notation used here, the reference distribution satisfies the kinetic equation, Eq. (9), and the full, nonlinear balance equations, Eqs. (7). Using the definitions given above, these translate into

$$(\partial_t^{(0)} + \mathbf{v}\cdot\nabla^{(0)})f_0(\mathbf{r},\mathbf{v},t;[\psi_0]) = J_0[\mathbf{r},\mathbf{v},t|f_0] \tag{17}$$

and the fields are solutions to the full, nonlinear balance equations

$$\partial_t^{(0)}n_0 + \mathbf{u}\cdot\nabla^{(0)}n_0 + n_0\nabla^{(0)}\cdot\mathbf{u}_0 = 0,$$

$$\partial_t^{(0)}u_{0,i} + \mathbf{u}_0\cdot\nabla^{(0)}u_{0.i} + (mn_0)^{-1}\partial_j^{(0)}P_{ij}^{(00)} = 0,$$

$$\partial_t^{(0)}T_0 + \mathbf{u}_0\cdot\nabla^{(0)}T_0 + \frac{2}{Dn_0k_B}(\nabla^{(0)}\cdot\mathbf{q}^{(00)} + P_{ij}^{(00)}\partial_j^{(0)}u_{0,i})$$
$$= -\zeta^{(00)}T_0, \tag{18}$$

where, e.g., $P_{ij}^{(00)}$ is the pressure tensor evaluated in the reference state, and

$$\partial_t^{(n)}\psi_0 = 0, \quad n > 0. \tag{19}$$

Thus, in the ordering scheme developed here, the reference state is an exact solution to the zeroth order perturbative equations.

For the standard case describing deviations from the equilibrium state, the hydrodynamic fields are constant in both space and time and $\zeta^{(00)}=0$ so that the balance equations just reduce to $\partial_t^{(0)}\psi_0=0$. The left-hand side of the kinetic equation therefore vanishes leaving $0=J_0[\mathbf{r},\mathbf{v},t|f_0]$ which is indeed satisfied by the equilibrium distribution. For a granular fluid, $\zeta^{(00)}\neq 0$ and the simplest solution that can be constructed consists of spatially homogeneous, but time dependent fields giving

$$\partial_t^{(0)}f_0(\mathbf{r},\mathbf{v},t;[\psi_0]) = J_0[\mathbf{r},\mathbf{v},t|f_0] \tag{20}$$

and

$$\partial_t^{(0)}n_0 = 0,$$

$$\partial_t^{(0)}u_{0,i} = 0,$$

$$\partial_t^{(0)}T_0 = -\zeta^{(00)}T_0 \tag{21}$$

so that the distribution depends on time through its dependence on the temperature. The balance equations, together with the assumption of normality, serve to define the meaning of the left-hand side of Eq. (20) giving

$$-\zeta^{(00)}T_0\frac{\partial}{\partial T}f_0(\mathbf{r},\mathbf{v},t;[\psi_0]) = J_0[\mathbf{r},\mathbf{v},t|f_0]. \tag{22}$$

Typically, this is solved by assuming a scaling solution of the form $f_0(\mathbf{r},\mathbf{v},t;[\psi_0])=\Phi(\mathbf{v}\sqrt{m\sigma^2/k_BT(t)})$.

D. The zeroth order Chapman-Enskog solution

As emphasized above, the Chapman-Enskog method is an expansion in gradients of the deviations of the hydrodynamic fields from the reference state. Using the ordering developed above, the zeroth order kinetic equation is

$$\partial_t^{(0)}f^{(0)}(\mathbf{r},\mathbf{v};\delta\psi(\mathbf{r},t),[\psi_0]) + \mathbf{v}\cdot\nabla^{(0)}f^{(0)}(\mathbf{r},\mathbf{v};\delta\psi(\mathbf{r},t),[\psi_0])$$
$$= J_0[\mathbf{r},\mathbf{v},t|f_0] \tag{23}$$

and the zeroth order balance equations are

$$\partial_t^{(0)}n + \mathbf{u}\cdot\nabla n_0 + n\nabla\cdot\mathbf{u}_0 = 0,$$

$$\partial_t^{(0)}u_i + \mathbf{u}\cdot\nabla u_{0.i} + (mn)_j^{-1}\nabla^{(0)}P_{ij}^{(0)} = 0,$$

$$\partial_t^{(0)}T + \mathbf{u}\cdot\nabla T_0 + \frac{2}{Dnk_B}(\nabla^{(0)}\cdot\mathbf{q}^{(0)} + P_{ij}^{(0)}\partial_j u_{0,i}) = -\zeta^{(0)}T. \tag{24}$$

Making use of the balance equations satisfied by the reference fields, Eq. (18), this can be written in terms of the deviations as

$$\partial_t^{(0)}\delta n + \delta\mathbf{u}\cdot\nabla n_0 + \delta n\,\nabla\cdot\mathbf{u}_0 = 0,$$

$$\partial_t^{(0)}\delta u_i + \delta\mathbf{u}\cdot\nabla u_{0,i} + (mn)^{-1}\nabla_j^{(0)}P_{ij}^{(0)} - (mn_0)^{-1}\nabla_j P_{ij}^{(00)} = 0,$$

$$\partial_t^{(0)}\delta T + \delta\mathbf{u}\cdot\nabla T_0 + \frac{2}{Dnk_B}(\nabla^{(0)}\cdot\mathbf{q}^{(0)} + P_{ij}^{(0)}\nabla_j u_{0,i})$$

$$-\frac{2}{Dn_0k_B}(\nabla\cdot\mathbf{q}^{(00)} + P_{ij}^{(00)}\nabla_j u_{0,i}) = -\zeta^{(0)}T + \zeta^{(00)}T_0. \tag{25}$$

Since the zeroth order distribution is a *function* of $\delta\psi$ but a *functional* of the reference fields, the time derivative in Eq. (23) is evaluated using

$$\partial_t^{(0)}f^{(0)} = \sum_\alpha (\partial_t^{(0)}\delta\psi_\alpha(\mathbf{r},t))\frac{\partial}{\partial\delta\psi_\alpha(\mathbf{r},t)}f^{(0)} + \sum_\alpha\int d\mathbf{r}'(\partial_t^{(0)}\psi_{0,\alpha}(\mathbf{r}',t))\frac{\delta}{\delta\psi_{0,\alpha}(\mathbf{r}',t)}f^{(0)} \tag{26}$$

and these equations must be solved subject to the additional boundary condition

$$\lim_{\delta\psi\to 0} f^{(0)}(\mathbf{r},\mathbf{v},t;\delta\psi(\mathbf{r},t),[\psi_0]) = f_0(\mathbf{r},\mathbf{v},t;[\psi_0]). \tag{27}$$

There are several important points to be made here. First, it must be emphasized that the reference fields $\psi_0(\mathbf{r},t)$ and the deviations $\delta\psi(\mathbf{r},t)$ are playing different roles in these equations. The former are fixed and assumed known whereas the latter are independent variables. The result of a solution of these equations will be the zeroth order distribution as a function of the variables $\delta\psi$. For any given physical problem, the deviations will be determined by solving the balance equations, Eqs. (25), subject to appropriate boundary conditions and only then is the distribution completely specified. Second, nothing is said here about the solution of Eqs. (23)–(26) which, in general, constitute a complicated functional equation in terms of the reference state variables $\psi_{0,\alpha}(\mathbf{r},t)$. The only obvious exceptions, and perhaps the only practical cases, are when the reference state is either time-independent, so that $\partial_t^{(0)}\psi_{0,\alpha}=0$, or spatially homogeneous so that $f^{(0)}$ is a function, and not a functional, of the reference fields. The equilibrium state is both, the homogeneous cooling state is a spatially homogeneous state, and time-independent flow states such as uniform shear flow or Pouseille flow with thermalizing walls are important examples of time-independent, spatially inhomogeneous states. Third, since Eqs. (23) and (24) are the lowest order equations in a gradient expansion, they are to be solved for *arbitrarily large* deviations of the fields, $\delta\psi$. There is no sense in which the deviations should be considered to be small. The fourth observation, and perhaps the most important, is that there is no conceptual connection between the zeroth order distribution $f^{(0)}(\mathbf{v};\delta\psi(\mathbf{r},t),\nabla_0^{(\infty)}\psi_0(\mathbf{r},t))$ and the reference distribution $f_0(\mathbf{v};\nabla_0^{(\infty)}\psi_0(\mathbf{r},t))$ except for the limit given in Eq. (27). In particular, it will almost always be the case that

$$f^{(0)}(\mathbf{v};\delta\psi(\mathbf{r},t),\nabla_0^{(\infty)}\psi_0(\mathbf{r},t)) \neq f_0(\mathbf{v};\nabla_0^{(\infty)}(\psi_0(\mathbf{r},t)+\delta\psi(\mathbf{r},t))). \tag{28}$$

A rare exception for which this inequality is reversed is when the reference state is the equilibrium state. In that case, the density, temperature, and velocity fields are uniform and the reference distribution is just a Gaussian

$$f_0(\mathbf{r},\mathbf{v};\nabla_0^{(\infty)}\psi_0) = \phi_0(\mathbf{v};n_0,T_0,\mathbf{U}_0) \tag{29}$$

and the solution to the zeroth order equations is the local equilibrium distribution

$$\begin{aligned} f^{(0)}(\mathbf{v};\delta\psi(\mathbf{r},t),\nabla_0^{(\infty)}\psi_0(\mathbf{r},t)) &= \phi_0(\mathbf{v};n+\delta n(\mathbf{r},\mathbf{t}),T+\delta T(\mathbf{r},\mathbf{t}),\mathbf{U}+\delta\mathbf{U}(\mathbf{r},\mathbf{t})) \\ &= f_0(\mathbf{v};\nabla_0^{(\infty)}(\psi_0(\mathbf{r},t)+\delta\psi(\mathbf{r},t))). \end{aligned} \tag{30}$$

For steady states, as will be illustrated in the next section, it is not the case that $f^{(0)}$ is obtained from the steady-state distribution via a "localization" along the lines of that shown in Eq. (30). On the other hand, Eqs. (23) and (24) are the same whether they are solved for the general field $\delta\psi(\mathbf{r},t)$ or for the spatially homogeneous field $\delta\psi(t)$ with the subsequent localization $\delta\psi(t)\rightarrow\delta\psi(\mathbf{r},t)$. Furthermore, these equations are identical to those one would solve in order to obtain an exact normal solution to the full kinetic equation, Eq. (17), and balance equations, Eqs. (18), for the fields $\psi_0(\mathbf{r},t)+\delta\psi(t)$. In other words, the zeroth order Chapman-Enskog distribution is the localization of the exact distribution for homogeneous deviations from the reference state. Again, only in the case of the equilibrium reference state is it true that this corresponds to the localization of the reference state itself.

E. First order Chapman-Enskog

In the following, the equations for the first order terms will also be needed. Collecting terms in Eq. (17), the first order distribution function is found to satisfy

$$\begin{aligned} &\partial_t^{(0)}f^{(1)}(\mathbf{v};\delta\psi(\mathbf{r},t),[\psi_0]) + \mathbf{v}\cdot\nabla^{(0)}f^{(1)}(\mathbf{v};\delta\psi(\mathbf{r},t),[\psi_0]) \\ &\quad = J_0[\mathbf{r},\mathbf{v},t|f_1] + J_1[\mathbf{r},\mathbf{v},t|f_0] - (\partial_t^{(1)}f^{(0)}(\mathbf{v};\delta\psi(\mathbf{r},t),[\psi_0]) \\ &\qquad + \mathbf{v}\cdot\nabla^{(1)}f^{(0)}(\mathbf{v};\delta\psi(\mathbf{r},t),[\psi_0])) \end{aligned} \tag{31}$$

and the first order balance equations become

$$\partial_t^{(1)}\delta n + \mathbf{u}\cdot\nabla\delta n + n\,\nabla\cdot\delta\mathbf{u} = 0,$$

$$\partial_T^{(1)}\delta u_i + \mathbf{u}\cdot\nabla\delta u_i + (mn)^{-1}\nabla_j^{(1)}P_{ij}^{(0)} + (mn)^{-1}\nabla_J^{(0)}P_{ij}^{(1)} = 0,$$

$$\partial_t^{(1)}\delta T + \mathbf{u}\cdot\nabla\delta T + \frac{2}{Dnk_B}(\nabla^{(1)}\cdot\mathbf{q}^{(0)} + P_{ij}^{(0)}\nabla_j\delta u_i)$$

$$+\frac{2}{Dnk_B}(\nabla^{(0)}\cdot\mathbf{q}^{(1)} + P_{ij}^{(1)}\nabla_j u_{0,i}) = -\zeta^{(1)}T. \tag{32}$$

III. APPLICATION TO UNIFORM SHEAR FLOW OF GRANULAR FLUIDS

Uniform shear flow (USF) is a macroscopic state that is characterized by a constant density, a uniform temperature, and a simple shear with the local velocity field given by

$$u_i = a_{ij} r_j, \quad a_{ij} = a\delta_{ix}\delta_{jy}, \tag{33}$$

where a is the *constant* shear rate. If one assumes that the pressure tensor, heat flux vector, and cooling rate are also spatially uniform, the reference-state balance equations, Eqs. (18), become

$$\partial_t^{(0)} n_0 = 0,$$

$$\partial_t^{(0)} u_{0,i} + a u_{0,y}\delta_{ix} = 0,$$

$$\partial_t^{(0)} T_0 + \frac{2}{Dn_0 k_B} a P_{xy}^{(00)} = -\zeta^{(00)} T_0. \tag{34}$$

The question of whether or not these assumptions of spatial homogeneity are true depends on the detailed form of the collision operator: in Ref. [23] it is shown that only for the linear velocity profile, Eq. (33), this assumption is easily verified for the Enskog kinetic theory (and hence for simpler approximations to it such as the Boltzmann and BGK theories). This linear velocity profile is generated by Lee-Edwards boundary conditions [24], which are simply periodic boundary conditions in the local Lagrangian frame. For elastic gases, $\zeta^{(00)} = 0$ and the temperature grows in time due to viscous heating and so a steady state is not possible unless an external (artificial) thermostat is introduced [17]. However, for inelastic gases, the temperature changes in time due to the competition between two (opposite) mechanisms: on the one hand, viscous (shear) heating and on the other hand, energy dissipation in collisions. A steady state occurs when both mechanisms cancel each other at which point the balance equation for temperature becomes

$$\frac{2}{Dn_0 k_B} a P_{xy}^{(00)} = -\zeta^{(00)} T_0. \tag{35}$$

Note that both the pressure tensor and the cooling rate are in general functions of the two control parameters, the shear rate and the coefficient of restitution, and the hydrodynamic variables, the density and the temperature, so that this relation fixes any one of these in terms of the other three: for example, it could be viewed as giving the steady state temperature as a function of the other variables.

At a microscopic level, the one-body distribution for USF will clearly be inhomogeneous since Eqs. (5) and (33) imply that the steady state distribution must give

$$ay\hat{\mathbf{x}} = \frac{1}{n_0}\int d\mathbf{v}\mathbf{v} f_0(\mathbf{r},\mathbf{v}). \tag{36}$$

However, it can be shown, at least up to the Enskog theory [23], that for the Lee-Edwards boundary conditions, the state of USF possesses a modified translational invariance whereby the steady state distribution, when expressed in terms of the local rest-frame velocities $V_i = v_i - a_{ij} r_j$, does not have any explicit dependence on position. In terms of these variables, and assuming a steady state, the kinetic equation becomes

$$-aV_y \frac{\partial}{\partial V_x} f(\mathbf{V}) = J[\mathbf{V}|f,f]. \tag{37}$$

The solution of this equation has been considered in some detail for the BGK-type models [16–19], the Boltzmann equation [13], and the Enskog equation [25,26,23].

A. The model kinetic theory

Here, for simplicity, attention will be restricted to a particularly simple kinetic theory which nevertheless gives realistic results that can be compared to experiment. The kinetic theory used is the kinetic model of Brey, Dufty, and Santos [18], which is a relaxation type model where the operator $J[f,f]$ is approximated as

$$J[f,f] \to -\nu^*(\alpha)\nu(\psi)(f-\phi_0) + \frac{1}{2}\zeta^*(\alpha)\nu(\psi)\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{C}f). \tag{38}$$

The right-hand side involves the peculiar velocity $\mathbf{C} = \mathbf{v} - \mathbf{u} = \mathbf{V} - \delta\mathbf{u}$ and the local equilibrium distribution, Eq. (1). The parameters in this relaxation approximation are taken so as to give agreement with the results from the Boltzmann theory of the homogeneous cooling state as discussed in Ref. [18]. Defining the collision rate for elastic hard spheres in the Boltzmann approximation as

$$\nu(\psi) = \frac{8\pi^{(D-2)/2}}{(D+2)\Gamma(D/2)} n\sigma^D \sqrt{\frac{\pi k_B T}{m\sigma^2}}, \tag{39}$$

the correction for the effect of the inelasticity is chosen to reproduce the Navier-Stokes shear viscosity coefficient of an inelastic gas of hard spheres in the Boltzmann approximation [27,12] giving

$$\nu^*(\alpha) = \frac{1}{4D}(1+\alpha)[(D-1)\alpha + D + 1]. \tag{40}$$

The second term in Eq. (38) accounts for the collisional cooling and the coefficient is chosen so as to give the same cooling rate for the homogeneous cooling state as the Boltzmann kinetic theory [27,12],

$$\zeta^*(\alpha) = \frac{D+2}{4D}(1-\alpha^2). \tag{41}$$

In this case, the expressions for the pressure tensor, heat-flux vector, and cooling rate take particularly simple forms typical of the Boltzmann description [4]

$$P_{ij} = m\int d\mathbf{C} C_i C_j f(\mathbf{r},\mathbf{C},t),$$

$$q_i = \frac{1}{2}m\int d\mathbf{C} C_i C^2 f(\mathbf{r},\mathbf{C},t), \tag{42}$$

while the cooling rate can be calculated directly from Eqs. (38) and (8) with the result $\zeta(\psi) = \nu(\psi)\zeta^*(\alpha)$.

B. The steady state

Before proceeding with the Chapman-Enskog solution of the kinetic equation, it is useful to describe the steady state for which the distribution satisfies Eq. (37) which now becomes

$$-aV_y\frac{\partial}{\partial V_x}f(\mathbf{V}) = -\nu^*(\alpha)\nu(\psi_0)(f-\phi_0) + \frac{1}{2}\zeta^*(\alpha)\nu(\psi_0)\frac{\partial}{\partial\mathbf{V}}\cdot(\mathbf{V}f). \tag{43}$$

The balance equations reduce to

$$2aP^{ss}_{xy} = -\zeta^*(\alpha)\nu(\psi)Dn_0k_BT_0. \tag{44}$$

An equation for the pressure tensor is obtained by multiplying Eq. (43) through by mV_iV_j and integrating giving

$$aP^{ss}_{iy}\delta_{jx} + aP^{ss}_{jy}\delta_{ix} = -\nu^*(\alpha)\nu(\psi_0)(P^{ss}_{ij} - n_0k_BT_0\delta_{ij}) - \zeta^*(\alpha)\nu(\psi_0)P^{ss}_{ij}.$$

This set of algebraic equations is easily solved giving the only nonzero components of the pressure tensor as

$$P^{ss}_{ii} = \frac{\nu^*(\alpha) + \delta_{ix}D\zeta^*(\alpha)}{\nu^*(\alpha) + \zeta^*(\alpha)}n_0k_BT_0,$$

$$P^{ss}_{xy} = -\frac{a^*_{ss}}{\nu^*(\alpha) + \zeta^*(\alpha)}P_{yy}, \tag{45}$$

where $a^*_{ss}=a_{ss}/\nu(\psi_0)$ satisfies the steady state condition, Eq. (44)

$$\frac{a^{*2}_{ss}\nu^*(\alpha)}{[\nu^*(\alpha) + \zeta^*(\alpha)]^2} = \frac{D}{2}\zeta^*(\alpha). \tag{46}$$

For fixed control parameters, α and a, this is a relation constraining the state variables n_0 and T_0. The steady state distribution can be given explicitly, see, e.g., [28].

C. Zeroth order Chapman-Enskog

Since the only spatially varying reference field is the velocity and since it is linear in the spatial coordinate, the zeroth order kinetic equation, Eq. (23), becomes

$$\partial_t^{(0)}f^{(0)} + \mathbf{v}\cdot(\nabla^{(0)}u_{0i})\frac{\partial}{\partial u_{0i}}f^{(0)} = -\nu(\psi)(f^{(0)} - \phi_0) + \frac{1}{2}\zeta^*(\alpha)\nu(\psi)\frac{\partial}{\partial\mathbf{v}}\cdot(\mathbf{C}f^{(0)}) \tag{47}$$

or, writing the derivative on the left in terms of the peculiar velocity,

$$\partial_t^{(0)}f^{(0)} + v_y\partial_y^0f^{(0)} - av_y\frac{\partial}{\partial C_x}f^{(0)} = -\nu(\psi)(f - \phi_0) + \frac{1}{2}\zeta^*(\alpha)\nu(\psi)\frac{\partial}{\partial\mathbf{v}}\cdot(\mathbf{C}f^{(0)}). \tag{48}$$

Here, the second term on the left accounts for any explicit dependence of the distribution on the coordinate y, aside from the implicit dependence coming from $\mathbf{C}$. Since it is a zero order derivative, it does not act on the deviations $\delta\psi$. In terms of the peculiar velocity, this becomes

$$\partial_t^{(0)}f^{(0)} + (C_y + \delta u_y)\partial_y^0f^{(0)} - aC_y\frac{\partial}{\partial C_x}f^{(0)} - a\delta u_y\frac{\partial}{\partial C_x}f^{(0)} = -\nu(\psi)(f - \phi_0) + \frac{1}{2}\zeta^*(\alpha)\nu(\psi)\frac{\partial}{\partial\mathbf{C}}\cdot(\mathbf{C}f^{(0)}). \tag{49}$$

The first term on the left is evaluated using Eq. (26) and the zeroth order balance equations

$$\partial_t^{(0)}n = 0,$$

$$\partial_t^{(0)}u_i + a\delta u_y\delta_{ix} = 0,$$

$$\partial_t^{(0)}T + \frac{2}{Dnk_B}aP^{(0)}_{xy} = -\zeta^*(\alpha)\nu(\psi)T, \tag{50}$$

and the assumption of normality

$$\partial_t^{(0)}f^{(0)} = (\partial_t^{(0)}\delta n)\left(\frac{\partial}{\partial\delta n}f^{(0)}\right) + (\partial_t^{(0)}\delta T)\left(\frac{\partial}{\partial\delta T}f^{(0)}\right) + (\partial_t^{(0)}\delta u_i)\left(\frac{\partial}{\partial\delta u_i}f^{(0)}\right)$$

to give

$$\left(-\zeta^*(\alpha)\nu(\psi)T - \frac{2}{Dnk_B}aP^{(0)}_{xy}\right)\frac{\partial}{\partial T}f^{(0)} - aC_y\frac{\partial}{\partial C_x}f^{(0)} - a\delta u_y\left(\frac{\partial}{\partial C_x}f^{(0)} + \frac{\partial}{\partial\delta u_x}f^{(0)}\right) = -\nu^*(\alpha)\nu(\psi)(f^{(0)} - \phi_0) + \frac{1}{2}\zeta^*(\alpha)\nu(\psi)\frac{\partial}{\partial\mathbf{C}}\cdot(\mathbf{C}f^{(0)}), \tag{51}$$

where the temperature derivative is understood to be evaluated at constant density. Here, the second term on the left in Eq. (49) has been dropped as neither Eq. (49) nor the balance equations contain explicit reference to the velocity field u_0, and so no explicit dependence on the coordinate y, thus justifying the assumption that such dependence does not occur in $f^{(0)}$. One can also assume that $f^{(0)}$ depends on δu_i only through the peculiar velocity, since in that case the term proportional to δu_y vanishes as well and there is no other explicit dependence on δu_y.

Equation (51) is closed once the pressure tensor is specified. Since the primary goal here is to develop the transport equations for deviations from the reference state, attention will be focused on the determination of the pressure tensor and the heat flux vector. It is a feature of the simple kinetic model used here that these can be calculated without determining the explicit form of the distribution.

1. The zeroth order pressure tensor

An equation for the pressure tensor can be obtained by multiplying this equation through by mC_iC_j and integrating over velocities. Using the definition given in Eq. (42),

$$\left(-\zeta^*(\alpha)\nu(\psi)T-\frac{2}{Dnk_B}aP_{xy}^{(0)}\right)\frac{\partial}{\partial T}P_{ij}^{(0)}+a\delta_{ix}P_{jy}^{(0)}+a\delta_{jx}P_{iy}^{(0)}$$
$$=-\nu^*(\alpha)\nu(\psi)(P_{ij}^{(0)}-\delta_{ij}nk_BT)-\zeta^*(\alpha)\nu(\psi)P_{ij}^{(0)}, \qquad (52)$$

and of course there is the constraint that by definition $\mathrm{Tr}(\mathsf{P})=Dnk_BT$. It is interesting to observe that Eqs. (50)–(52) are identical with their steady state counterparts when the steady state condition, $\zeta^{(0)}T=2/Dnk_BaP_{xy}^{(0)}$, is fulfilled. However, here the solution of these equations is needed for arbitrary values of δT, δn, and $\delta\mathbf{u}$. Another point of interest is that these equations are local in the deviations $\delta\psi$ so that they are exactly the same equations as describe spatially homogeneous deviations from the reference state. As mentioned above, this is the meaning of the zeroth order solution to the Chapman-Enskog expansion: it is the exact solution to the problem of uniform deviations from the reference state. It is this exact solution which is "localized" to give the zeroth order Chapman-Enskog approximation and not the reference distribution, f_0, except in the rare cases, such as equilibrium, when they coincide.

To complete the specification of the distribution, Eqs. (51) and (52) must be supplemented by boundary conditions. The relevant dimensionless quantity characterizing the strength of the nonequilibrium state is the dimensionless shear rate defined as

$$a^*\equiv a/\nu=a\frac{(D+2)\Gamma(D/2)}{8\pi^{(D-1)/2}n\sigma^D}\sqrt{\frac{m\sigma^2}{k_BT}}. \qquad (53)$$

It is clear that for a uniform system, the dimensionless shear rate becomes smaller as the temperature rises so that we expect that in the limit of infinite temperature, the system will behave as an inelastic gas without any shear, i.e., in the homogeneous cooling state, giving the boundary condition

$$\lim_{T\to\infty}\frac{1}{nk_BT}P_{ij}=\delta_{ij}, \qquad (54)$$

and in this limit, the distribution must go to the homogeneous cooling state distribution. These boundary conditions can be implemented equivalently by rewriting Eqs. (60) in terms of the inverse temperature, or more physically the variable a^*, and the dimensionless pressure tensor $P_{ij}^{(*)}=1/nk_BTP_{ij}^{(0)}$ giving

$$\left(\frac{1}{2}\zeta^*(\alpha)+\frac{1}{D}a^*P_{xy}^{(*)}\right)a^*\frac{\partial}{\partial a^*}P_{ij}^{(*)}$$
$$=\frac{2}{D}a^*P_{xy}^{(*)}P_{ij}^{(*)}-a^*\delta_{ix}P_{jy}^{(*)}-a^*\delta_{jx}P_{iy}^{(*)}-\nu^*(\alpha)(P_{ij}^{(*)}-\delta_{ij}) \qquad (55)$$

and writing $f^{(0)}(\mathbf{C};\psi)=n(m/2\pi k_BT)^{D/2}g(\sqrt{m/k_BT}\mathbf{C};a^*)$

$$\left(\zeta^*(\alpha)+\frac{2}{D}a^*P_{xy}^{(*)}\right)a^*\frac{\partial}{\partial a^*}g+\frac{1}{D}a^*P_{xy}^{(*)}C_i\frac{\partial}{\partial C_i}g+a^*P_{xy}^{(*)}g$$
$$-a^*C_y\frac{\partial}{\partial C_x}g=-\nu^*(\alpha)[g-\exp(-mC^2/k_BT)], \qquad (56)$$

with boundary condition $\lim_{a^*\to 0}P_{ij}^{(*)}=\delta_{ij}$ and $\lim_{a^*\to 0}g=\exp(-mC^2/k_BT)$. For practical calculations, it is more convenient to introduce a fictitious time variable, s, and to express these equations as

$$\frac{da^*}{ds}=\frac{1}{2}a^*\zeta^*(\alpha)+\frac{1}{D}a^{*2}P_{xy}^{(*)},$$

$$\frac{\partial}{\partial s}P_{ij}^{(0)}=\frac{2}{D}a^*P_{xy}^{(*)}P_{ij}^{(*)}-a^*\delta_{ix}P_{jy}^{(*)}-a^*\delta_{jx}P_{iy}^{(*)}$$
$$-\nu^*(\alpha)(P_{ij}^{(*)}-\delta_{ij}) \qquad (57)$$

where the boundary condition is then $P_{ij}^{(*)}(s=0)=\delta_{ij}$, and $a^*(s=0)=0$. The distribution then satisfies

$$\frac{\partial}{\partial s}g=-\frac{1}{D}a^*P_{xy}^{(*)}C_i\frac{\partial}{\partial C_i}g-a^*P_{xy}^{(*)}g+a^*C_y\frac{\partial}{\partial C_x}g$$
$$-\nu^*(\alpha)[g-\exp(-mC^2/k_BT)] \qquad (58)$$

with $\lim_{s\to 0}g=\exp(-mC^2/k_BT)$. These are to be solved simultaneously to give $P_{ij}^{(*)}(s)$, $a^*(s)$, and $f^{(0)}(s)$ from which the desired curves $P_{ij}^{(*)}(a^*)$ and $f^{(0)}(a^*)$ are obtained.

Physically, if the gas starts at a very high temperature, it would be expected to cool until it reached the steady state. It is easy to see that the right-hand sides of Eqs. (57) do in fact vanish in the steady state so that the steady state represents a critical point of this system of differential equations [29]. In order to fully specify the curve $P_{ij}(T)$ and the distribution $f^{(0)}$ it is necessary to integrate as well from a temperature below the steady state temperature. Clearly, in the case of *zero* temperature, one expects that the pressure tensor goes to zero since this corresponds to the physical situation in which the atoms stream at exactly the velocities predicted by their positions and the macroscopic flow field. (Note that if the atoms have finite size, this could still lead to collisions. However, the BGK kinetic theory used here is properly understood as an approximation to the Boltzmann theory appropriate for a low density gas in which the finite size of the grains is of no importance.) Thus the expectation is that the zero-temperature limit will give

$$\lim_{T\to 0}P_{ij}^{(0)}=0. \qquad (59)$$

Then, in terms of fictitious time parameters, one has

$$\frac{dT}{ds}=-\zeta^*(\alpha)\nu(\psi)T-\frac{2}{D}aTP_{xy}^{(*)},$$

$$\frac{\partial}{\partial s}P_{ij}^{(*)}=a\frac{2}{D}P_{xy}^{(*)}P_{ij}^{(*)}-a\delta_{ix}P_{jy}^{(*)}-a\delta_{jx}P_{iy}^{(*)}$$
$$-\nu^*(\alpha)\nu(\psi)(P_{ij}^{(*)}-\delta_{ij}) \qquad (60)$$

and for the distribution

$$\frac{\partial}{\partial s}f^{(0)} = aC_y\frac{\partial}{\partial C_x}f^{(0)} - \nu^*(\alpha)\nu(\psi)(f^{(0)} - \phi_0) + \frac{1}{2}\zeta^*(\alpha)\nu(\psi)\frac{\partial}{\partial \mathbf{C}}\cdot(\mathbf{C}f^{(0)}). \tag{61}$$

A final point is that the solution of these equations requires more than the boundary condition $P_{ij}^{(0)}(s=0)=0$ since evaluation of the right-hand side of Eq. (60) requires a statement about $P_{ij}^{(*)}(s=0)$ as well. A straightforward series solution of Eq. (52) in the vicinity of $T=0$ gives $P_{xy}^* \sim a^{*-1/3}$ and $P_{ii}^* \sim a^{*-2/3}$ so that the correct boundary condition here is $P_{ij}^{(*)}(s=0)=0$. The solution of these equations can then be performed as discussed in Ref. [14] with the boundary conditions given here.

It will also prove useful below to know the behavior of the pressure tensor near the steady state. This is obtained by making a series solution to Eq. (55) in the variable $(a^* - a_{ss}^*)$ where a_{ss}^* is the reduced shear in the steady state. Details are given in Appendix A and the result is that

$$P_{ij}^{(0)} = P_{ij}^{ss}\left(1 + A_{ij}^*(\alpha)\left(\frac{a^*}{a_{ss}^*} - 1\right) + \cdots\right), \tag{62}$$

with the coefficients

$$A_{xy}^*(\alpha) = -2\frac{\Delta(\alpha) + \zeta^*(\alpha)}{\zeta^*(\alpha)},$$

$$(1-\delta_{ix})A_{ii}^*(\alpha) = -2\left(\frac{\nu^*(\alpha) + \zeta^*(\alpha)}{\Delta(\alpha) + \nu^*(\alpha) + \frac{1}{2}\zeta^*(\alpha)}\right)(1-\delta_{ix}),$$

$$A_{xx}^*(\alpha) = -2D\frac{\left[\Delta(\alpha) + \frac{1}{D}\nu^*(\alpha) + \frac{1}{2}\zeta^*(\alpha)\right][\nu^*(\alpha) + \zeta^*(\alpha)]}{\left[\Delta(\alpha) + \nu^*(\alpha) + \frac{1}{2}\zeta^*(\alpha)\right][\nu^*(\alpha) + D\zeta^*(\alpha)]}, \tag{63}$$

where $\Delta(\alpha)$ is the real root of

$$4\Delta^3 + 8[\nu^*(\alpha) + \zeta^*(\alpha)]\Delta^2 + [4\nu^{*2}(\alpha) + 14\nu^*(\alpha)\zeta^*(\alpha) + 7\zeta^{*2}(\alpha)]\Delta + \zeta^*(\alpha)[2\nu^{*2}(\alpha) - \nu^*(\alpha)\zeta^*(\alpha) - 2\zeta^{*2}(\alpha)] = 0. \tag{64}$$

2. Higher order moments: the zeroth order heat flux vector

Determination of the heat flux vector requires consideration of the full tensor of third order moments. Since fourth order moments will also be needed later, it is easiest to consider the equations for the general Nth order moment. Defining the s order contribution to the Nth moment as

$$M_{i_1\cdots i_N}^{(s)}(\mathbf{r},t) = m\int d\mathbf{v}C_{i_1}\cdots C_{i_N}f^{(s)}(\mathbf{r},\mathbf{C},t), \tag{65}$$

the goal here will be to give expressions for the zeroth order contribution, $M_{i_1\cdots i_N}^{(0)}(\mathbf{r},t)$. To simplify the equations, a more compact notation will be used for the indices whereby a collection of numbered indices, such as $i_1\cdots i_N$, will be written more compactly as I_N so that capital letters denote collections of indices and the subscript on the capital indicates the number of indices in the collection. Some examples of this are

$$M_{I_N}^{(0)} = M_{i_1\cdots i_N}^{(0)},$$

$$M_{I_2}^{(0)} = M_{i_1 i_2}^{(0)},$$

$$M_{I_2 y}^{(0)} = M_{i_1 i_2 y}^{(0)}. \tag{66}$$

In terms of the general moments, the heat flux vector is

$$q_i^{(0)}(\mathbf{r},t) = \frac{1}{2}\sum_j M_{ijj}^{(0)}(\mathbf{r},t) = \frac{1}{2}M_{ijj}^{(0)}(\mathbf{r},t), \tag{67}$$

where the second equality introduces the Einstein summation convention whereby repeated indices are summed. The pressure tensor is just the second moment $P_{ij}^{(0)}=M_{ij}^{(0)}$. The local equilibrium moments are easily shown to be zero for odd N while the result for even N is

$$M_{I_N}^{(le)} = mn\left(\frac{2k_BT}{m}\right)^{N/2}2^{N/2}\frac{\Gamma\left(\frac{N+1}{2}\right)\Gamma\left(\frac{N+2}{2}\right)}{\sqrt{\pi}\Gamma(N)} \times \mathcal{P}_{I_N}\delta_{i_1 i_2}\delta_{i_3 i_4}\cdots\delta_{i_{N-1}i_N}, \tag{68}$$

where the operator $\mathcal{P}_{ijk\cdots}$ indicates the sum over distinct permutations of the indices $ijk\cdots$ and has no effect on any other indices. (For example, $\mathcal{P}_{I_4}\delta_{i_1 i_2}\delta_{i_3 i_4} = \delta_{i_1 i_2}\delta_{i_3 i_4} + \delta_{i_1 i_3}\delta_{i_2 i_4} + \delta_{i_1 i_4}\delta_{i_2 i_3}$). An equation for the general Nth order moment can be obtained from Eq. (47) with the result

$$\left(-\zeta^*(\alpha) - \frac{2}{D}a^*P_{xy}^{(*)}\right)T\frac{\partial}{\partial T}M_{I_N}^{(0)} + \left(\nu^*(\alpha) + \frac{N}{2}\zeta^*(\alpha)\right)M_{I_N}^{(0)} + a^*\mathcal{P}_{I_N}\delta_{xi_N}M_{I_{N-1}y}^{(0)} = \nu^*(\alpha)M_{I_N}^{(le)}. \tag{69}$$

Writing $M_{I_N}^{(0)}=mn(2k_BT/m)^{N/2}M_{I_N}^*$ gives

$$-\left(\zeta^*(\alpha) + \frac{2}{D}a^*P_{xy}^{(*)}\right)T\frac{\partial}{\partial T}M_{I_N}^* + \left(\nu^*(\alpha) - \frac{N}{D}a^*P_{xy}^{(*)}\right)M_{I_N}^* + a^*\mathcal{P}_{I_N}\delta_{xi_N}M_{I_{N-1}y}^* = \nu^*(\alpha)M_{I_N}^{(le*)}. \tag{70}$$

Notice that the moments are completely decoupled order by order in N. Since the source on the right vanishes for odd N it is natural to assume that $M_{I_N}^*=0$ for odd N. This is certainly true for temperatures above the steady state temperature since the appropriate boundary condition in this case, based on the discussion above, is that $\lim_{T\to\infty}M_{I_N}^* = M_{I_N}^{(le^*)}=0$. In the opposite limit, $T\to 0$, as mentioned above, one has that $P_{xy}^* \sim a^{*-1/3} \sim T^{1/6}$ and there are two cases to consider depending on whether or not the third term on the left contributes. If it does, i.e., if one or more indices is equal to x, then a series solution near $T=0$ gives $M_{I_N}^* \sim a^{*-1} \sim T^{1/2}$ while if no index is equal to x then $M_{I_N}^* \sim a^{*-2/3} \sim T^{1/3}$ giving in both cases the boundary condition $\lim_{T\to 0}M^{*}_{I_N}=0$. In particular, this shows that the odd moments vanish for all temperatures. From this, it immediately follows that

$$q_i^{(0)}(\mathbf{r},t)=0. \tag{71}$$

D. First order Chapman-Enskog: General formalism

The equation for the first order distribution, Eq. (31), becomes

$$\begin{aligned}\partial_t^{(0)}f^{(1)}+av_y\frac{\partial}{\partial u_{0x}}f^{(1)}&=-\nu(\psi)f^{(1)}+\frac{1}{2}\zeta^{*}(\alpha)\nu(\psi)\frac{\partial}{\partial\mathbf{v}}\cdot(\mathbf{C}f^{(1)})\\&\quad-(\partial_t^{(1)}f^{(0)}+\mathbf{v}\cdot\nabla_1 f^{(0)}),\end{aligned}\tag{72}$$

and the operator $\partial_t^{(1)}$ is defined via the corresponding balance equations which are now

$$\begin{gathered}\partial_t^{(1)}\delta n+\mathbf{u}\cdot\nabla\delta n+n\nabla\cdot\delta\mathbf{u}=0,\\ \partial_t^{(1)}\delta u_i+\mathbf{u}\cdot\nabla\delta u_i+(mn)^{-1}\partial_j^{(1)}P_{ij}^{(0)}+(mn)^{-1}\partial_y^{(0)}P_{iy}^{(1)}=0,\\ \partial_t^{(1)}\delta T+\mathbf{u}\cdot\nabla\delta T+\frac{2}{Dnk_B}(P_{ij}^{(0)}\nabla_j\delta u_i+\nabla^{(0)}\cdot\mathbf{q}^{(1)}+aP_{xy}^{(1)})=0.\end{gathered}\tag{73}$$

Writing the kinetic equation in the form

$$\begin{aligned}&\partial_t^{(0)}f^{(1)}+a\frac{\partial}{\partial u_{0x}}v_yf^{(1)}+\nu^{*}(\alpha)\nu(\psi)f^{(1)}\\&\quad-\frac{1}{2}\zeta^{*}(\alpha)\nu(\psi)\frac{\partial}{\partial\mathbf{v}}\cdot(\mathbf{C}f^{(1)})=-(\partial_t^{(1)}n+u_l\partial_l^1n)\frac{\partial}{\partial n}f^{(0)}\\&\quad-(\partial_t^{(1)}T+u_l\partial_l^1T)\frac{\partial}{\partial T}f^{(0)}-(\partial_t^{(1)}\delta u_j+u_l\partial_l^1\delta u_j)\frac{\partial}{\partial\delta u_j}f^{(0)}\\&\quad-(\partial_l^1u_l)f^{(0)}-\partial_l^1C_lf^{(0)},\end{aligned}\tag{74}$$

equations for the Nth moment can be obtained by multiplying through by $C_{i_1}\cdots C_{i_N}$ and integrating over velocity. The first two terms on the left contribute

$$\begin{aligned}&\int C_{i_1}\cdots C_{i_N}\left(\partial_t^{(0)}f^{(1)}+a\frac{\partial}{\partial u_{0x}}v_yf^{(1)}\right)d\mathbf{v}\\&\quad=\partial_t^{(0)}M_{I_N}^{(1)}+\mathcal{P}_{I_N}(\partial_t^{(0)}\delta u_{i_N})M_{I_{N-1}}^{(1)}+a\frac{\partial}{\partial u_{0x}}(M_{I_Ny}^{(1)}+\delta u_yM_{I_N}^{(1)})\\&\qquad+a\mathcal{P}_{I_N}\delta_{xi_N}(M_{I_{N-1}y}^{(1)}+\delta u_yM_{I_{N-1}}^{(1)})\\&\quad=\partial_t^{(0)}M_{I_N}^{(1)}+a\frac{\partial}{\partial u_{0x}}(M_{I_Ny}^{(1)}+\delta u_yM_{I_N}^{(1)})+a\mathcal{P}_{I_N}\delta_{xi_N}M_{I_{N-1}y}^{(1)},\end{aligned}\tag{75}$$

where the last line follows from using the zeroth order balance equation $\partial_t^{(0)}\delta u_{i_N}=-a\delta_{i_Nx}\delta u_y$. The evaluation of the right-hand side is straightforward with the only difficult term being

$$\int C_{i_1}\cdots C_{i_N}\left(\frac{\partial}{\partial\delta u_j}f^{(0)}\right)d\mathbf{v}=\frac{\partial}{\partial\delta u_j}M_{I_N}^{(0)}+\mathcal{P}_{I_N}\delta_{i_Nj}M_{I_{N-1}}^{(0)},\tag{76}$$

and from Eq. (70) it is clear that $M_{I_N}^{(0)}$ is independent of δu_j so that the first term on the right vanishes. Thus

$$\begin{aligned}&\partial_t^{(0)}M_{I_N}^{(1)}+a\frac{\partial}{\partial u_{0x}}(M_{I_Ny}^{(1)}+\delta u_yM_{I_N}^{(1)})+a\mathcal{P}_{I_N}\delta_{xi_N}M_{I_{N-1}y}^{(1)}\\&\quad+\left(\nu^{*}(\alpha)+\frac{N}{2}\zeta^{*}(\alpha)\right)\nu(\psi)M_{I_N}^{(1)}\\&\quad=-(\partial_t^{(1)}n+u_l\partial_l^1n)\frac{\partial}{\partial n}M_{I_N}^{(0)}-(\partial_t^{(1)}T+u_l\partial_l^1T)\frac{\partial}{\partial T}M_{I_N}^{(0)}\\&\qquad-(\partial_t^{(1)}\delta u_j+u_l\partial_l^1\delta u_j)\mathcal{P}_{I_N}\delta_{i_Nj}M_{I_{N-1}}^{(0)}-(\partial_l^1u_l)M_{I_N}^{(0)}\\&\qquad-\mathcal{P}_{I_N}(\partial_l^1u_{i_N})M_{I_{N-1}l}^{(0)}-\partial_l^1M_{I_Nl}^{(0)}.\end{aligned}\tag{77}$$

Superficially, it appears that the right-hand side depends explicitly on the reference field, since $u_l=u_{0,l}+\delta u_l$, which would in turn generate an explicit dependence of the moments on the y coordinate. However, when the balance equations are used to eliminate $\partial_t^{(1)}$ this becomes

$$\begin{aligned}&\partial_t^{(0)}M_{I_N}^{(1)}+a\frac{\partial}{\partial u_{0x}}(M_{I_Ny}^{(1)}+\delta u_yM_{I_N}^{(1)})+a\mathcal{P}_{I_N}\delta_{xi_N}M_{I_{N-1}y}^{(1)}\\&\quad+\left(\nu^{*}(\alpha)+\frac{N}{2}\zeta^{*}(\alpha)\right)\nu(\psi)M_{I_N}^{(1)}\\&\quad=(\partial_l^{(1)}\delta u_l)n\frac{\partial}{\partial n}M_{I_N}^{(0)}+\frac{2}{Dnk_B}(M_{lk}^{(0)}\partial_l^{(1)}\delta u_k+aM_{xy}^{(1)})\frac{\partial}{\partial T}M_{I_N}^{(0)}\\&\qquad+\frac{1}{mn}\mathcal{P}_{I_N}(\partial_l^{(1)}P_{li_N}^{(0)}+\partial_y^{(0)}P_{yi_N}^{(1)})M_{I_{N-1}}^{(0)}-(\partial_l^1\delta u_l)M_{I_N}^{(0)}\\&\qquad-\mathcal{P}_{I_N}(\partial_l^1\delta u_{i_N})M_{I_{N-1}l}^{(0)}-\partial_l^1M_{I_Nl}^{(0)}.\end{aligned}\tag{78}$$

Then, assuming that the first order moments are independent of the reference field, $\mathbf{u}_0$, gives

$$\begin{aligned}&\partial_t^{(0)}M_{I_N}^{(1)}+a\mathcal{P}_{I_N}\delta_{xi_N}M_{I_{N-1}y}^{(1)}+\left(\nu^{*}(\alpha)+\frac{N}{2}\zeta^{*}(\alpha)\right)\nu(\psi)M_{I_N}^{(1)}\\&\quad-\left(\frac{2a}{Dnk_B}\frac{\partial}{\partial T}M_{I_N}^{(0)}\right)M_{xy}^{(1)}=\left[\delta_{ab}\left(n\frac{\partial}{\partial n}M_{I_N}^{(0)}-M_{I_N}^{(0)}\right)\right.\\&\qquad\left.+\frac{2}{Dnk_B}P_{ab}^{(0)}\frac{\partial}{\partial T}M_{I_N}^{(0)}-\mathcal{P}_{I_N}\delta_{bi_N}M_{I_{N-1}a}^{(0)}\right](\partial_a^{(1)}\delta u_b)\\&\qquad+\left[\frac{1}{mn}\mathcal{P}_{I_N}\left(\frac{\partial}{\partial\delta n}P_{li_N}^{(0)}\right)M_{I_{N-1}}^{(0)}-\frac{\partial}{\partial\delta n}M_{I_Nl}^{(0)}\right](\partial_l^{(1)}\delta n)\\&\qquad+\left[\frac{1}{mn}\mathcal{P}_{I_N}\left(\frac{\partial}{\partial\delta T}P_{li_N}^{(0)}\right)M_{I_{N-1}}^{(0)}-\frac{\partial}{\partial\delta T}M_{I_Nl}^{(0)}\right](\partial_l^{(1)}\delta T),\end{aligned}\tag{79}$$

which is consistent since no factors of $\mathbf{u}_0$ appear and since the zeroth order moments are known to be independent of the reference velocity field.

The moment equations are linear in gradients in the deviation fields, so generalized transport coefficients are introduced via the definition

$$M_{I_N}^{(1)} = -\lambda_{I_N ab}\frac{\partial \delta\psi_a}{\partial r_b} = -\mu_{I_N a}\frac{\partial \delta n}{\partial r_a} - \kappa_{I_N a}\frac{\partial \delta T}{\partial r_a} - \eta_{I_N ab}\frac{\partial \delta u_a}{\partial r_b}, \tag{80}$$

where the transport coefficients for different values of N have the same name but can always be distinguished by the number of indices they carry. The zeroth order time derivative is evaluated using

$$\begin{aligned}
\partial_t^{(0)}\lambda_{I_N ab}\frac{\partial \delta\psi_a}{\partial r_b} &= (\partial_t^{(0)}\lambda_{I_N ab})\frac{\partial \delta\psi_a}{\partial r_b} + \lambda_{I_N ab}\partial_t^{(0)}\frac{\partial \delta\psi_a}{\partial r_b}\\
&= \left[(\partial_t^{(0)}T)\frac{\partial \lambda_{I_N ab}}{\partial T} + (\partial_t^{(0)}\delta u_j)\frac{\partial \lambda_{I_N ab}}{\partial \delta u_j}\right]\frac{\partial \delta\psi_a}{\partial r_b}\\
&\quad + \lambda_{I_N ab}\frac{\partial}{\partial r_b}(\partial_t^{(0)}\delta\psi_a)\\
&= (\partial_t^{(0)}T)\frac{\partial \lambda_{I_N ab}}{\partial T}\frac{\partial \delta\psi_a}{\partial r_b} + \lambda_{I_N ab}\frac{\partial \delta\psi_c}{\partial r_b}\frac{\partial(\partial_t^{(0)}\delta\psi_a)}{\partial \delta\psi_c}\\
&= \left[(\partial_t^{(0)}T)\frac{\partial \lambda_{I_N ab}}{\partial T} + \lambda_{I_N cb}\frac{\partial(\partial_t^{(0)}\delta\psi_c)}{\partial \delta\psi_a}\right]\frac{\partial \delta\psi_a}{\partial r_b},
\end{aligned} \tag{81}$$

where the third line follows from (a) the fact that the transport coefficients will have no explicit dependence on the velocity field, as may be verified from the structure of Eq. (79), and (b) the fact that the gradient here is a first order gradient ∇_1 so that it only contributes via gradients of the deviations of the fields thus giving the last term on the right. Since the fields are independent variables, the coefficients of the various terms $\partial\delta\psi_b/\partial r_a$ must vanish independently. For the coefficients of the velocity gradients, this gives

$$\begin{aligned}
&(\partial_t^{(0)}T)\frac{\partial}{\partial T}\eta_{I_N ab} + \eta_{I_N cb}\frac{\partial(\partial_t^{(0)}\delta u_c)}{\partial \delta u_a} + a\mathcal{P}_{I_N}\delta_{xi_N}\eta_{I_{N-1}yab}\\
&\quad + \left(\nu^*(\alpha) + \frac{N}{2}\zeta^*(\alpha)\right)\nu(\psi)\eta_{I_N ab} - \left(\frac{2a}{Dnk_B}\frac{\partial}{\partial T}M_{I_N}^{(0)}\right)\eta_{xyab}\\
&\quad = -\delta_{ab}\left(n\frac{\partial}{\partial n}M_{I_N}^{(0)} - M_{I_N}^{(0)}\right) - \frac{2}{Dnk_B}M_{ab}^{(0)}\frac{\partial}{\partial T}M_{I_N}^{(0)}\\
&\qquad + \mathcal{P}_{I_N}\delta_{bi_N}M_{I_{N-1}a}^{(0)}.
\end{aligned} \tag{82}$$

The vanishing of the coefficients of the density gradients gives

$$\begin{aligned}
&(\partial_t^{(0)}T)\frac{\partial}{\partial T}\mu_{I_N a} + \kappa_{I_N a}\frac{\partial(\partial_t^{(0)}T)}{\partial n} + a\mathcal{P}_{I_N}\delta_{xi_N}\mu_{I_{N-1}ya}^{N}\\
&\quad + \left(\nu^*(\alpha) + \frac{N}{2}\zeta^*(\alpha)\right)\nu(\psi)\mu_{I_N a} - \left(\frac{2a}{Dnk_B}\frac{\partial}{\partial T}M_{I_N}^{(0)}\right)\mu_{xya}\\
&\quad = -\frac{1}{mn}\mathcal{P}_{I_N}\left(\frac{\partial}{\partial \delta n}P_{ai_N}^{(0)}\right)M_{I_{N-1}}^{(0)} + \frac{\partial}{\partial \delta n}M_{I_N a}^{(0)},
\end{aligned} \tag{83}$$

while the vanishing of the coefficient of the temperature gradient gives

$$\begin{aligned}
&(\partial_t^{(0)}T)\frac{\partial}{\partial T}\kappa_{I_N a} + \frac{\partial(\partial_t^{(0)}T)}{\partial T}\kappa_{I_N a} + a\mathcal{P}_{I_N}\delta_{xi_N}\kappa_{I_{N-1}ya}^{N}\\
&\quad + \left(\nu^*(\alpha) + \frac{N}{2}\zeta^*(\alpha)\right)\nu(\psi)\kappa_{I_N a} - \left(\frac{2a}{Dnk_B}\frac{\partial}{\partial T}M_{I_N}^{(0)}\right)\kappa_{xya}\\
&\quad = -\frac{1}{mn}\mathcal{P}_{I_N}\left(\frac{\partial}{\partial \delta T}P_{ai_N}^{(0)}\right)M_{I_{N-1}}^{(0)} + \frac{\partial}{\partial \delta T}M_{I_N a}^{(0)}.
\end{aligned} \tag{84}$$

Notice that for even moments, the source terms for the density and temperature transport coefficients all vanish (as they involve odd zeroth order moments) and it is easy to verify that the boundary conditions are consistent with $\mu_{I_N a} = \kappa_{I_N a} = 0$ and only the velocity gradients contribute. For odd values of N, the opposite is true and $\eta_{I_N ab} = 0$ while the others are in general nonzero.

E. Navier-Stokes transport

1. The first order pressure tensor

Specializing to the case $N=2$ gives the transport coefficients appearing in the pressure tensor

$$P_{I_N}^{(1)} = -\eta_{I_N ab}\frac{\partial \delta u_a}{\partial r_b} \tag{85}$$

where the generalized viscosity satisfies

$$\begin{aligned}
&(\partial_t^{(0)}T)\frac{\partial}{\partial T}\eta_{ijab} - a\eta_{ijxb}\delta_{ay} + a\delta_{xi}\eta_{jyab} + a\delta_{xj}\eta_{iyab}\\
&\quad + (\nu^*(\alpha) + \zeta^*(\alpha))\nu(\psi)\eta_{ijab} - \left(\frac{2a}{Dnk_B}\frac{\partial}{\partial T}P_{ij}^{(0)}\right)\eta_{xyab}\\
&\quad = -\delta_{ab}\left(n\frac{\partial}{\partial n}P_{ij}^{(0)} - P_{ij}^{(0)}\right) - \frac{2}{Dnk_B}P_{ab}^{(0)}\frac{\partial}{\partial T}P_{ij}^{(0)}\\
&\qquad + \delta_{bi}P_{ja}^{(0)} + \delta_{bj}P_{ia}^{(0)}.
\end{aligned} \tag{86}$$

2. First order third moments and the heat flux vector

For the third moments, the contribution of density gradients to the heat flux is well-known in the theory of granular fluids and the transport coefficient is here the solution of

$$\begin{aligned}
&(\partial_t^{(0)}T)\frac{\partial}{\partial T}\mu_{ijka} + \frac{\partial(\partial_t^{(0)}T)}{\partial n}\kappa_{ijka} + \left(\nu^*(\alpha) + \frac{3}{2}\zeta^*(\alpha)\right)\nu(\psi)\mu_{ijka}\\
&\quad + a\delta_{xk}\mu_{ijya} + a\delta_{xi}\mu_{kjya} + a\delta_{xj}\mu_{ikya}\\
&\quad = -\frac{1}{mn}\left(\frac{\partial}{\partial n}P_{ak}^{(0)}\right)P_{ij}^{(0)} - \frac{1}{mn}\left(\frac{\partial}{\partial n}P_{ai}^{(0)}\right)P_{kj}^{(0)}\\
&\qquad - \frac{1}{mn}\left(\frac{\partial}{\partial n}P_{aj}^{(0)}\right)P_{ik}^{(0)} + \frac{\partial}{\partial n}M_{ijka}^{(0)},
\end{aligned} \tag{87}$$

and the generalized thermal conductivity is determined from

$$\begin{aligned}(\partial_t^{(0)}T)\frac{\partial}{\partial T}\kappa_{ijka} &+ \frac{\partial(\partial_t^{(0)}T)}{\partial T}\kappa_{ijka} + \left(\nu^*(\alpha)+\frac{3}{2}\zeta^*(\alpha)\right)\nu(\psi)\kappa_{ijka}\\ &+ a\delta_{xk}\kappa_{ijya} + a\delta_{xi}\kappa_{kjya} + a\delta_{xj}\kappa_{ikya}\\ &= -\frac{1}{mn}\left(\frac{\partial}{\partial T}P^{(0)}_{ak}\right)P^{(0)}_{ij} - \frac{1}{mn}\left(\frac{\partial}{\partial T}P^{(0)}_{ai}\right)P^{(0)}_{kj}\\ &\quad -\frac{1}{mn}\left(\frac{\partial}{\partial T}P^{(0)}_{aj}\right)P^{(0)}_{ik} + \frac{\partial}{\partial T}M^{(0)}_{ijka}. \end{aligned} \tag{88}$$

Note that both of these require knowledge of the zeroth order fourth velocity moment $M^{(0)}_{ijka}$. The heat flux vector is

$$q_i^{(1)} = -\bar{\mu}_{ia}\frac{\partial \delta n}{\partial r_a} - \bar{\kappa}_{ia}\frac{\partial \delta T}{\partial r_a} \tag{89}$$

where

$$\begin{aligned}\bar{\mu}_{ia} &= \mu_{ijja},\\ \bar{\kappa}_{ia} &= \kappa_{ijja}.\end{aligned} \tag{90}$$

F. The second order transport equations

In this section, the results obtained so far are put together so as to give the Navier-Stokes equations for deviations from the steady state. The Navier-Stokes equations result from the sum of the balance equations. To first order, this takes the form

$$\begin{aligned}&\partial_t n + \mathbf{u}\cdot\nabla\delta n + n\nabla\cdot\delta\mathbf{u} = 0,\\ &\partial_t u_i + \mathbf{u}\cdot\nabla\delta u_i + a\delta_{ix}\delta u_y + (mn)^{-1}\partial_j^{(1)}P^{(0)}_{ij} = 0,\\ &\partial_t T + \mathbf{u}\cdot\nabla\delta T + \frac{2}{Dnk_B}\left(P^{(0)}_{ij}\nabla_j\delta u_i + aP^{(0)}_{xy} + aP^{(1)}_{xy}\right)\\ &\quad = -\zeta^*(\alpha)\nu(\psi)T,\end{aligned} \tag{91}$$

where $\partial_t = \partial_t^{(0)} + \partial_t^{(1)}$. By analogy with the analysis of an equilibrium system, these will be termed the Euler approximation. Summing to second order to get the Navier-Stokes approximation gives

$$\begin{aligned}&\partial_t n + \mathbf{u}\cdot\nabla\delta n + n\nabla\cdot\delta\mathbf{u} = 0,\\ &\partial_t u_i + \mathbf{u}\cdot\nabla\delta u_i + a\delta_{ix}\delta u_y + (mn)^{-1}\partial_j^{(1)}P^{(0)}_{ij} + (mn)^{-1}\partial_y^{(1)}P^{(1)}_{iy}\\ &\quad + (mn)^{-1}\partial_j^{(0)}P^{(2)}_{ij} = 0,\\ &\partial_t T + \mathbf{u}\cdot\nabla\delta T + \frac{2}{Dnk_B}\left(\nabla^{(1)}\cdot\mathbf{q}^{(1)} + \nabla^{(0)}\cdot\mathbf{q}^{(2)} + P^{(0)}_{ij}\nabla_j\delta u_i\right.\\ &\quad \left. + P^{(1)}_{ij}\nabla_j\delta u_i + aP^{(0)}_{xy} + aP^{(1)}_{xy} + aP^{(2)}_{xy}\right) = -\zeta^*(\alpha)\nu(\psi)T,\end{aligned} \tag{92}$$

where now $\partial_t = \partial_t^{(0)} + \partial_t^{(1)} + \partial_t^{(2)}$ but this expression is problematic. Based on the results so far, it seems reasonable to expect that $\partial_j^{(0)}P^{(2)}_{ij} = \nabla^{(0)}\cdot\mathbf{q}^{(2)} = 0$. However, to consistently write the equations to third order requires knowledge of $P^{(2)}_{xy}$ which is not available without extending the solution of the kinetic equation to third order. The reason this problem arises here, and not in the analysis about equilibrium, is that the shear rate, a, arises from a gradient of the reference field. In the usual analysis, such a term would be first order and $aP^{(2)}_{xy} = (\partial_i u_{0j})P^{(2)}_{ij}$ would be of third order and therefore neglected here. This is unfortunate and shows that this method of analysis does not completely supplant the need to go beyond the second order solution in order to study shear flow. However, this problem is not unique. In fact, in calculations of the transport coefficients for the homogeneous cooling state of a granular gas, a similar problem occurs in the calculation of the cooling rate: the true Navier-Stokes expression requires going to third order in the solution of the kinetic equation [30,12]. (This is because the source does not appear under a gradient, as can be seen in the equations above.) This suggests that the same type of approximation be accepted here, namely that the term $aP^{(2)}_{xy}$ is neglected, so that the total pressure tensor and heat flux vectors are

$$\begin{aligned}P_{ij} &= P^{(0)}_{ij} + P^{(1)}_{ij},\\ q_i &= q_i^{(0)} + q_i^{(1)}\end{aligned} \tag{93}$$

and the transport equations can be written as

$$\begin{aligned}&\partial_t n + \nabla\cdot(n\mathbf{u}) = 0,\\ &\partial_t u_i + \mathbf{u}\cdot\nabla u_i + (mn)^{-1}\partial_j P_{ij} = 0,\\ &\partial_t T + \mathbf{u}\cdot\nabla T + \frac{2}{Dnk_B}\left(\nabla\cdot\mathbf{q} + P_{ij}\nabla_j u_i\right) = -\zeta^*(\alpha)\nu(\psi)T,\end{aligned} \tag{94}$$

which is the expected form of the balance equations. The total fluxes are given terms of the generalized transport coefficients

$$\begin{aligned}P_{ij} &= P^{(0)}_{ij} - \eta_{ijab}\frac{\partial \delta u_a}{\partial r_b},\\ q_i &= -\mu_{ijja}\frac{\partial \delta n}{\partial r_a} - \kappa_{ijja}\frac{\partial \delta T}{\partial r_a}.\end{aligned} \tag{95}$$

G. Linearized second order transport

Some simplification occurs if attention is restricted to linear deviations from the steady state. In particular, since the dissipitive contributions to the fluxes are already of first order in the deviations from the steady state, see Eq. (95), the transport coefficients are only needed to lowest, i.e., zero, order in the deviations. The defining expressions for the transport coefficients therefore simplify since the factor $\partial_t^{(0)}T$ occuring in Eqs. (86) and (87) is at least of first order in the deviations from the steady state (since it vanishes in the steady state) thus implying that the temperature derivative can be neglected. The differential equations for the transport coefficients thus become coupled algebraic equations greatly simplifying the determination of the transport coefficients. Taking this into account, the total fluxes are

$$P_{ij}=P_{ij}^{(ss)}+\left(\frac{\partial P_{ij}^{(0)}}{\partial \delta n}\right)_{ss}\delta n+\left(\frac{\partial P_{ij}^{(0)}}{\partial \delta T}\right)_{ss}\delta T-\eta_{ijab}^{ss}\frac{\partial \delta u_a}{\partial r_b},$$

$$q_i=-\bar{\mu}_{ia}^{ss}\frac{\partial \delta n}{\partial r_a}-\bar{\kappa}_{ia}^{ss}\frac{\partial \delta T}{\partial r_a}, \tag{96}$$

where the superscript on the transport coefficients, and subscript on the derivatives, indicate that they are evaluated to zeroth order in the deviations from the steady state $\{\psi_0\}$, e.g.,

$$\left(\frac{\partial P_{ij}^{(0)}}{\partial \delta n}\right)_{ss}\equiv\lim_{\psi\to\psi_0}\frac{\partial P_{ij}^{(0)}(n,T;a)}{\partial \delta n}=\lim_{n\to n_0}\frac{\partial P_{ij}^{(0)}(n,T_0;a)}{\partial \delta n},$$

$$\left(\frac{\partial P_{ij}^{(0)}}{\partial \delta T}\right)_{ss}\equiv\lim_{\psi\to\psi_0}\frac{\partial P_{ij}^{(0)}(n,T;a)}{\partial \delta T}=\lim_{T\to T_0}\frac{\partial P_{ij}^{(0)}(n_0,T;a)}{\partial \delta n}. \tag{97}$$

The equation for the viscosity becomes

$$\begin{aligned}
&-a_{ss}^{*}\eta_{ijxb}^{ss}\delta_{ay}+a_{ss}^{*}\delta_{xi}\eta_{jyab}^{ss}+a_{ss}^{*}\delta_{xj}\eta_{iyab}^{ss}+[\nu^{*}(\alpha)+\zeta^{*}(\alpha)]\eta_{ijab}^{ss}\\
&\quad-\frac{2a_{ss}^{*}}{Dn_0k_B}\left(\frac{\partial}{\partial T}P_{ij}^{(0)}\right)_{ss}\eta_{xyab}^{ss}\\
&\quad=-\nu^{-1}(\psi_0)\delta_{ab}\left[n_0\left(\frac{\partial}{\partial n}P_{ij}^{(0)}\right)_{ss}-P_{ij}^{(ss)}\right]\\
&\qquad-\frac{2\nu^{-1}(\psi_0)}{Dn_0k_B}P_{ab}^{(ss)}\left(\frac{\partial}{\partial T}P_{ij}^{(0)}\right)_{ss}\\
&\qquad+\nu^{-1}(\psi_0)(\delta_{bi}P_{ja}^{(ss)}+\delta_{bj}P_{ia}^{(ss)}),
\end{aligned} \tag{98}$$

where a_{ss}^{*} was defined in Eq. (46). The generalized heat conductivities will be given by the simplified equations

$$\begin{aligned}
&\nu^{-1}(\psi_0)\left(\frac{\partial(\partial_t^{(0)}T)}{\partial n}\right)_{ss}\kappa_{ijka}^{ss}+\left(\nu^{*}(\alpha)+\frac{3}{2}\zeta^{*}(\alpha)\right)\mu_{ijka}^{ss}\\
&\quad+a_{ss}^{*}\mathcal{P}_{ijk}\delta_{xk}\mu_{ijya}^{ss}\\
&\quad=-\frac{\nu^{-1}(\psi_0)}{mn_0}\mathcal{P}_{ijk}\left(\frac{\partial}{\partial n}P_{ak}^{(0)}\right)_{ss}P_{ij}^{(ss)}+\nu^{-1}(\psi_0)\left(\frac{\partial}{\partial n}M_{ijka}^{(0)}\right)_{ss}
\end{aligned} \tag{99}$$

and

$$\begin{aligned}
&\nu^{-1}(\psi_0)\left(\frac{\partial(\partial_t^{(0)}T)}{\partial T}\right)_{ss}\kappa_{ijka}^{ss}+\left(\nu^{*}(\alpha)+\frac{3}{2}\zeta^{*}(\alpha)\right)\kappa_{ijka}^{ss}\\
&\quad+a_{ss}^{*}\mathcal{P}_{ijk}\delta_{xk}\kappa_{ijya}^{ss}\\
&\quad=-\frac{\nu^{-1}(\psi_0)}{mn_0}\mathcal{P}_{ijk}\left(\frac{\partial}{\partial T}P_{ak}^{(0)}\right)_{ss}P_{ij}^{(ss)}+\nu^{-1}(\psi_0)\left(\frac{\partial}{\partial T}M_{ijka}^{(0)}\right)_{ss}.
\end{aligned} \tag{100}$$

In these equations, the hydrodynamic variables ψ_0 must satisfy the steady state balance condition, Eq. (46). The derivatives of the pressure tensor and fourth order moment tensor, $M_{ijka}^{(0)}$, in the steady state are given explicitly in Appendix A and the explicit solution of Eqs. (98)–(100) are given in Appendix B. The linearized transport equations become

$$\partial_t\delta n+ay\frac{\partial}{\partial x}\delta n+n_0\nabla\cdot\delta\mathbf{u}=0,$$

$$\begin{aligned}
&\partial_t\delta u_i+ay\frac{\partial}{\partial x}\delta u_i+a\,\delta u_y\delta_{ix}+(mn_0)^{-1}\\
&\quad\times\left[\left(\frac{\partial P_{ij}^{(0)}}{\partial n}\right)_{ss}\frac{\partial \delta n}{\partial r_j}+\left(\frac{\partial P_{ij}^{(0)}}{\partial T}\right)_{ss}\frac{\partial \delta T}{\partial r_j}+\eta_{ijab}^{ss}\frac{\partial^2\delta u_a}{\partial r_j\partial r_b}\right]=0,
\end{aligned}$$

$$\begin{aligned}
&\partial_t\delta T+ay\frac{\partial}{\partial x}\delta T+\frac{2}{Dn_0k_B}\left(\bar{\mu}_{ia}^{ss}\frac{\partial^2\delta n}{\partial r_i\partial r_a}+\bar{\kappa}_{ia}^{ss}\frac{\partial^2\delta T}{\partial r_i\partial r_a}+P_{ij}^{(ss)}\frac{\partial \delta u_i}{\partial r_j}\right.\\
&\quad\left.+a\,\eta_{xyab}^{ss}\frac{\partial \delta u_a}{\partial r_b}\right)+\frac{2a}{Dn_0^2k_B}\left[n_0\left(\frac{\partial P_{xy}^{(0)}}{\partial \delta n}\right)_{ss}-P_{xy}^{(ss)}\right]\delta n\\
&\quad+\frac{2a}{Dn_0k_B}\left(\frac{\partial P_{xy}^{(0)}}{\partial \delta T}\right)_{ss}\delta T\\
&\quad=-\frac{3}{2}\zeta^{*}(\alpha)\nu(\psi_0)\delta T-\zeta^{*}(\alpha)\nu(\psi_0)T_0\frac{\delta n}{n_0},
\end{aligned} \tag{101}$$

where the fact that $\nu(\psi)\sim nT^{1/2}$ has been used. These equations have recently been used by Garzó to study the stability of the granular fluid under uniform shear flow [15].

IV. CONCLUSIONS

In this paper, the extension of the Chapman-Enskog method to arbitrary reference states has been presented. One of the key ideas is the separation of the gradient operator into "zeroth" and "first" order operators that help to organize the expansion. It is also important that the zeroth order distribution be recognized as corresponding to the exact distribution for *arbitrary uniform* deviations of *all* hydrodynamic fields from the reference state. This distribution does not in general have anything to do with the distribution in the reference state, except in the very special case that the reference state itself is spatially uniform.

The method was illustrated by application to the paradigmatic nonuniform system of a fluid undergoing uniform shear flow. In particular, the fluid was chosen to be a granular fluid which therefore admits of a steady state. The analysis was based on a particularly simple kinetic theory in order to allow for illustration of the general concepts without the technical complications involved in, e.g., using the Boltzmann equation. Nevertheless, it should be emphasized that the difference between the present calculation and that using the Boltzmann equation would be no greater than in the case of an equilibrium fluid. The main difference is that with the simplified kinetic theory, it is possible to obtain closed equations for the velocity moments without having to explicitly solve for the distribution. When solving the Boltzmann equation, the moment equations are not closed and it is necessary to resort to expansions in orthogonal polynomials. In that case, the calculation is usually organized somewhat differently: attention is focused on solving directly for the distribution but this is only a technical point. (In fact, Chapman originally developed his version of the Chapman-Enskog method using Maxwell's moment equations while Enskog based his on the Boltzmann equation [4]. The methods are of course equivalent.)

It is interesting to compare the hydrodynamic equations derived here to the "standard" equations for fluctuations about a uniform granular fluid. As might be expected, the hydrodynamic equations describing fluctuations about the state of uniform shear flow are more complex in some ways than are the usual Navier-Stokes equations for a granular fluid, but the similarities with the simpler case are perhaps more surprising. The complexity arises from the fact that the transport coefficients do not have the simple spatial symmetries present in the homogeneous fluid where, e.g., there is a single thermal conductivity rather than the vector quantity that occurs here. However, just as in the homogeneous system, the heat flux vector still only couples to density and temperature gradients and the pressure tensor to velocity gradients so that the hydrodynamics equations, Eqs. (94), have the same structure as the Navier-Stokes equations for the homogeneous system.

An additional complication in the general analysis presented here is that the zeroth order pressure tensor and the transport coefficients are obtained as the solution to partial differential equations in the temperature rather than as simple algebraic functions. This requires that appropriate boundary conditions be supplied which will, in general, depend on the particular problem being solved. Here, in the high-temperature limit, the nonequilibrium effects are of no importance and the appropriate boundary condition on all quantities is that they approach their equilibrium values. Boundary conditions must also be given at low temperature as the two domains are separated by the steady state which represents a critical point. At low temperatures, there are no collisions and no deviations from the macroscopic state so that all velocity moments go to zero thus giving the necessary boundary conditions. A particularly simple case occurs when the hydrodynamic equations are linearized about the reference state as would be appropriate for a linear stability analysis. Then, the transport properties are obtained as the solution to simple algebraic equations. The solutions of those equations have been explicitly formulated in Appendix B. It is to be hoped that the existence of these expressions will lead to new empirical work, either experimental or computer simulation, to test these results.

A particular simplifying feature of uniform shear flow is that the flow field has a constant first gradient and, as a result, the moments do not explicitly depend on the flow field. This will not be true for more complex, nonlinear flow fields. However, the application of the methods discussed in Sec. II should make possible an analysis similar to that given here for the simple case of uniform shear flow.

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APPENDIX A: DERIVATIVES OF THE ZEROTH ORDER MOMENTS IN THE STEADY STATE

1. The pressure

Recall that in the steady state

$$P_{ii}^{*ss} = \frac{\nu^*(\alpha) + \delta_{ix} D \zeta^*(\alpha)}{\nu^*(\alpha) + \zeta^*(\alpha)},$$

$$P_{xy}^{*ss} = -\frac{a_{ss}^* \nu^*(\alpha)}{[\nu^*(\alpha) + \zeta^*(\alpha)]^2}, \tag{A1}$$

and the explicit form of the steady state condition, Eq. (44) giving the value of the reduced shear in the steady state, a_{ss}^*, is

$$\frac{a_{ss}^{*2} \nu^*(\alpha)}{[\nu^*(\alpha) + \zeta^*(\alpha)]^2} = \frac{D}{2} \zeta^*(\alpha). \tag{A2}$$

Assume that the stresses are analytic in a^* so that near the singularity

$$P_{ij}^* = P_{ii}^{*ss} + A_{ij}^*(a^* - a_{ss}^*) + \cdots . \tag{A3}$$

They satisfy Eq. (55),

$$\left(\frac{1}{2}\zeta^*(\alpha) + \frac{1}{D} a^* P_{xy}^*\right) a^* \frac{\partial}{\partial a^*} P_{ij}^* = \frac{2}{D} a^* P_{xy}^* P_{ij}^* - a^* \delta_{ix} P_{jy}^* - a^* \delta_{jx} P_{iy}^* - \nu^*(\alpha)(P_{ij}^* - \delta_{ij}). \tag{A4}$$

Substituting Eq. (A3) into this and equating terms order by order in $(a^* - a_{ss}^*)$ gives

$$\left(\frac{1}{2}\zeta^*(\alpha) + \frac{1}{D} a_{ss}^* P_{xy}^{*ss}\right) a_{ss}^* A_{ij}^* = \frac{2}{D} a_{ss}^* P_{xy}^{*ss} P_{ij}^{*ss} - a_{ss}^* \delta_{ix} P_{jy}^{*ss} - a_{ss}^* \delta_{jx} P_{iy}^{*ss} - \nu^*(\alpha)(P_{ij}^{*ss} - \delta_{ij}) \tag{A5}$$

and

$$\left(\frac{1}{D} P_{xy}^{(*ss)} + \frac{1}{D} a_{ss}^* A_{xy}\right) a_{ss}^* A_{ij}^* = \frac{2}{D} P_{xy}^{*ss} P_{ij}^{*ss} + \frac{2}{D} a_{ss}^* A_{xy}^* P_{ij}^{*ss} + \frac{2}{D} a_{ss}^* P_{xy}^{*ss} A_{ij}^* - \delta_{ix} P_{jy}^{*ss} - \delta_{jx} P_{iy}^{*ss} - a_{ss}^* \delta_{ix} A_{jy}^* - a_{ss}^* \delta_{jx} A_{iy}^* - \nu^*(\alpha) A_{ij}. \tag{A6}$$

The first of these is satisfied identically by Eqs. (A1) and (A2) while the second gives

$$\frac{1}{D} a_{ss}^{*3} A_{xy} A_{ij} + a_{ss}^* \left(\nu^*(\alpha) + \frac{1}{2}\zeta^*(\alpha)\right) A_{ij} - \frac{2}{D} a_{ss}^{*2} A_{xy} P_{ij}^{*(ss)} + a_{ss}^{*2} \delta_{ix} A_{jy} + a_{ss}^{*2} \delta_{jx} A_{iy} = \nu^*(\alpha)(P_{ij}^{ss(*)} - \delta_{ij}). \tag{A7}$$

In component form this is

$$\frac{1}{D}a_{ss}^{*3}A_{xy}A_{yy} + a_{ss}^{*}\left(\nu^{*}(\alpha) + \frac{1}{2}\zeta^{*}(\alpha)\right)A_{yy} - \frac{2}{D}a_{ss}^{*2}A_{xy}P_{yy}^{*ss} = \nu^{*}(\alpha)(P_{yy}^{*ss} - 1),$$

$$\frac{1}{D}a_{ss}^{*3}A_{xy}^{2} + a_{ss}^{*}\left(\nu^{*}(\alpha) + \frac{1}{2}\zeta^{*}(\alpha)\right)A_{xy} - \frac{2}{D}a_{ss}^{*2}A_{xy}P_{xy}^{*ss} + a_{ss}^{*2}A_{yy} = \nu^{*}(\alpha)P_{xy}^{*ss}. \tag{A8}$$

Substituting

$$A_{xy} = DC/a_{ss}^{*2},$$

$$(1-\delta_{ix})A_{ii} = (1-\delta_{ix})DB/a_{ss}^{*},$$

$$\sum_{ii} A_{ii} = 0 \tag{A9}$$

gives

$$BC + \left(\nu^{*}(\alpha) + \frac{1}{2}\zeta^{*}(\alpha)\right)B - \frac{2}{D}CP_{yy}^{*ss} = \frac{1}{D}\nu^{*}(\alpha)(P_{yy}^{*ss} - 1),$$

$$C^{2} + \left(\nu^{*}(\alpha) + \frac{1}{2}\zeta^{*}(\alpha)\right)C - \frac{2}{D}a_{ss}^{*}CP_{xy}^{*ss} + a_{ss}^{*2}B = \frac{1}{D}\nu^{*}(\alpha)a_{ss}^{*}P_{xy}^{*ss} \tag{A10}$$

and the steady state condition makes this

$$BC + \left(\nu^{*}(\alpha) + \frac{1}{2}\zeta^{*}(\alpha)\right)B - \frac{2}{D}CP_{yy}^{*ss} = \frac{1}{D}\nu^{*}(\alpha)(P_{yy}^{*ss} - 1),$$

$$C^{2} + \left(\nu^{*}(\alpha) + \frac{3}{2}\zeta^{*}(\alpha)\right)C + a_{ss}^{*2}B = -\frac{1}{2}\nu^{*}(\alpha)\zeta^{*}(\alpha). \tag{A11}$$

The solution is

$$B = P_{yy}^{*ss}\frac{1}{D}\frac{2C(\alpha) - \zeta^{*}(\alpha)}{C(\alpha) + \nu^{*}(\alpha) + \frac{1}{2}\zeta^{*}(\alpha)} \tag{A12}$$

with $C(\alpha)$ being the real root of

$$4C^{3} + 8[\nu^{*}(\alpha) + \zeta^{*}(\alpha)]C^{2} + [4\nu^{*2}(\alpha) + 14\nu^{*}(\alpha)\zeta^{*}(\alpha) + 7\zeta^{*2}(\alpha)]C + \zeta^{*}(\alpha)[2\nu^{*2}(\alpha) - \nu^{*}(\alpha)\zeta^{*}(\alpha) - 2\zeta^{*2}(\alpha)] = 0. \tag{A13}$$

In summary, near the steady state, the pressure takes the form

$$P_{ij} = nk_{B}T[P_{ij}^{*ss} + A_{ij}^{*}(a^{*} - a_{ss}^{*}) + \cdots]$$

with

$$A_{xy}^{*} = DC/a_{ss}^{*2} = \left(\frac{\nu^{*}(\alpha)}{[\nu^{*}(\alpha) + \zeta^{*}(\alpha)]^{2}}\right)\frac{2C(\alpha)}{\zeta^{*}(\alpha)} = -\frac{1}{a_{ss}^{*}}P_{xy}^{*ss}\frac{2C(\alpha)}{\zeta^{*}(\alpha)},$$

$$(1-\delta_{ix})A_{ii}^{*} = DB/a_{ss}^{*} = \frac{1}{a_{ss}^{*}}P_{yy}^{*ss}\frac{2C(\alpha) - \zeta^{*}(\alpha)}{C(\alpha) + \nu^{*}(\alpha) + \frac{1}{2}\zeta^{*}(\alpha)},$$

$$A_{xx}^{*} = -(D-1)\frac{1}{a_{ss}^{*}}P_{yy}^{*ss}\frac{2C(\alpha) - \zeta^{*}(\alpha)}{C(\alpha) + \nu^{*}(\alpha) + \frac{1}{2}\zeta^{*}(\alpha)}. \tag{A14}$$

This gives

$$\lim_{T\to T_0}\left(\frac{\partial P_{ij}}{\partial T}\right) = n_0 k_B\left(P_{ij}^{*ss} - \frac{1}{2}a_{ss}^{*}A_{ij}^{*}\right),$$

$$\lim_{T\to T_0}\left(\frac{\partial P_{ij}}{\partial n}\right) = k_B T_0(P_{ij}^{*ss} - a_{ss}^{*}A_{ij}^{*}). \tag{A15}$$

2. Higher order moments

The general moment equations are

$$\frac{1}{2}\left(\zeta^{*}(\alpha) + \frac{2}{D}a^{*}P_{xy}^{(*)}\right)a^{*}\frac{\partial}{\partial a^{*}}M_{I_N}^{*} + \left(\nu^{*}(\alpha) - \frac{N}{D}a^{*}P_{xy}^{(*)}\right)M_{I_N}^{*} + a^{*}\mathcal{P}_{I_N}\delta_{i_N x}M_{I_{N-1}y}^{*} = \nu^{*}(\alpha)M_{I_N}^{(le*)}. \tag{A16}$$

In the linear approximation, writing

$$M_{I_N}^{*} = M_{I_N}^{*ss} + 2A_{I_N}^{*}(a^{*} - a_{ss}^{*}) + \cdots, \tag{A17}$$

where the factor of 2 is introduced to agree with the notation for the pressure, gives

$$\left(\nu^{*}(\alpha) - \frac{N}{D}a^{*}P_{xy}^{(*)}\right)M_{I_N}^{*ss} + a^{*}\mathcal{P}_{I_N}\delta_{xi_N}M_{I_{N-1}y}^{*ss} = \nu^{*}(\alpha)M_{I_N}^{(le*)} \tag{A18}$$

and

$$\frac{1}{D}(P_{xy}^{(*ss)} + a_{ss}^{*}A_{xy}^{(*)})a_{ss}^{*}A_{I_N}^{*} + \left(\nu^{*}(\alpha) + \frac{N}{2}\zeta^{*}(\alpha)\right)A_{I_N}^{*} + a_{ss}^{*}\mathcal{P}_{I_N}\delta_{i_N x}A_{I_{N-1}y}^{*} = \frac{1}{2}\left(\frac{N}{D}a_{ss}^{*}A_{xy}^{(*)} + \frac{N}{D}P_{xy}^{(*ss)}\right)M_{I_N}^{*ss} - \frac{1}{2}\mathcal{P}_{I_N}\delta_{i_N x}M_{I_{N-1}y}^{*ss}. \tag{A19}$$

For $N > 2$ these are simple linear equations for $M_{I_N}^{*ss}$ and $A_{I_N}^{*}$. They have the common form

$$Z_{I_N}^{*} + a_{ss}^{*}X_N\mathcal{P}_{I_n}\delta_{i_N x}Z_{I_{N-1}y}^{*} = R_{I_N} \tag{A20}$$

where for $Z_{I_N}^{*}=M_{I_N}^{*ss}$ one has

$$X_N = \left(\nu^{*}(\alpha) - \frac{N}{D}a^{*}P_{xy}^{(*)}\right)^{-1},$$

$$R_{I_N} = \nu^{*}(\alpha)M_{I_N}^{(le*)}, \tag{A21}$$

whereas for $Z_{I_N}^{*}=A_{I_N}^{*ss}$ one has

$$X_N=\left(\frac{1}{D}a_{ss}^{*2}A_{xy}^{*}+\nu^{*}(\alpha)+\frac{(N-1)}{2}\zeta^{*}(\alpha)\right)^{-1}$$
$$=\left(C(\alpha)+\nu^{*}(\alpha)+\frac{(N-1)}{2}\zeta^{*}(\alpha)\right)^{-1},$$

$$R_{I_N}=\frac{1}{2}X_N\left(\frac{N}{D}(a_{ss}^{*}A_{xy}^{*}+P_{xy}^{*ss})M_{I_N}^{*ss}-\mathcal{P}_{I_N}\delta_{i_Nx}M_{I_{N-1}y}^{*ss}\right). \tag{A22}$$

The solution of Eq. (A20) is straightforward and the result is

$$\begin{aligned}Z_{I_N}^{*}=R_{I_N}&-Xa_{ss}^{*}\mathcal{P}_{I_n}(\delta_{i_Nx}R_{yI_{N-1}})\\&+(2!)(Xa_{ss}^{*})^{2}\mathcal{P}_{I_n}(\delta_{i_Nx}\delta_{i_{N-1}x}R_{yyI_{N-2}})\\&-(3!)(Xa_{ss}^{*})^{3}\mathcal{P}_{I_n}(\delta_{i_Nx}\delta_{i_{N-1}x}\delta_{i_{N-2}x}R_{yyyI_{N-3}})+\cdots\\&+(N!)(Xa_{ss}^{*})^{N}R_{y\cdots y}\delta_{i_1x}\cdots\delta_{i_Nx},\end{aligned} \tag{A23}$$

thus giving explicit expressions for both $M_{I_N}^{*ss}$ and $A_{I_N}^{*}$.

Using these results, derivatives of the moments in the steady state can easily be evaluated. Using

$$\frac{\partial a^{*}}{\partial n_0}=-\frac{a^{*}}{n_0},$$
$$\frac{\partial a^{*}}{\partial T_0}=-\frac{a^{*}}{2T_0} \tag{A24}$$

gives

$$\lim_{T\to T_0}\left(\frac{\partial M_{I_n}}{\partial n}\right)=m\left(\frac{2k_BT_0}{m}\right)^{N/2}\left(M_{I_N}^{*ss}-\frac{1}{2}a^{*}A_{I_N}^{*}\right),$$
$$\lim_{T\to T_0}\left(\frac{\partial M_{I_n}}{\partial T}\right)=mn_0T_0^{-1}\left(\frac{2k_BT_0}{m}\right)^{N/2}\left(\frac{N}{2}M_{I_N}^{*ss}-\frac{1}{4}a^{*}A_{I_N}^{*}\right). \tag{A25}$$

It is also useful to calculate

$$\lim_{T\to T_0}\left(\frac{\partial\nu(\psi)T\left(-\zeta^{*}(\alpha)-\frac{2}{D}a^{*}P_{xy}^{*}\right)}{\partial\psi}\right)$$
$$=-\frac{2}{D}\nu(\psi_0)T_0(P_{xy}^{*ss}+a_{ss}^{*}A_{xy}^{*})\lim_{T\to T_0}\frac{\partial a^{*}}{\partial\psi} \tag{A26}$$

so that

$$\lim_{T\to T_0}\left(\frac{\partial(\partial_t^{(0)}T)}{\partial T}\right)=\frac{1}{D}\nu(\psi_0)a_{ss}^{*}(P_{xy}^{*ss}+a_{ss}^{*}A_{xy}^{*})$$
$$=\frac{1}{2}\nu(\psi_0)[2C(\alpha)-\zeta^{*}(\alpha)],$$

$$\lim_{T\to T_0}\left(\frac{\partial(\partial_t^{(0)}T)}{\partial n}\right)=\frac{2}{D}\nu(\psi_0)\frac{T_0}{n_0}a_{ss}^{*}(P_{xy}^{*ss}+a_{ss}^{*}A_{xy}^{*})$$
$$=\nu(\psi_0)\frac{T_0}{n_0}[2C(\alpha)-\zeta^{*}(\alpha)]. \tag{A27}$$

APPENDIX B: TRANSPORT COEFFICIENTS IN THE STEADY STATE

In this appendix, the equations for the transport coefficients in the steady state are solved. Note that throughout this appendix, the summation convention is *not* used.

1. Viscosity

The viscosities are determined by Eq. (98) which is

$$\begin{aligned}&-a_{ss}^{*}\eta_{ijxb}^{ss}\delta_{ay}+a_{ss}^{*}\delta_{xi}\eta_{jyab}^{ss}+a_{ss}^{*}\delta_{xj}\eta_{iyab}^{ss}+[\nu^{*}(\alpha)+\zeta^{*}(\alpha)]\eta_{ijab}^{ss}\\&\quad-\frac{2a_{ss}^{*}}{Dn_0k_B}\left(\frac{\partial}{\partial T}P_{ij}^{(0)}\right)_{ss}\eta_{xyab}^{ss}\\&\quad=-\nu^{-1}(\psi_0)\delta_{ab}\left[n_0\left(\frac{\partial}{\partial n}P_{ij}^{(0)}\right)_{ss}-P_{ij}^{(ss)}\right]\\&\qquad-\frac{2\nu^{-1}(\psi_0)}{Dn_0k_B}P_{ab}^{(ss)}\left(\frac{\partial}{\partial T}P_{ij}^{(0)}\right)_{ss}\\&\qquad+\nu^{-1}(\psi_0)(\delta_{bi}P_{ja}^{(ss)}+\delta_{bj}P_{ia}^{(ss)}).\end{aligned} \tag{B1}$$

Using Eq. (A15), this becomes

$$\begin{aligned}&-a_{ss}^{*}\eta_{ijxb}^{ss}\delta_{ay}+a_{ss}^{*}\delta_{xi}\eta_{jyab}^{ss}+a_{ss}^{*}\delta_{xj}\eta_{iyab}^{ss}+[\nu^{*}(\alpha)+\zeta^{*}(\alpha)]\eta_{ijab}^{ss}\\&\quad-\frac{2a_{ss}^{*}}{D}\left(P_{ij}^{*ss}-\frac{1}{2}a_{ss}^{*}A_{ij}^{*}\right)\eta_{xyab}^{ss}\\&\quad=n_0k_BT_0\nu^{-1}(\psi_0)\left[a_{ss}^{*}A_{ij}^{*}\left(\delta_{ab}+\frac{1}{D}a_{ss}^{*}P_{ab}^{*ss}\right)\right.\\&\qquad\left.-\frac{2}{D}P_{ij}^{*ss}P_{ab}^{*ss}+\delta_{bi}P_{ja}^{*ss}+\delta_{bj}P_{ia}^{*ss}\right].\end{aligned} \tag{B2}$$

It is convenient to introduce a scaled viscosity

$$\eta_{ijab}^{*}=\frac{\nu(\psi_0)}{n_0k_BT_0}\eta_{ijab}^{ss} \tag{B3}$$

and the abbreviations

$$Y=[\nu^{*}(\alpha)+\zeta^{*}(\alpha)]^{-1},$$

$$\begin{aligned}R_{ijab}=Y\bigg[&a_{ss}^{*}A_{ij}^{*}\left(\delta_{ab}+\frac{1}{D}a_{ss}^{*}P_{ab}^{*ss}\right)-\frac{2}{D}P_{ij}^{*ss}P_{ab}^{*ss}\\&+\delta_{bi}P_{ja}^{*ss}+\delta_{bj}P_{ia}^{*ss}\bigg]\end{aligned} \tag{B4}$$

so that the defining equation can be written as

$$\begin{aligned}\eta_{ijab}^{*}=R_{ijab}&+a_{ss}^{*}Y\eta_{ijxb}^{*}\delta_{ay}-a_{ss}^{*}Y\delta_{xi}\eta_{jyab}^{*}-a_{ss}^{*}Y\delta_{xj}\eta_{iyab}^{*}\\&+\frac{2a_{ss}^{*}}{D}\left(P_{ij}^{*ss}-\frac{1}{2}a_{ss}^{*}A_{ij}^{*}\right)Y\eta_{xyab}^{*}.\end{aligned} \tag{B5}$$

It is easiest to separate this into two cases. For $a\neq y$, the second term on the right drops out giving

$$\begin{aligned}\eta_{ijab}^{*}=R_{ijab}&-a_{ss}^{*}Y\delta_{xi}\eta_{jyab}^{*}-a_{ss}^{*}Y\delta_{xj}\eta_{iyab}^{*}\\&+\frac{2a_{ss}^{*}}{D}\left(P_{ij}^{*ss}-\frac{1}{2}a_{ss}^{*}A_{ij}^{*}\right)Y\eta_{xyab}^{*},\quad a\neq y\end{aligned} \tag{B6}$$

and the indices a and b are passive in the sense that the equations are invariant with respect to their values so that the only important indices are i and j. To solve this requires knowledge of $\eta^*_{iyab}=\eta^*_{yiab}$ and η^*_{xyab} which satisfy

$$\eta^*_{iyab}=R_{iyab}-a^*_{ss}Y\delta_{xi}\eta^*_{yyab}+\frac{2a^*_{ss}}{D}\left(P^{*ss}_{iy}-\frac{1}{2}a^*_{ss}A^*_{iy}\right)Y\eta^*_{xyab},\quad a\neq y,$$

$$\eta^*_{xyab}=R_{xyab}-a^*_{ss}Y\eta^*_{yyab}+\frac{2a^*_{ss}}{D}\left(P^{*ss}_{xy}-\frac{1}{2}a^*_{ss}A^*_{xy}\right)Y\eta^*_{xyab},\quad a\neq y. \tag{B7}$$

Finally, these in turn require knowledge of η^*_{yyab} which satisfies

$$\eta^*_{yyab}=R_{yyab}+\frac{2a^*_{ss}}{D}\left(P^{*ss}_{yy}-\frac{1}{2}a^*_{ss}A^*_{yy}\right)Y\eta^*_{xyab},\quad a\neq y. \tag{B8}$$

Using this to eliminate η^*_{yyab} in the previous equation gives an explicit solution for η^*_{xyab},

$$\left[1-\frac{2a^*_{ss}}{D}\left(P^{*ss}_{xy}-\frac{1}{2}a^*_{ss}A^*_{xy}\right)Y+\frac{2a^{*2}_{ss}}{D}\left(P^{*ss}_{yy}-\frac{1}{2}a^*_{ss}A^*_{yy}\right)Y^2\right]\eta^*_{xyab}=R_{xyab}-a^*_{ss}YR_{yyab},\quad a\neq y. \tag{B9}$$

Using Eqs. (A1), (A2), (A9), and (A12), the prefactor is

$$\left[1-\frac{2a^*_{ss}}{D}\left(P^{*ss}_{xy}-\frac{1}{2}a^*_{ss}A^*_{xy}\right)Y+\frac{2a^{*2}_{ss}}{D}\left(P^{*ss}_{yy}-\frac{1}{2}a^*_{ss}A^*_{yy}\right)Y^2\right]=\left(1+\frac{\zeta^*(\alpha)+C(\alpha)}{\nu^*(\alpha)+\zeta^*(\alpha)}+\frac{\zeta^*(\alpha)}{C(\alpha)+\nu^*(\alpha)+\frac{1}{2}\zeta^*(\alpha)}\right) \tag{B10}$$

so

$$\eta^*_{xyab}=\left(1+\frac{\zeta^*(\alpha)+C(\alpha)}{\nu^*(\alpha)+\zeta^*(\alpha)}+\frac{\zeta^*(\alpha)}{C(\alpha)+\nu^*(\alpha)+\frac{1}{2}\zeta^*(\alpha)}\right)^{-1}\times(R_{xyab}-a^*_{ss}YR_{yyab}),\quad a\neq y,$$

$$\eta^*_{yyab}=R_{yyab}+\frac{2a^*_{ss}}{D}\left(P^{*ss}_{yy}-\frac{1}{2}a^*_{ss}A^*_{yy}\right)Y\eta^*_{xyab},\quad a\neq y,$$

$$\eta^*_{iyab}=R_{iyab}-a^*_{ss}Y\delta_{xi}\eta^*_{yyab}+\frac{2a^*_{ss}}{D}\left(P^{*ss}_{iy}-\frac{1}{2}a^*_{ss}A^*_{iy}\right)Y\eta^*_{xyab},\quad a\neq y,$$

$$\eta^*_{ijab}=R_{ijab}-a^*_{ss}Y\delta_{xi}\eta^*_{jyab}-a^*_{ss}Y\delta_{xj}\eta^*_{iyab}+\frac{2a^*_{ss}}{D}\left(P^{*ss}_{ij}-\frac{1}{2}a^*_{ss}A^*_{ij}\right)Y\eta^*_{xyab},\quad a\neq y. \tag{B11}$$

Then, the remaining unknown viscosities correspond to the case $a=y$ and satisfy

$$\eta^*_{ijyb}=R_{ijyb}+a^*_{ss}Y\eta^*_{ijxb}-a^*_{ss}Y\delta_{xi}\eta^*_{jyyb}-a^*_{ss}Y\delta_{xj}\eta^*_{iyyb}+\frac{2a^*_{ss}}{D}\left(P^{*ss}_{ij}-\frac{1}{2}a^*_{ss}A^*_{ij}\right)Y\eta^*_{xyyb}. \tag{B12}$$

This has the same structure as Eq. (B6) with $R_{ijab}\rightarrow R_{ijyb}+a^*_{ss}Y\eta^*_{ijxb}$ where η^*_{ijxb} is now known. The solution is therefore analogous

$$\eta^*_{xyyb}=\left(1+\frac{\zeta^*(\alpha)+C(\alpha)}{\nu^*(\alpha)+\zeta^*(\alpha)}+\frac{\zeta^*(\alpha)}{C(\alpha)+\nu^*(\alpha)+\frac{1}{2}\zeta^*(\alpha)}\right)^{-1}\times(R_{xyyb}+a^*_{ss}Y\eta^*_{xyxb}-a^*_{ss}YR_{yyyb}-a^{*2}_{ss}Y^2\eta^*_{yyxb}),$$

$$\eta^*_{yyyb}=R_{yyab}+a^*_{ss}Y\eta^*_{yyxb}+\frac{2a^*_{ss}}{D}\left(P^{*ss}_{yy}-\frac{1}{2}a^*_{ss}A^*_{yy}\right)Y\eta^*_{xyyb},$$

$$\eta^*_{iyyb}=R_{iyyb}+a^*_{ss}Y\eta^*_{iyxb}-a^*_{ss}Y\delta_{xi}\eta^*_{yyyb}+\frac{2a^*_{ss}}{D}\left(P^{*ss}_{iy}-\frac{1}{2}a^*_{ss}A^*_{iy}\right)Y\eta^*_{xyyb},$$

$$\eta^*_{ijyb}=R_{ijyb}+a^*_{ss}Y\eta^*_{ijxb}-a^*_{ss}Y\delta_{xi}\eta^*_{jyyb}-a^*_{ss}Y\delta_{xj}\eta^*_{iyyb}+\frac{2a^*_{ss}}{D}\left(P^{*ss}_{ij}-\frac{1}{2}a^*_{ss}A^*_{ij}\right)Y\eta^*_{xyyb}. \tag{B13}$$

2. Thermal conductivity

From Eq. (100), the generalized heat conductivities κ^{ss}_{ijka} are solutions of

$$\nu^{-1}(\psi_0)\left(\frac{\partial(\partial_t^{(0)}T)}{\partial T}\right)_{ss}\kappa^{ss}_{ijka}+\left(\nu^*(\alpha)+\frac{3}{2}\zeta^*(\alpha)\right)\kappa^{ss}_{ijka}+a^*_{ss}\mathcal{P}_{ijk}\delta_{xk}\kappa^{ss}_{ijya}=-\frac{\nu^{-1}(\psi_0)}{mn_0}\mathcal{P}_{ijk}\left(\frac{\partial}{\partial T}P^{(0)}_{ak}\right)_{ss}P^{ss}_{ij}+\nu^{-1}(\psi_0)\left(\frac{\partial}{\partial T}M^{(0)}_{ijka}\right)_{ss}. \tag{B14}$$

Using Eqs. (A15), (A27), and (A25) this becomes

$$
\begin{aligned}
&[C(\alpha)+\nu^*(\alpha)+\zeta^*(\alpha)]\kappa^{ss}_{ijka}+a^*_{ss}\mathcal{P}_{ijk}\delta_{xk}\kappa^{ss}_{ijya}\\
&\quad=\frac{n_0T_0k_B^2}{m\nu(\psi_0)}\left[-\mathcal{P}_{ijk}\left(P^{*ss}_{ka}-\frac{1}{2}a^*_{ss}A^*_{ka}\right)P^{*ss}_{ij}\right.\\
&\qquad\left.+4\left(2M^{*ss}_{ijka}-\frac{1}{4}a^*A^*_{ijka}\right)\right]. \qquad \text{(B15)}
\end{aligned}
$$

Introducing the scaled transport coefficient

$$
\kappa^*_{ijka}=\frac{m\nu(\psi_0)}{n_0k_B^2T_0}\kappa^{ss}_{ijka} \qquad \text{(B16)}
$$

and the definitions

$$
X=[C(\alpha)+\nu^*(\alpha)+\zeta^*(\alpha)]^{-1},
$$

$$
Q_{ijka}=X\left[-\mathcal{P}_{ijk}\left(P^{*ss}_{ka}-\frac{1}{2}a^*_{ss}A^*_{ka}\right)P^{*ss}_{ij}+8M^{*ss}_{ijka}-a^*A^*_{ijka}\right] \qquad \text{(B17)}
$$

the equation to be solved becomes

$$
\kappa^*_{ijka}+a^*_{ss}X\mathcal{P}_{ijk}\delta_{xi}\kappa^*_{kjya}=Q_{ijka}. \qquad \text{(B18)}
$$

This is of the same form as Eq. (A20) [note that the index a plays no role in this equation and that Eqs. (65) and (80) imply that κ^{ss}_{ijka} is symmetric in the first three indices, i, j, and k] so the solution is obtained from Eq. (A23) and is

$$
\begin{aligned}
\kappa^*_{ijka}&=Q_{ijka}-a^*_{ss}X\mathcal{P}_{ijk}\delta_{xk}Q_{ijya}+2(a^*_{ss}X)^2\mathcal{P}_{ijk}\delta_{xj}\delta_{xk}Q_{iyya}\\
&\quad-6(a^*_{ss}X)^3\delta_{xi}\delta_{xj}\delta_{xk}Q_{yyya}. \qquad \text{(B19)}
\end{aligned}
$$

From this, the thermal conductivities entering the Navier-Stokes equations are

$$
\begin{aligned}
\bar{\kappa}^*_{ia}&=\sum_j(Q_{ijja}-a^*_{ss}X\delta_{xi}Q_{jjya})-2a^*_{ss}XQ_{ixya}+2(a^*_{ss}X)^2Q_{iyya}\\
&\quad+4(a^*_{ss}X)^2\delta_{xi}Q_{xyya}-6(a^*_{ss}X)^3\delta_{xi}Q_{yyya}. \qquad \text{(B20)}
\end{aligned}
$$

3. Coupling of density gradients to the heat flux

The transport coefficient governing the contributions of density gradients to the heat flux satisfies

$$
\begin{aligned}
&\nu^{-1}(\psi_0)\left(\frac{\partial(\partial_t^{(0)}T)}{\partial n}\right)_{ss}\kappa^{ss}_{ijka}+\left(\nu^*(\alpha)+\frac{3}{2}\zeta^*(\alpha)\right)\mu^{ss}_{ijka}\\
&\quad+a^*_{ss}\mathcal{P}_{ijk}\delta_{xk}\mu^{ss}_{ijya}\\
&\quad=-\frac{\nu^{-1}(\psi_0)}{mn_0}\mathcal{P}_{ijk}\left(\frac{\partial}{\partial n}P^{(0)}_{ak}\right)_{ss}P^{ss}_{ij}+\nu^{-1}(\psi_0)\left(\frac{\partial}{\partial n}M^{(0)}_{ijka}\right)_{ss}. \qquad \text{(B21)}
\end{aligned}
$$

Using Eqs. (A15), (A27), and (A25) this becomes

$$
\begin{aligned}
&\left(\nu^*(\alpha)+\frac{3}{2}\zeta^*(\alpha)\right)\mu^{ss}_{ijka}+a^*_{ss}\mathcal{P}_{ijk}\delta_{xk}\mu^{ss}_{ijya}\\
&\quad=\frac{\nu^{-1}(\psi_0)}{m}(k_BT_0)^2\left[-\mathcal{P}_{ijk}(P^{*ss}_{ka}-a^*_{ss}A^*_{ka})P^{*ss}_{ij}\right.\\
&\qquad\left.+4\left(M^{*ss}_{ijka}-\frac{1}{2}a^*_{ss}A^*_{ijka}\right)-[2C(\alpha)-\zeta^*(\alpha)]\nu(\psi_0)\kappa^*_{ijka}\right], \qquad \text{(B22)}
\end{aligned}
$$

so that defining

$$
\mu^{ss}_{ijka}=\frac{\nu^{-1}(\psi_0)}{m}(k_BT_0)^2\mu^*_{ijka} \qquad \text{(B23)}
$$

the equation for μ^{ss}_{ijka} becomes

$$
\begin{aligned}
&\left(\nu^*(\alpha)+\frac{3}{2}\zeta^*(\alpha)\right)\mu^*_{ijka}+a^*_{ss}\mathcal{P}_{ijk}\delta_{xk}\mu^*_{ijya}\\
&\quad=-\mathcal{P}_{ijk}(P^{*ss}_{ka}-a^*_{ss}A^*_{ka})P^{*ss}_{ij}+4\left(M^{*ss}_{ijka}-\frac{1}{2}a^*_{ss}A^*_{ijka}\right)\\
&\qquad-[2C(\alpha)-\zeta^*(\alpha)]\nu(\psi_0)\kappa^*_{ijka}. \qquad \text{(B24)}
\end{aligned}
$$

Again, noting that the index a is passive in this equation and that μ^{ss}_{ijka} is symmetric in i, j, and k, this is again of the same form as Eq. (A20) so that

$$
\begin{aligned}
\mu^*_{ijka}&=Q_{ijka}-a^*_{ss}X\mathcal{P}_{ijk}\delta_{xk}Q_{ijya}+2(a^*_{ss}X)^2\mathcal{P}_{ijk}\delta_{xj}\delta_{xk}Q_{iyya}\\
&\quad-6(a^*_{ss}X)^3\delta_{xi}\delta_{xj}\delta_{xk}Q_{yyya}, \qquad \text{(B25)}
\end{aligned}
$$

with

$$
X=\left(\nu^*(\alpha)+\frac{3}{2}\zeta^*(\alpha)\right)^{-1},
$$

$$
\begin{aligned}
Q_{ijka}&=X(-\mathcal{P}_{ijk}(P^{*ss}_{ij}-a^*_{ss}A^*_{ij})P^{*ss}_{ij}+4M^{*ss}_{ijka}-2a^*_{ss}A^*_{ijka}\\
&\quad-[2C(\alpha)-\zeta^*(\alpha)]\nu(\psi_0)\kappa^*_{ijka}). \qquad \text{(B26)}
\end{aligned}
$$

The transport coefficients required for the Navier-Stokes order equations are

$$
\begin{aligned}
\bar{\mu}^*_{ia}&=\sum_j(Q_{ijja}-a^*_{ss}X\delta_{xi}Q_{jjya})-2a^*_{ss}XQ_{ixya}+2(a^*_{ss}X)^2Q_{iyya}\\
&\quad+4(a^*_{ss}X)^2\delta_{xi}Q_{xyya}-6(a^*_{ss}X)^3\delta_{xi}Q_{yyya}. \qquad \text{(B27)}
\end{aligned}
$$

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