Two-loop field theory and nonasymptotic properties of the dynamical model for the λ transition in ^{3}He-^{4}He mixtures

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Model F′ introduced by Siggia and Nelson [Phys. Rev. B **15**, 1427 (1977)] describes the critical dynamics of ^{3}He-^{4}He mixtures near the superfluid transition. Using the minimal subtraction scheme this model is renormalized within dynamical field theory. The dynamic ζ functions needed for the nonasymptotic flow properties are presented in two-loop order. The fixed points are discussed and the stable fixed points are identified. The transition to limiting models contained in model F′ is shown analytically by performing the corresponding limits and numerically by calculating the nonlinear flow. These results are the basis for further experimental comparison of the transport coefficents in ^{3}He-^{4}He mixtures at higher concentrations including the tricritical point.

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I. INTRODUCTION

In order to describe the critical dynamics of ^{3}He-^{4}He mixtures near the superfluid transition one has to extend the model appropriate for the superfluid transition in pure ^{4}He [1] (model F). Such a model (we call it model F′) was developed by Siggia and Nelson [2] consisting of dynamical equations for the superfluid order parameter (OP) ψ_0 and two conserved densities m_{10} and m_{20} connected to the entropy density and concentration. It is thought that this model describes the superfluid transition in the whole plane of the thermodynamic space of temperature T, pressure P, and difference Δ, of the chemical potentials of ^{4}He and ^{3}He. This plane ends in a line of tricritical points. Whereas the asymptotic dynamical singular behavior of the superfluid transition in the thermodynamic space for mixtures is strongly related to that of pure ^{4}He the tricritical asymptotic dynamics belong to a quite different universality class of multicritical dynamical behavior. This is most prominently seen in the nonuniversal nonasymptotic behavior of different transport coefficients [3].

Comparing the transport properties of the dynamical behavior of model F′ for different mole fractions X of ^{3}He in ^{4}He one may distinguish different regions between special values of X (at saturated vapor pressure). These special values are (i) $X=0$ for pure ^{4}He, (ii) the decoupling point at $X_D=0.37$ [4], and (iii) the tricritical point at $X=0.67$. All these points and regions between are described by certain fixed points of the renormalization group (RG) equations derived here within the field theoretical version of the model F′. Whereas in case (i) one conserved density is decoupled, in (ii) both conserved densities couple to the OP, but one is decoupled asymptotically. The tricritical point (iii) is in this model described by Gaussian static fluctuations (i.e., the fourth-order static coupling of the OP fluctuations is set to zero) [5].

One-loop RG results have first been derived for the superfluid transition in Ref. [5] and for the tricritical point in Ref. [2]. From the superfluid transition in pure ^{4}He it turned out that the asymptotic dynamics is not properly described in one-loop order. The slow transient behavior of the time scale ratio between the order parameter and the entropy density [6] appears only in two-loop order. It became clear that an analysis in two-loop order is also necessary in mixtures. Additionally the pure ^{4}He case revealed that a dynamic model without static coupling between OP and the secondary density (model E) [6,7] has to be extended to a model incorporating this coupling (model F) [8,9]. This step made the two-loop calculation rather complex and even more for the case when two secondary densities couple as in model F′.

A first attempt to obtain the critical dynamical behavior in mixtures in two-loop order has been made in Ref. [10] by considering a reduced model, called model E′, where the static couplings between the OP and the secondary densities (entropy density and concentration) are neglected. This is correct in the very asymptotic region above T_λ. In the accessible temperature region above T_λ, and especially below T_λ, deviations from the true temperature behavior of the transport coefficients appear, which cannot be resolved within model E′. Most prominently this can be seen in the temperature independent thermal conductivity below T_λ [11] not reproducible by model E′ where a temperature dependent thermal conductivity below T_λ is generated [12]. The reason for this is the neglect of the static couplings between OP and the secondary densities, which leads to temperature independent thermodynamic derivatives (specific heat, concentration, susceptibility, and so on) within the model. In order to incorporate the static couplings between OP and secondary densities, and thus a temperature dependent thermodynamics at least in first order, an analysis was performed [14] by considering a combined model where the full model F′ has been calculated in one-loop order and the slow transient was taken into account by adding the two-loop model E′ contributions to the corresponding functions. Within the combined model an improvement in the critical behavior of the transport coefficients has been achieved especially concerning the constant thermal conductivity below T_λ and it is consistent with the plateau value above T_λ. Here we present a complete field theoretic calculation of model F′ in two-loop order. An earlier short presentation [13] contained an error in the order parameter field theoretic function, which was corrected later [15]. An analysis of the critical and tricritical dynamics in lowest order of $\epsilon=4-d$ (d is the dimension of space) concentrating on the asymptotic behavior of the transport coefficients has been performed by Onuki [4].

The paper is organized as follows. In Sec. II the definition of the dynamical model and variants necessary for the calculation are introduced. Section III deals with the definitions of correlation and vertex functions. Then in Sec. IV the dynamical models are renormalized and the field theoretic functions necessary for the nonasymptotic flow equations of the model parameters are presented in Sec. V. The explicit two loop expressions for the field theoretic functions are calculated in Sec. VI. This is followed by Sec. VII discussing various limits to submodels contained in model F′. The fixed points are determined in Sec. VIII and the behavior of the flow at special decoupling points is discussed in Sec. IX. A conclusion gives an outlook on further tasks to be treated within model F′. Technical details of the field theoretic calculations are deferred to appendixes. Appendix A treats the OP vertex function, Appendix B the vertex functions of the conserved densities, Appendix C contains the expressions of the ϵ-expanded integrals appearing in the calculation, and finally Appendix D presents the renormalization factors.

II. EQUATIONS OF MODEL F′

We consider a system including a nonconserved complex order parameter $\psi_0(x,t)=\psi_0'(x,t)+i\psi_0''(x,t)$ and two conserved real secondary densities $m_{10}(x,t)$ and $m_{20}(x,t)$. In order to perform an explicit two-loop calculation, we need the model equations for two different sets of secondary densities, which are connected by an orthogonal transformation.

The first set of secondary densities which we write as a two-dimensional vector

$$\boldsymbol{m}_0(x,t)=\begin{pmatrix} m_{10}(x,t) \\ m_{20}(x,t)\end{pmatrix} \tag{1}$$

represents the basic model in which all physical quantities will be calculated for comparisons with experiments and in which also the flow of the model parameters will be presented. The basic model is characterized by the property that only one of the secondary densities couples to the OP within statics [4,13]. This has the advantage that a scalar renormalization scheme can be used quite analogous to model F in ^{4}He leading to scalar dynamic ζ functions.

The second set of secondary densities introduced as

$$\bar{\boldsymbol{m}}_0(x,t)=\begin{pmatrix} \bar{m}_{10}(x,t) \\ \bar{m}_{20}(x,t)\end{pmatrix} \tag{2}$$

corresponds to a dynamically diagonalized model, which is characterized by the property that the matrix of the kinetic coefficients of the secondary densities is diagonal [10,13]. This is necessary to make the two-loop contributions calculable at the expense of two static couplings to the OP leading to a more complex matrix renormalization scheme and matrices of dynamic ζ functions. The parameters of this model are connected to the corresponding ones in the basic model by an orthogonal transformation given later in the text. The two-loop contributions to the functions in question will be first calculated explicitly in the dynamically diagonalized model and then transformed back to the basic model.

A. Basic model

The nonreversible part of the equations is relaxational for the nonconserved OP, while it is diffusive for the conserved secondary densities. This leads to the dynamic equations [2]

$$\frac{\partial\psi_0}{\partial t}=-2\mathring{\Gamma}\frac{\delta H}{\delta\psi_0^+}+i\psi_0\mathring{\boldsymbol{g}}\cdot\frac{\delta H}{\delta\boldsymbol{m}_0}+\theta_\psi, \tag{3}$$

$$\frac{\partial\psi_0^+}{\partial t}=-2\mathring{\Gamma}^+\frac{\delta H}{\delta\psi_0}-i\psi_0^+\mathring{\boldsymbol{g}}\cdot\frac{\delta H}{\delta\boldsymbol{m}_0}+\theta_\psi^+, \tag{4}$$

$$\frac{\partial\boldsymbol{m}_0}{\partial t}=\mathring{\boldsymbol{\Lambda}}\cdot\nabla^2\frac{\delta H}{\delta\boldsymbol{m}_0}+\mathring{\boldsymbol{g}}\,\mathrm{Im}[\psi_0^+\nabla^2\psi_0]+\boldsymbol{\theta}_m. \tag{5}$$

The superscript + denotes complex conjugated quantities. The kinetic coefficient of the OP $\mathring{\Gamma}=\mathring{\Gamma}'+i\mathring{\Gamma}''$ is assumed to be a complex quantity. The mode couplings $\mathring{g}_i$ and the stochastic forces θ_{m_i} of the secondary densities are represented by vectors

$$\mathring{\boldsymbol{g}}=\begin{pmatrix}\mathring{g}_1\\ \mathring{g}_2\end{pmatrix},\qquad \boldsymbol{\theta}_m=\begin{pmatrix}\theta_{m_1}\\ \theta_{m_2}\end{pmatrix}, \tag{6}$$

while the kinetic coefficients of the secondary densities are represented by the matrix

$$\mathring{\boldsymbol{\Lambda}}=\begin{pmatrix}\mathring{\lambda} & \mathring{L}\\ \mathring{L} & \mathring{\mu}\end{pmatrix}. \tag{7}$$

The dot in Eqs. (3)–(5) denotes a scalar product of the two-dimensional vectors or matrices.

The stochastic forces θ_{α_i} satisfy the relations

$$\langle\theta_\psi(x,t)\theta_\psi^+(x',t')\rangle=4\mathring{\Gamma}'\,\delta(x-x')\,\delta(t-t'), \tag{8}$$

$$\langle\boldsymbol{\theta}_m(x,t)\otimes\boldsymbol{\theta}_m(x',t')\rangle=-2\mathring{\boldsymbol{\Lambda}}\nabla^2\delta(x-x')\,\delta(t-t'), \tag{9}$$

with $\otimes$ as a two-dimensional tensor product. The critical behavior of the thermodynamic derivatives follows from the static functional

$$H\{\psi_0,\boldsymbol{m}_0\}=H_\psi\{\psi_0\}+H_m\{\psi_0,\boldsymbol{m}_0\} \tag{10}$$

with an OP functional

$$H_\psi\{\psi_0\}=\int d^dx\left(\frac{1}{2}\mathring{\tau}\psi_0^+\psi_0+\frac{1}{2}\vec{\nabla}\psi_0^+\vec{\nabla}\psi_0+\frac{\mathring{\tilde{u}}}{4!}(\psi_0^+\psi_0)^2\right) \tag{11}$$

and a secondary density functional

$$H_m\{\psi_0,\boldsymbol{m}_0\} = \int d^dx\left(\frac{1}{2}\boldsymbol{m}_0\cdot\boldsymbol{m}_0 + \frac{1}{2}\mathring{\gamma}m_{20}\psi_0^+\psi_0 - \mathring{h}m_{20}\right). \tag{12}$$

The static functional for the secondary densities (12) is of course not of general form for model F′ but contains already several special features. First, as preliminarily remarked, only one coupling $\mathring{\gamma}$ and one external field $\mathring{h}$, corresponding to the second secondary density m_{20}, are present, while the first secondary density m_{10} does not couple to the OP and therefore appears in a Gaussian form. Second, the whole Gaussian part of the secondary densities is diagonal and the static susceptibilities are normalized to 1. This structure is obtained by a suitable transformation of the general form of the static functional, which usually appears [13]. A subsequent scaling of the densities leads to the static functional in Eq. (12). Details about these transformations are described in Ref. [16].

B. Dynamically diagonal model

Because the dynamic perturbation expansion gets extremely extensive when a nondiagonal kinetic coefficient L is present, it is absolutely necessary to diagonalize matrix (7), which appears in the dynamic equation (5) [10,13]. The eigenvalues of this matrix are

$$\mathring{\lambda}_1 = \frac{1}{2}(\mathring{\lambda} + \mathring{\mu} + \mathring{K}), \quad \mathring{\lambda}_2 = \frac{1}{2}(\mathring{\lambda} + \mathring{\mu} - \mathring{K}), \tag{13}$$

with

$$\mathring{K} = \sqrt{(\mathring{\lambda} - \mathring{\mu})^2 + 4\mathring{L}^2}. \tag{14}$$

The diagonal dynamic coefficient matrix is then obtained by

$$\mathring{\bar{\boldsymbol{\Lambda}}} \equiv \begin{pmatrix} \mathring{\lambda}_1 & 0 \\ 0 & \mathring{\lambda}_2 \end{pmatrix} = \boldsymbol{R}^T\cdot\mathring{\boldsymbol{\Lambda}}\cdot\boldsymbol{R}. \tag{15}$$

The superscript T denotes the transposed matrix. The transformation matrix $\boldsymbol{R}$ is obtained from the eigenvectors corresponding to Eq. (13). It is an orthogonal matrix ($\boldsymbol{R}^{-1}=\boldsymbol{R}^T$) and has the structure

$$\boldsymbol{R} = \begin{pmatrix} R_{11} & -R_{21} \\ R_{21} & R_{11} \end{pmatrix} \tag{16}$$

with

$$R_{11} = \sqrt{\frac{\mathring{\lambda} - \mathring{\mu} + \mathring{K}}{2\mathring{K}}}, \quad R_{21} = \sqrt{\frac{\mathring{\mu} - \mathring{\lambda} + \mathring{K}}{2\mathring{K}}}. \tag{17}$$

Application to Eq. (5) leads to the dynamic equations

$$\frac{\partial\psi_0}{\partial t} = -2\mathring{\Gamma}\frac{\delta H}{\delta\psi_0^+} + i\psi_0\mathring{\bar{\boldsymbol{g}}}\cdot\frac{\delta H}{\delta\bar{\boldsymbol{m}}_0} + \theta_\psi, \tag{18}$$

$$\frac{\partial\psi_0^+}{\partial t} = -2\mathring{\Gamma}^+\frac{\delta H}{\delta\psi_0} - i\psi_0^+\mathring{\bar{\boldsymbol{g}}}\cdot\frac{\delta H}{\delta\bar{\boldsymbol{m}}_0} + \theta_\psi^+, \tag{19}$$

$$\frac{\partial\bar{\boldsymbol{m}}_0}{\partial t} = \mathring{\bar{\boldsymbol{\Lambda}}}\cdot\nabla^2\frac{\delta H}{\delta\bar{\boldsymbol{m}}_0} - 2\mathring{\bar{\boldsymbol{g}}}\,\mathrm{Im}(\psi_0^+\nabla^2\psi_0) + \boldsymbol{\theta}_{\bar{m}}, \tag{20}$$

with transformed secondary densities and mode couplings

$$\bar{\boldsymbol{m}}_0 \equiv \begin{pmatrix} \bar{m}_{10} \\ \bar{m}_{20} \end{pmatrix} = \boldsymbol{R}^T\cdot\boldsymbol{m}_0, \tag{21}$$

$$\mathring{\bar{\boldsymbol{g}}} \equiv \begin{pmatrix} \mathring{\bar{g}}_1 \\ \mathring{\bar{g}}_2 \end{pmatrix} = \boldsymbol{R}^T\cdot\mathring{\boldsymbol{g}}. \tag{22}$$

The static functional Eq. (12) also has to be transformed to the secondary densities $\bar{m}_{i0}$. Inserting Eq. (21) into Eq. (12) one immediately obtains

$$H_m\{\psi_0,\bar{\boldsymbol{m}}_0\} = \int d^dx\left(\frac{1}{2}\bar{\boldsymbol{m}}_0\cdot\bar{\boldsymbol{m}}_0 + \frac{1}{2}\mathring{\bar{\boldsymbol{\gamma}}}\cdot\bar{\boldsymbol{m}}_0\psi_0^+\psi_0 - \mathring{\bar{\boldsymbol{h}}}\cdot\bar{\boldsymbol{m}}_0\right) \tag{23}$$

with transformed static couplings and fields

$$\mathring{\bar{\boldsymbol{\gamma}}} \equiv \begin{pmatrix} \mathring{\bar{\gamma}}_1 \\ \mathring{\bar{\gamma}}_2 \end{pmatrix} = \boldsymbol{R}^T\cdot\begin{pmatrix} 0 \\ \mathring{\gamma} \end{pmatrix}, \tag{24}$$

$$\mathring{\bar{\boldsymbol{h}}} \equiv \begin{pmatrix} \mathring{\bar{h}}_1 \\ \mathring{\bar{h}}_2 \end{pmatrix} = \boldsymbol{R}^T\cdot\begin{pmatrix} 0 \\ \mathring{h} \end{pmatrix}. \tag{25}$$

The simpler dynamic structure with a diagonal matrix $\mathring{\bar{\boldsymbol{\Lambda}}}$ has been reached at the expense of two static couplings $\mathring{\bar{\gamma}}_i$, which will influence the renormalization scheme considerably.

C. Ginzburg-Landau-Wilson functional

Both static functionals (11) with (12) or (23) may be reduced to the Ginzburg-Landau-Wilson (GLW) functional with complex OP by integrating out the secondary densities. One obtains

$$H_{GLW} = \int d^dx\left(\frac{1}{2}\mathring{r}\psi_0^+\psi_0 + \frac{1}{2}\vec{\boldsymbol{\nabla}}\psi_0^+\vec{\boldsymbol{\nabla}}\psi_0 + \frac{\mathring{u}}{4!}(\psi_0^+\psi_0)^2\right). \tag{26}$$

The parameters $\mathring{r}$ and $\mathring{u}$ in Eq. (26) are related to the corresponding parameters of the extended static functionals by

$$\mathring{r} = \mathring{\tau} + \mathring{\gamma}\mathring{h} = \mathring{\tau} + \mathring{\bar{\boldsymbol{\gamma}}}\cdot\mathring{\bar{\boldsymbol{h}}}, \tag{27}$$

$$\mathring{u} = \mathring{\tilde{u}} - 3\mathring{\gamma}^2 = \mathring{\tilde{u}} - 3\mathring{\bar{\boldsymbol{\gamma}}}\cdot\mathring{\bar{\boldsymbol{\gamma}}}. \tag{28}$$

The ability to eliminate the secondary density part (12) or (23) in (10) also leads to relations between the correlation functions of the secondary densities and the OP correlation function [17]. For the first and second cumulants one obtains

$$\langle m_{20}(x)\rangle = \mathring{h} - \mathring{\gamma}\left\langle\tfrac{1}{2}|\psi_0(x)|^2\right\rangle, \tag{29}$$

$$\langle m_{20}(x)m_{20}(0)\rangle_c = 1 + \mathring{\gamma}^2\left\langle \tfrac{1}{2}|\psi_0(x)|^2\tfrac{1}{2}|\psi_0(0)|^2\right\rangle_c \tag{30}$$

when Eq. (12) is used or

$$\langle \bar{\boldsymbol{m}}_0(x)\rangle = \mathring{\bar{\boldsymbol{h}}} - \mathring{\bar{\boldsymbol{\gamma}}}\left\langle \tfrac{1}{2}|\psi_0(x)|^2\right\rangle, \tag{31}$$

$$\langle \bar{\boldsymbol{m}}_0(x)\otimes\bar{\boldsymbol{m}}_0(0)\rangle_c = \mathbf{1} + \mathring{\bar{\boldsymbol{\gamma}}}\otimes\mathring{\bar{\boldsymbol{\gamma}}}\left\langle \tfrac{1}{2}|\psi_0(x)|^2\tfrac{1}{2}|\psi_0(0)|^2\right\rangle_c \tag{32}$$

when Eq. (23) is used. The subscript c in Eqs. (30) and (32) denotes the cumulant $\langle AB\rangle_c \equiv \langle AB\rangle - \langle A\rangle\langle B\rangle$. Note that the angular brackets in Eqs. (29) and (30) or (31) and (32) have to be calculated with a probability density $\exp(-H)/\mathcal{N}$ on the left hand side, and with $\exp(-H_{GLW})/\mathcal{N}'$ on the right hand side, where $\mathcal{N}$ and $\mathcal{N}'$ are appropriate normalization factors. The correlation function $\left\langle\frac{1}{2}\left|\psi_0(x)\right|^2\frac{1}{2}\left|\psi_0(0)\right|^2\right\rangle_c$ in Eqs. (30) and (32) determines the specific heat in the GLW model. In order to perform the static perturbation expansion in the usual manner, the external fields are chosen to eliminate the finite expectation values of the corresponding secondary densities. This is fulfilled if the external field satisfies the condition

$$\mathring{h} = \mathring{\gamma}\left\langle \tfrac{1}{2}|\psi_0(x)|^2\right\rangle \tag{33}$$

or

$$\mathring{\bar{\boldsymbol{h}}} = \mathring{\bar{\boldsymbol{\gamma}}}\left\langle \tfrac{1}{2}|\psi_0(x)|^2\right\rangle, \tag{34}$$

respectively.

III. DYNAMIC CORRELATION AND VERTEX FUNCTIONS

The dynamic correlation function of the OP is usually expressed as

$$\mathring{C}_{\psi\psi^+}(\xi,x,t) = \int \frac{d^dk}{(2\pi)^d}\int\frac{d\omega}{2\pi}e^{ikx-i\omega t}\mathring{C}_{\psi\psi^+}(\xi,k,\omega). \tag{35}$$

Within the the field theoretic approach which is used in this work [18], the dynamic correlation function is connected to dynamic vertex functions via

$$\mathring{C}_{\psi\psi^+}(\xi,k,\omega) = -\frac{\mathring{\Gamma}_{\tilde{\psi}\tilde{\psi}^+}(\xi,k,\omega)}{|\mathring{\Gamma}_{\psi\tilde{\psi}^+}(\xi,k,\omega)|^2} \tag{36}$$

where the functions appearing in the above expression have to be calculated within the perturbation expansion. A closer examination of the loop expansion reveals that the dynamic response vertex function $\mathring{\Gamma}_{\psi\tilde{\psi}^+}$ has the general structure

$$\mathring{\Gamma}_{\psi\tilde{\psi}^+}(\xi,k,\omega) = -i\omega\mathring{\Omega}_{\psi\tilde{\psi}^+}(\xi,k,\omega) + \mathring{\Gamma}_{\psi\psi^+}(\xi,k)\mathring{\Gamma}^{(d)}_{\psi\tilde{\psi}^+}(\xi,k,\omega) \tag{37}$$

where $\mathring{\Gamma}_{\psi\psi^+}(\xi,k)$ is the well known static two-point vertex function of the GLW model with a complex OP. It determines also the correlation length ξ by

$$\xi^2(\mathring{r},\mathring{u}) = \left.\frac{\partial \ln \mathring{\Gamma}_{\psi\psi^+}}{\partial k^2}\right|_{k=0}. \tag{38}$$

$\mathring{\Omega}_{\psi\tilde{\psi}^+}$ and $\mathring{\Gamma}^{(d)}_{\psi\tilde{\psi}^+}$ are purely dynamic functions. The explicit expressions of these functions are given in Appendix A 1. They determine also the second dynamic vertex function $\mathring{\Gamma}_{\tilde{\psi}\tilde{\psi}^+}$ in Eq. (36). A proper rearrangement of the perturbative contributions shows that the relation

$$\mathring{\Gamma}_{\tilde{\psi}\tilde{\psi}^+}(\xi,k,\omega) = -2\,\mathrm{Re}[\mathring{\Omega}_{\psi\tilde{\psi}^+}(\xi,k,\omega)\mathring{\Gamma}^{(d)}_{\psi\tilde{\psi}^+}(\xi,k,\omega)] \tag{39}$$

holds.

Analogous to Eq. (35) the dynamic correlation functions of the secondary densities are

$$\mathring{C}_{m_im_j}(\xi,x,t) = \int \frac{d^dk}{(2\pi)^d}\int\frac{d\omega}{2\pi}e^{ikx-i\omega t}\mathring{C}_{m_im_j}(\xi,k,\omega) \tag{40}$$

which are the components of a matrix. The connection to the dynamic vertex functions is more complex than in the case of the OP. The matrix components read

$$\mathring{C}_{m_im_i} = -\frac{p^{(ij\neq i)}}{\det(\mathring{\Gamma}_{m\tilde{m}})}, \quad \mathring{C}_{m_im_{j\neq i}} = \frac{q^{(ij\neq i)}}{\det(\mathring{\Gamma}_{m\tilde{m}})} \tag{41}$$

with

$$p^{(ij\neq i)} = \mathring{\Gamma}_{\tilde{m}_i\tilde{m}_i}\mathring{\Gamma}_{m_j\tilde{m}_j}\mathring{\Gamma}_{\tilde{m}_jm_j} + \mathring{\Gamma}_{\tilde{m}_j\tilde{m}_j}\mathring{\Gamma}_{m_j\tilde{m}_i}\mathring{\Gamma}_{\tilde{m}_im_j} - \mathring{\Gamma}_{\tilde{m}_i\tilde{m}_j}\mathring{\Gamma}_{m_j\tilde{m}_i}\mathring{\Gamma}_{\tilde{m}_jm_j} - \mathring{\Gamma}_{\tilde{m}_j\tilde{m}_i}\mathring{\Gamma}_{m_j\tilde{m}_j}\mathring{\Gamma}_{\tilde{m}_im_j}, \tag{42}$$

$$q^{(ij\neq i)} = \mathring{\Gamma}_{\tilde{m}_i\tilde{m}_i}\mathring{\Gamma}_{m_i\tilde{m}_j}\mathring{\Gamma}_{\tilde{m}_jm_j} + \mathring{\Gamma}_{\tilde{m}_j\tilde{m}_j}\mathring{\Gamma}_{m_i\tilde{m}_i}\mathring{\Gamma}_{\tilde{m}_im_j} - \mathring{\Gamma}_{\tilde{m}_i\tilde{m}_j}\mathring{\Gamma}_{m_i\tilde{m}_i}\mathring{\Gamma}_{\tilde{m}_jm_j} - \mathring{\Gamma}_{\tilde{m}_j\tilde{m}_i}\mathring{\Gamma}_{m_i\tilde{m}_j}\mathring{\Gamma}_{\tilde{m}_im_j}. \tag{43}$$

The above expressions are valid for two secondary densities. The indices i and j have to be always different, that is, if $i=1$ then $j=2$, and vice versa. The dynamic response vertex functions of the secondary densities have the general structure

$$\mathring{\Gamma}_{m_i\tilde{m}_j}(\xi,k,\omega) = -i\omega\mathring{\Omega}_{m_i\tilde{m}_j}(\xi,k,\omega) + \sum_l \mathring{\Gamma}_{m_im_l}(\xi,k)\mathring{\Gamma}^{(d)}_{m_l\tilde{m}_j}(\xi,k,\omega) \tag{44}$$

where $\mathring{\Gamma}_{m_im_l}(\xi,k)$ are components of the matrix of the static two-point vertex functions calculated with the extended static functional (10). The sum covers the number of secondary densities. A relation corresponding to Eq. (39) holds also for the dynamic vertex functions of the secondary densities. We have

$$\mathring{\Gamma}_{\tilde{m}_i\tilde{m}_j}(\xi,k,\omega) = -2\sum_l \mathrm{Re}[\mathring{\Omega}_{m_l\tilde{m}_i}(\xi,k,\omega)\mathring{\Gamma}^{(d)}_{m_l\tilde{m}_j}(\xi,k,\omega)]. \tag{45}$$

The dynamic vertex functions from above are related to the transport coefficients as presented in Ref. [13].

IV. RENORMALIZATION OF MODEL F′

A. Static renormalization

The renormalization of the GLW functional (26) is well known and has been extensively discussed in the literature (see, e.g., Ref. [19]). Details for the renormalization of the extended functional can be found in several papers [17,20]. This subsection is intended to give a survey over our definitions of the renormalization factors. Proofs of relations between the renormalization factors can be found elsewhere. We restrict our considerations to the minimal subtraction scheme for renormalization [21], which means that only the singular terms are collected in the renormalization factors.

1. GLW model

For the OP ψ we introduce the following renormalization factor:

$$\psi_0 = Z_\psi^{1/2}\psi, \quad \psi_0^+ = Z_\psi^{1/2}\psi^+ \tag{46}$$

where Z_ψ is a real quantity. The renormalization of the parameters r and u appearing in Eq. (26) is defined as usual,

$$\mathring{r} = Z_\psi^{-1} Z_r r, \quad \mathring{u} = \kappa^\epsilon Z_\psi^{-2} Z_u u A_d^{-1}. \tag{47}$$

κ represents a free wave number scale and $\epsilon=4-d$. The geometry factor A_d is chosen as [21]

$$A_d = \Gamma\left(1-\frac{\epsilon}{2}\right)\Gamma\left(1+\frac{\epsilon}{2}\right)\frac{\Omega_d}{(2\pi)^d} \tag{48}$$

with d the spatial dimension, Ω_d the surface of the d-dimensional unit sphere, and $\Gamma(x)$ the Euler Γ function. To complete the renormalization of Eq. (26) a Z factor

$$\frac{1}{2}|\psi_0|^2 = Z_{\psi^2}\frac{1}{2}|\psi|^2 \tag{49}$$

is necessary to renormalize correlation functions containing $\frac{1}{2}|\psi|^2$ insertions. Within the minimal subtraction scheme Z_r is connected to the renormalization of the $\frac{1}{2}|\psi|^2$ insertions by the relation $Z_{\psi^2}=Z_\psi^{-1}Z_r$. Finally the correlation function $\langle\frac{1}{2}|\psi_0|^2\frac{1}{2}|\psi_0|^2\rangle_c$, which represents the specific heat within the model, needs an additive renormalization A_{ψ^2}.

2. Basic model

The extended static functional (10) with (12) renormalizes analogous to model C due to the absence of a static coupling between m_{10} and the OP. The secondary density m_{20} and the coupling parameter γ between oder parameter and secondary density, which guarantees a nontrivial static critical behavior of the thermodynamic derivatives, will be renormalized by

$$m_{20} = Z_m m_2, \tag{50}$$

$$\mathring{\gamma} = \kappa^{\epsilon/2} Z_\psi^{-1} Z_m^{-1} Z_\gamma \gamma A_d^{-1/2}. \tag{51}$$

Note that we have introduced the Z factor Z_m instead of $Z_m^{1/2}$ contrary to most of the definitions in the literature (see, e.g., Ref. [8]). Our definition is more convenient when one wants to maintain consistency with the definitions in the dynamically diagonalized model where a matrix $\boldsymbol{Z}_m$ has to be introduced (see next section). The secondary density m_{10} does not need to be renormalized; thus we have $m_{10}=m_1$.

Since the extended static functional is a Gaussian extension of the Ginzburg-Landau-Wilson model, no new poles appear. Thus relations between the Z factors of the Ginzburg-Landau-Wilson model and the extended static functional exist. First, the renormalization factor of the coupling γ is determined by

$$Z_\gamma = Z_m^2 Z_\psi Z_{\psi^2} \tag{52}$$

leading with Eq. (51) to

$$\mathring{\gamma} = \kappa^{\epsilon/2} Z_{\psi^2} Z_m \gamma A_d^{-1/2}. \tag{53}$$

Second, from Eq. (30) it follows that the renormalization factor Z_m of the secondary density is determined by the additive renormalization $A_{\psi^2}(u)$ of the specific heat in the GLW model. This gives

$$Z_m^{-2}(u,\gamma) = 1 + \gamma^2 A_{\psi^2}(u). \tag{54}$$

3. Dynamically diagonal model

In the case of the static functional of the dynamically diagonal model (10) with (12) the renormalization is getting more complex. The renormalization has to be performed in such a way that relations (27), (28) and also (32) are preserved. The renormalized counterparts of the parameters have to satisfy the same relations. With two static couplings $\mathring{\bar{\gamma}}_i$ this cannot be achieved with scalar renormalization factors as in the basic model. A matrix renormalization scheme has to be introduced. The secondary densities $\bar{\boldsymbol{m}}_0$ and the coupling parameters $\mathring{\bar{\boldsymbol{\gamma}}}$ between oder parameter and secondary densities will now renormalize as

$$\bar{\boldsymbol{m}}_0 = \boldsymbol{Z}_{\bar{m}} \cdot \bar{\boldsymbol{m}}, \tag{55}$$

$$\mathring{\bar{\boldsymbol{\gamma}}} = \kappa^{\epsilon/2} Z_\psi^{-1} \boldsymbol{Z}_{\bar{m}}^{-1} \cdot \boldsymbol{Z}_\gamma \cdot \bar{\boldsymbol{\gamma}} A_d^{-1/2}. \tag{56}$$

The renormalization factor of the coupling $\bar{\boldsymbol{\gamma}}$ is now determined by

$$\boldsymbol{Z}_\gamma = \boldsymbol{Z}_{\bar{m}}^2 Z_\psi Z_{\psi^2} \tag{57}$$

which leads with Eq. (56) to

$$\mathring{\bar{\boldsymbol{\gamma}}} = \kappa^{\epsilon/2} Z_{\psi^2} \boldsymbol{Z}_{\bar{m}} \cdot \bar{\boldsymbol{\gamma}} A_d^{-1/2}. \tag{58}$$

From Eq. (32) it follows that the renormalization matrix $\boldsymbol{Z}_{\bar{m}}$ is related to the additive renormalization $A_{\psi^2}(u)$ of the specific heat in the GLW model by

$$\boldsymbol{Z}_{\bar{m}}^{-2}(u,\bar{\gamma}_i) = \boldsymbol{1} + \bar{\boldsymbol{\gamma}} \otimes \bar{\boldsymbol{\gamma}} A_{\psi^2}(u). \tag{59}$$

B. Dynamic renormalization

First we will consider the renormalizations that are independent of the dynamic diagonalization procedure in their explicit form. This addresses quantities exclusively connected to the OP.

In the dynamic functional an OP auxiliary density $\tilde{\psi}_0$ and auxiliary densities $\tilde{m}_{i0}$ to the corresponding secondary densities are introduced. The OP auxiliary density renormalizes like

$$\tilde{\psi}_0 = Z_{\tilde{\psi}}^{1/2} \tilde{\psi} \tag{60}$$

while for the OP kinetic coefficient Γ a renormalization

$$\mathring{\Gamma} = Z_\Gamma \Gamma \tag{61}$$

is introduced. The above Z factor contains the singular contributions of the two dynamic functions $\mathring{\Omega}_{\psi\tilde{\psi}^+}$ and $\mathring{\Gamma}^{(d)}_{\psi\tilde{\psi}^+}$, which appear in Eq. (37), as well as static contributions. Therefore we can write

$$Z_\Gamma = Z_\psi^{1/2} Z_{\tilde{\psi}^+}^{-1/2} Z_\Gamma^{(d)}. \tag{62}$$

$Z_{\tilde{\psi}^+}^{-1/2}$ contains the poles of $\mathring{\Omega}_{\psi\tilde{\psi}^+}$ and $Z_\Gamma^{(d)}$ the poles of $\mathring{\Gamma}^{(d)}_{\psi\tilde{\psi}^+}$.

The secondary densities are conserved; therefore no independent renormalization factors are needed for the corresponding auxiliary densities. Their renormalization is determined by the renormalization factor of the secondary densities and therefore depends on the model considered.

1. Basic model

For the secondary auxiliary densities $\tilde{m}_{10}$ and $\tilde{m}_{20}$ we introduce a renormalization

$$\tilde{m}_{i0} = Z_{\tilde{m}_i} \tilde{m}_i, \quad i = 1,2. \tag{63}$$

The Z factors in the above equation are determined by the Z factors of the corresponding densities m_{i0}; thus we have

$$Z_{\tilde{m}_1} = 1, \quad Z_{\tilde{m}_2} = Z_m^{-1}. \tag{64}$$

The renormalization factors of kinetic coefficients of the secondary densities are introduced as

$$\mathring{\lambda} = Z_\lambda \lambda, \quad \mathring{L} = Z_L L, \quad \mathring{\mu} = Z_\mu \mu. \tag{65}$$

The above Z factors contain static contributions which will be separated. We can write

$$Z_\lambda = Z_\lambda^{(d)}, \quad Z_L = Z_m Z_L^{(d)}, \quad Z_\mu = Z_m^2 Z_\mu^{(d)} \tag{66}$$

where the renormalization factors with superscript (d) contain only the dynamic poles of the k^2 derivative of the dynamic vertex functions which correspond to the secondary densities. Due to a Ward identity [22] the mode coupling coefficients need no independent renormalization, so we simply have

$$\mathring{g}_1 = \kappa^{\epsilon/2} g_1 A_d^{-1/2}, \quad \mathring{g}_2 = \kappa^{\epsilon/2} Z_m g_2 A_d^{-1/2}. \tag{67}$$

2. Dynamically diagonal model

The auxiliary densities corresponding to the secondary densities renormalize as

$$\tilde{\bar{\boldsymbol{m}}}_0 = \boldsymbol{Z}_{\tilde{\bar{m}}} \cdot \tilde{\bar{\boldsymbol{m}}} \tag{68}$$

where the renormalization matrix is determined by

$$\boldsymbol{Z}_{\tilde{\bar{m}}} = \boldsymbol{Z}_{\bar{m}}^{-1}. \tag{69}$$

The renormalization of the kinetic coefficients of the secondary densities is within the dynamically diagonal model more complex than in the basic model presented previously. The separation into static and dynamic contributions is given by

$$\mathring{\bar{\boldsymbol{\Lambda}}} = \boldsymbol{Z}_{\bar{m}} \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \boldsymbol{Z}_{\bar{\Lambda}}^{(d)} \cdot \boldsymbol{Z}_{\bar{m}}. \tag{70}$$

The mode coupling parameters renormalize analogous to Eq. (67) with

$$\mathring{\bar{\boldsymbol{g}}} = \kappa^{\epsilon/2} \boldsymbol{Z}_{\bar{m}} \cdot \bar{\boldsymbol{g}} A_d^{-1/2}. \tag{71}$$

V. FLOW EQUATIONS OF MODEL F′

The flow equations, which determine the critical temperature dependence of the model parameters, require the definition of ζ and β functions. In order to simplify the handling of the renormalization group functions in the flow equations we will use the uniform definition

$$\zeta_{a_i}(\{\alpha_j\}) = \frac{d \ln Z_{a_i}^{-1}}{d \ln \kappa} \tag{72}$$

for all ζ functions in statics and dynamics, regardless of whether we consider the basic model or the dynamically diagonalized model. $\{\alpha_j\}$ denotes the set of static and dynamic model parameters which include the static couplings u and γ (or $\bar{\gamma}_i$), the mode couplings g_i (or $\bar{g}_i$), and all kinetic coefficients Γ, Γ^+,λ,L,μ (or λ_i, respectively). The ζ function ζ_{a_i} is calculated from the renormalization factor Z_{a_i} introduced in the previous section. The critical behavior of the model parameters α_i is determined by the flow equations

$$l\frac{d\alpha_i}{dl} = \beta_{\alpha_i}(\{\alpha_j\}) \tag{73}$$

with β functions

$$\beta_{\alpha_i}(\{\alpha_j\}) = \alpha_i[-c_i + \zeta_{\alpha_i}(\{\alpha_j\})]. \tag{74}$$

c_i is the naive dimension of the corresponding parameter α_i obtained by power counting. For the static couplings $\alpha_i=u$ and γ the cutoff dimension c_i has the values ϵ and $\epsilon/2$, respectively. For the mode couplings $\alpha_i=g_i$ we have $c_i=\epsilon/2$. In the case that α_i is one of the kinetic coefficients Γ, Γ^+, λ, L, or μ we have $c_i=0$ because they are dimensionless quantities. The same naive dimensions are true in the dynamically diagonalized model.

A. Static flow equations

The flow equations of the static couplings u in Eq. (26) and γ in Eq. (12) are

$$l\frac{du}{dl} = \beta_u(u), \tag{75}$$

$$l\frac{d\gamma}{dl} = \beta_\gamma(u,\gamma). \tag{76}$$

The corresponding β functions can be written as

$$\beta_u(u) = u[-\epsilon - 2\zeta_\psi(u) + \zeta_u(u)] \tag{77}$$

and

$$\beta_\gamma(u,\gamma) = \gamma\left(-\frac{\epsilon}{2} + \zeta_{\psi^2}(u) + \frac{1}{2}\gamma^2 B_{\psi^2}(u)\right) \tag{78}$$

where several relations between the demoralization factors discussed in the previous section already have been used. Details have been presented in Ref. [23]. The expressions of the ζ functions appearing in the β functions above are well known. For a complex OP the two-loop expressions are

$$\zeta_\psi(u) = -\frac{u^2}{18}, \tag{79}$$

$$\zeta_u(u) = \frac{5}{3}u\left(1 - \frac{16}{15}u\right), \tag{80}$$

$$\zeta_{\psi^2}(u) = \frac{2}{3}u\left(1 - \frac{5}{12}u\right). \tag{81}$$

In order to include the nonasymptotic behavior in the flow of the parameters, it is necessary to use the nonlinear (not ϵ-expanded) β functions in the flow equations. With the above two-loop expressions for the ζ functions arises a problem with Eqs. (75), (79), and (80) for u. The equation $\beta_u(u^\star)=0$, which determines the fixed points $u^\star$, has no real solution. The corresponding flow $u(l)$ would have no real fixed point that is approached in the asymptotic region. A nonlinear flow for u with a definite fixed point $u^\star/4!$ can only be obtained by using the Borel summed expression of the β function like

$$\beta_u(u) = -\frac{u}{4!} + 40\left(\frac{u}{4!}\right)^2 \frac{(1 + a_4 u/4!)}{(1 + a_5 u/4!)} \tag{82}$$

as is used in pure ^{4}He [24]. The ζ function $\zeta_{\psi^2}(u)$ necessary for the γ flow may be taken from either the two-loop expression (81) or the Borel summed expression

$$\zeta_{\psi^2}(u) = 16\frac{u}{4!}\left(1 - 10\frac{u}{4!}\right) + a_1\left(\frac{u}{4!}\right)^3 - a_2\left(\frac{u}{4!}\right)^4, \tag{83}$$

leading to a slightly different flow of γ in the noncritical background. The coefficients a_1, a_2, a_4, a_5 have been calculated in Ref. [24] (see Table 2 therein) for several n. In the present case ($n=2$) we have $a_1=4851$, $a_2=57\,309$, $a_4=15.11$, and $a_5=34.25$. The u contributions of the function $B_{\psi^2}(u)=1+O(u^2)$ are negligible at the current order of calculation thus we simply have to insert $B_{\psi^2}(u)=1$ into Eq. (78).

B. Dynamic flow equations

The dynamic flow equations will be considered within the scope of the basic model with a nondiagonal matrix of kinetic coefficients. All flows calculated in the following sections are also considered for this set of dynamical parameters. It is convenient to introduce quantities which may have finite fixed point values. This is obtained by (i) considering the time scale ratios

$$w_1 = \frac{\Gamma}{\lambda}, \quad w_2 = \frac{\Gamma}{\mu}, \tag{84}$$

$$w_3 = \frac{L}{\sqrt{\lambda\mu}} \tag{85}$$

in the following instead of the kinetic coefficients themselves, and (ii) using mode coupling parameters

$$f_1 = \frac{F_1}{\sqrt{w_1'}}, \quad f_2 = \frac{F_2}{\sqrt{w_2'}} \tag{86}$$

instead of the mode couplings g_i where we have introduced

$$F_1 = \frac{g_1}{\lambda}, \quad F_2 = \frac{g_2}{\mu}. \tag{87}$$

Note that $w_1=w_1'+iw_1''$ and $w_2=w_2'+iw_2''$ are complex quantities while w_3, f_1, and f_2 are real.

From the definition of the parameters in Eqs. (84), (85), and (87) and the renormalization (61), (65), and (67) we obtain together with Eq. (72) the flow equations

$$l\frac{dw_1}{dl} = w_1(\zeta_\Gamma - \zeta_\lambda), \tag{88}$$

$$l\frac{dw_2}{dl} = w_2(\zeta_\Gamma - \zeta_\mu), \tag{89}$$

$$l\frac{dw_3}{dl} = w_3\left(\zeta_L - \frac{1}{2}\zeta_\lambda - \frac{1}{2}\zeta_\mu\right), \tag{90}$$

$$l\frac{df_1}{dl} = -\frac{f_1}{2}\left[\epsilon + \zeta_\lambda + \mathrm{Re}\left(\frac{\Gamma}{\Gamma'}\zeta_\Gamma\right)\right], \tag{91}$$

$$l\frac{df_2}{dl} = -\frac{f_2}{2}\left[\epsilon + \zeta_\mu - 2\zeta_m + \mathrm{Re}\left(\frac{\Gamma}{\Gamma'}\zeta_\Gamma\right)\right]. \tag{92}$$

Using the relations in Eq. (66) we can separate the static contributions to the ζ functions of the kinetic coefficients. Thus we have

$$\zeta_\lambda = \zeta_\lambda^{(d)}, \quad \zeta_L = \zeta_m + \zeta_L^{(d)}, \quad \zeta_\mu = 2\zeta_m + \zeta_\mu^{(d)}. \tag{93}$$

Inserting into the flow equations leads to

$$l\frac{dw_1}{dl} = w_1(\zeta_\Gamma - \zeta_\lambda^{(d)}), \tag{94}$$

$$l\frac{dw_2}{dl} = w_2(\zeta_\Gamma - \zeta_\mu^{(d)} - 2\zeta_m), \tag{95}$$

$$l\frac{dw_3}{dl} = w_3\left(\zeta_L^{(d)} - \frac{1}{2}\zeta_\lambda^{(d)} - \frac{1}{2}\zeta_\mu^{(d)}\right), \tag{96}$$

$$l\frac{df_1}{dl} = -\frac{f_1}{2}\left[\epsilon + \zeta_\lambda^{(d)} + \mathrm{Re}\left(\frac{\Gamma}{\Gamma'}\zeta_\Gamma\right)\right], \tag{97}$$

$$l\frac{df_2}{dl} = -\frac{f_2}{2}\left[\epsilon + \zeta_\mu^{(d)} + \mathrm{Re}\left(\frac{\Gamma}{\Gamma'}\zeta_\Gamma\right)\right]. \tag{98}$$

By separating the static from the dynamic parts in the ζ functions one can take advantage of the general structures appearing in the purely dynamic ζ functions $\zeta_i^{(d)}$ as well as in the static ζ function ζ_m. From relation (54) it follows immediately that ζ_m can be written as

$$\zeta_m(u,\gamma) = \frac{1}{2}\gamma^2 B_{\psi^2}(u) \tag{99}$$

valid in all orders of perturbation expansion. From the diagrammatic structure of the dynamic perturbation theory it follows that [10]

$$\zeta_\lambda^{(d)} = -\frac{f_1^2}{2}(1+\mathcal{Q}), \tag{100}$$

$$\zeta_\mu^{(d)} = -\frac{f_2^2}{2}(1+\mathcal{Q}), \tag{101}$$

$$\zeta_L^{(d)} = -\frac{f_1 f_2}{2w_3}(1+\mathcal{Q}). \tag{102}$$

The real function $\mathcal{Q}$ contains all higher order contributions beginning with two-loop order. Knowing this function determines all three ζ functions. Setting $\mathcal{Q}=0$ in Eqs. (100)–(102) reproduces the one-loop expressions of these functions. We have defined seven dynamical parameters in Eqs. (84)–(86). In the model only five dynamical kinetic coefficients appear (Γ is complex) and the mode couplings g_1 and g_2 are static quantities; therefore two exact relations between the above parameters have to exist. The first one follows from the fact that the imaginary parts of w_1 and w_2 have their origin in the same parameter Γ''. This leads to the relation

$$w_1'' = \frac{w_1'}{w_2'}w_2''. \tag{103}$$

The second relation follows from the property that the mode couplings g_i and also their ratio g_1/g_2 are purely static quantities. This leads to

$$w_1' = \frac{g_2^2 f_1^2}{g_1^2 f_2^2}w_2' = \frac{g_2 F_1}{g_1 F_2}w_2'. \tag{104}$$

The remaining task is to calculate the explicit expressions of the dynamic functions ζ_Γ and $\mathcal{Q}$ in two-loop order.

VI. DYNAMIC FUNCTIONS IN TWO-LOOP ORDER

The perturbation expansion of the dynamic vertex functions and the structures therein are outlined in detail in Appendixes A and B. The results for the pole terms of the integrals necessary for the renormalization factors are presented in Appendix C. The outcoming expressions for the static and dynamic renormalization factors in two-loop order are presented in Appendix D. The calculation is performed within the dynamic diagonal model (18)–(20). Relation (62) between the Z factors implies the relation between the corresponding ζ functions

$$\zeta_\Gamma = \zeta_\Gamma^{(d)} - \frac{1}{2}\zeta_{\tilde{\psi}^+} + \frac{1}{2}\zeta_\psi. \tag{105}$$

Inserting Eqs. (D1), (D5), and (D6) into Eqs. (72) and (105), we obtain the ζ function

$$\zeta_\Gamma = \sum_i \frac{\bar{D}_i^2}{\bar{w}_i(1+\bar{w}_i)} - \frac{2}{3}\sum_i \frac{u\bar{D}_i}{\bar{w}_i(1+\bar{w}_i)}A_i - \frac{1}{2}\sum_{i,j}\frac{\bar{D}_i\bar{D}_j}{\bar{w}_i(1+\bar{w}_i)\bar{w}_j(1+\bar{w}_j)}B_{ij} + \zeta_\Gamma^{(A^\star)}(u,\Gamma,\Gamma^+) \tag{106}$$

where we have introduced the coupling

$$\bar{D}_i = \bar{w}_i\bar{\gamma}_i - i\bar{F}_i. \tag{107}$$

The functions A_i and B_{ij} are defined as

$$A_i = \bar{w}_i\bar{\gamma}_i(1 - x_1 L_1) + i\bar{F}_i x_- x_1 L_1 - \bar{D}_i L_0, \tag{108}$$

$$B_{ij} = \bar{w}_i\bar{\gamma}_i\bar{w}_j\bar{\gamma}_j(1 - 2x_1L_1) - i\bar{F}_i i\bar{F}_j(2x_- L_1 + L_R) + (\bar{w}_i\bar{\gamma}_i i\bar{F}_j + \bar{w}_j\bar{\gamma}_j i\bar{F}_i)(1 + 2x_- x_1 L_1) - \bar{D}_i\bar{D}_j\Bigg(2L_0 + \frac{1}{1+\bar{w}_i}[\bar{w}_i + \bar{w}_j^2 l_{ij}^{(a)} - \bar{w}_i^2 l_{ji}^{(a)} + (1+\bar{w}_i - \bar{w}_j)(1+\bar{w}_i+\bar{w}_j)l_{ij}^{(s)}]\Bigg), \tag{109}$$

with

$$L_R = \left\{x_+ + \frac{\Gamma}{\Gamma^+} + x_+^2\left[x_+^2 + 2\left(\frac{\Gamma}{\Gamma^+}\right)^2\right]\right\}\frac{L_1}{x_+} - 3\frac{\Gamma}{\Gamma^+}. \tag{110}$$

In the above expressions we have used the following definitions:

$$x_\pm = 1 \pm \frac{\Gamma}{\Gamma^+}, \quad x_1 = 2 + \frac{\Gamma}{\Gamma^+}, \tag{111}$$

$$L_0 = 2\ln\frac{2}{1+\Gamma^+/\Gamma}, \quad L_1 = \ln\frac{(1+\Gamma^+/\Gamma)^2}{1+2\Gamma^+/\Gamma}, \tag{112}$$

$$l_{ij}^{(s)} = \ln\frac{(1+\bar{w}_i)(1+\bar{w}_j)}{1+\bar{w}_i+\bar{w}_j}, \quad l_{ij}^{(a)} = \ln\frac{1+\bar{w}_i}{1+\bar{w}_i/\bar{w}_j}. \tag{113}$$

$\zeta_\Gamma^{(A^\star)}(u,\Gamma,\Gamma^+)$ is the ζ function of model A*, model A with complex relaxation rate. It reads in two-loop order

$$\zeta_\Gamma^{(A^\star)}(u,\Gamma,\Gamma^+) = \frac{u^2}{9}\left(L_0 + x_1 L_1 - \frac{1}{2}\right). \quad (114)$$

The function $\mathcal{Q}$ in the dynamic ζ functions of the secondary densities (100)–(102) has the structure

$$\mathcal{Q} = \frac{1}{2}\,\mathrm{Re}(X_2), \quad (115)$$

from which it immediately follows that it is a real quantity. X_2 reads

$$X_2 = \sum_k \frac{\bar{D}_k}{\bar{w}_k'(1+\bar{w}_k)}\Bigg[\bar{D}_k\left(\frac{1}{2} + \ln\frac{1+\bar{w}_k}{1+\bar{w}_k^+}\right) + \bar{D}_k^+(1+\bar{w}_k) - (W_k^{(m)}\bar{\gamma}_k + \bar{w}_k i\bar{F}_k)W_k^{(m)}L_k^{(m)}\Bigg], \quad (116)$$

where we have introduced the definitions

$$L_k^{(m)} = \ln\left(1 + \frac{1}{W_k^{(m)}}\right), \quad (117)$$

$$W_k^{(m)} = \bar{w}_k + \bar{w}_k^+ + \bar{w}_k\bar{w}_k^+. \quad (118)$$

Each term in the sum over the index k contained in X_2 coincides with the corresponding function in model F in Ref. [8]. Because the functions ζ_Γ and $\mathcal{Q}$ are invariant under transformation (16) we obtain them as functions of the parameters of the dynamically nondiagonal model (basic model), as used in Sec. V B, by transforming the time scale ratios and couplings. From the transformation rules for the model parameters in Sec. II B it follows for the time scale ratios

$$\bar{w}_1 = \frac{2w_1w_2}{w_1+w_2+K_w}, \quad \bar{w}_2 = \frac{2w_1w_2}{w_1+w_2-K_w}, \quad (119)$$

with

$$K_w = \sqrt{(w_2-w_1)^2 + 4w_1w_2w_3^2}. \quad (120)$$

The mode coupling parameters transform as

$$\bar{F}_1 = \frac{2(w_2R_{11}F_1 + w_1R_{21}F_2)}{w_1+w_2+K_w},$$

$$\bar{F}_2 = \frac{2(-w_2R_{21}F_1 + w_1R_{11}F_2)}{w_1+w_2-K_w}. \quad (121)$$

For the static couplings we have

$$\bar{\gamma}_1 = R_{21}\gamma, \quad \bar{\gamma}_2 = R_{11}\gamma. \quad (122)$$

Introducing the time scale ratios the transformation matrix elements R_{11} and R_{21} in Eq. (17) can be rewritten as

$$R_{11} = \sqrt{\frac{w_2-w_1+K_w}{2K_w}}, \quad R_{21} = \sqrt{\frac{w_1-w_2+K_w}{2K_w}}. \quad (123)$$

In Ref. [13] (see Appendix B 1 there) it has been shown that no additional terms appear in the renormalization of $\boldsymbol{R}$ in comparison to model E′ [10].

VII. SUBCLASSES AND LIMITS OF MODEL F′

A. Reduced models contained in model F′

ζ_Γ in Eq. (106) includes the result of several reduced models. Some of them have been presented in the literature by several authors. The models differ in the existence of static couplings or mode couplings and may have a real or complex kinetic coefficient for the order parameter. In order to present an overview we have divided all these models into classes.

(i) Model A class. These models only describe the relaxation of a nonconserved OP without any coupling to secondary densities. Neglecting all static couplings ($\bar{\gamma}_i=0$) and mode couplings ($\bar{F}_i=0$) only the last term in Eq. (106) remains. The resulting function is equal to the corresponding ζ function in model A⋆ (complex Γ),

$$\zeta_\Gamma(u,\bar{\gamma}_i=0,\bar{w}_i,\bar{F}_i=0) = \zeta_\Gamma^{(A^\star)}(u,\Gamma,\Gamma^+), \quad (124)$$

as presented in Ref. [25]. Setting also the imaginary part of Γ equal to zero the kinetic coefficient Γ completely drops out [$L_0=0$ and $x_1L_1=3\ln(4/3)$ in this case] and we obtain

$$\zeta_\Gamma(u,\bar{\gamma}_i=0,\bar{w}_i',\bar{w}_i''=0,\bar{F}_i=0) = \zeta_\Gamma^{(A)}(u) \quad (125)$$

which has been denoted as model A (real Γ) in the literature.

(ii) Model C class. This model class only includes dissipative processes for the OP and one or two secondary densities. Setting the mode couplings $\bar{F}_i=0$, expression (106) reduces to the corresponding function in model C′⋆ (two couplings $\bar{\gamma}_i$ and a complex Γ),

$$\zeta_\Gamma(u,\bar{\gamma}_i,\bar{w}_i,\bar{F}_i=0) = \zeta_\Gamma^{(C'^\star)}(u,\bar{\gamma}_i,\bar{w}_i), \quad (126)$$

as presented in Ref. [16]. Of course all models like models C′ (two couplings $\bar{\gamma}_i$ and a real Γ), model C⋆ (one coupling $\bar{\gamma}$ and a complex Γ), and model C (one coupling $\bar{\gamma}$ and a real Γ), where the latter two have been discussed extensively in Ref. [23], are obtained by setting the imaginary part of Γ and/or one of the couplings $\bar{\gamma}_i$ equal to zero.

(ii) Model E class. This class of models includes the full dynamic equations (18)–(20) but without static couplings $\bar{\gamma}_i$ between the OP and the secondary densities leading to trivial thermodynamic properties (all thermodynamic functions are constants). Setting $\bar{\gamma}_i=0$ in Eq. (106) we obtain

$$\zeta_\Gamma(u,\bar{\gamma}_i=0,\bar{w}_i,\bar{F}_i) = \zeta_\Gamma^{(E'^\star)}(u,\bar{w}_i,\bar{F}_i), \quad (127)$$

which represents model E′⋆ (two mode couplings $\bar{F}_i$ and a complex Γ). The ζ function of model E′ (two mode couplings $\bar{F}_i$ and a real Γ) is obtained when additionally the imaginary part of Γ vanishes ($\bar{w}_i''=0$). We obtain

$$\zeta_\Gamma(u,\bar{\gamma}_i=0,\bar{w}_i',\bar{w}_i''=0,\bar{F}_i) = \zeta_\Gamma^{(E')}(u,\bar{w}_i',\bar{F}_i) \quad (128)$$

in agreement with the result presented in [10]. The functions for model E⋆ (one mode coupling $\bar{F}$ and a complex Γ) and model E (one mode coupling $\bar{F}$ and a real Γ), where the latter is already given in [26], are obtained by setting the imaginary part of Γ and/or one of the mode couplings $\bar{F}_i$ equal to zero. The result for model E⋆

$$\zeta_\Gamma^{(E^\star)}(u,w,F) = -\frac{F^2}{w(1+w)}\left(1+\frac{2}{3}u(L_0+x\_x_1L_1) - \frac{1}{2}\frac{F^2}{w(1+w)}b_{E^\star}\right) + \zeta_\Gamma^{(A^\star)}(u,\Gamma,\Gamma^+) \tag{129}$$

with

$$b_{E^\star} = 2(L_0+x\_L_1)+L_R+\frac{1}{1+w}\left(w+(1+2w)\ln\frac{(1+w)^2}{1+2w}\right) \tag{130}$$

can be immediately obtained from Eq. (106) by choosing $\bar{\gamma}_i=0$, $\bar{F}_1=0$, $\bar{w}_2=w$, $\bar{F}_2=F$. This expression has been confirmed recently by Dohm [9].

(iv) Model F. This model describes the critical dynamics of pure ^{4}He at the superfluid transition. In this case only one secondary density, and consequently one static coupling γ and one mode coupling g, is needed in the model equations. Setting the couplings of one of the two secondary densities in Eq. (106) equal to zero, for instance $\bar{\gamma}_1=0$, $\bar{F}_1=0$, and introducing $\bar{\gamma}_2=\gamma$, $\bar{w}_2=w$, $\bar{F}_2=F$ for the parameters of the other secondary density, we obtain from

$$\zeta_\Gamma(u,\bar{\gamma}_1=0,\bar{\gamma}_2,\bar{w}_i,\bar{F}_1=0,\bar{F}_2) = \zeta_\Gamma^{(F)}(u,\gamma,w,F) \tag{131}$$

the corresponding ζ function of model F. From Eq. (106) we obtain

$$\zeta_\Gamma^{(F)}(u,\gamma,w,F) = \frac{D^2}{w(1+w)} - \frac{2}{3}\frac{uD}{w(1+w)}A - \frac{1}{2}\frac{D^2}{w^2(1+w)^2}B + \zeta_\Gamma^{(A^\star)}(u,\Gamma,\Gamma^+). \tag{132}$$

The functions A and B follow from Eqs. (108) and (109) as

$$A = w\gamma(1-x_1L_1) + iFx\_x_1L_1 - DL_0, \tag{133}$$

$$B = w^2\gamma^2(1-2x_1L_1) - (iF)^2(2x\_L_1+L_R) + 2w\gamma iF(1+2x\_x_1L_1) - D^2\left[2L_0+\frac{1}{1+w}\left(w+(1+2w)\ln\frac{(1+w)^2}{1+2w}\right)\right]. \tag{134}$$

Agreement with our expression [14,15] has recently been obtained (see Ref. [9]).

B. Nonasymptotic limits of model F′

Since model F′ describes the critical dynamics of ^{3}He-^{4}He mixtures it is evident that the flow of the time scale ratios w_1, w_2, and w_3 and the mode couplings f_1 and f_2 depends on the mole fraction X of ^{3}He. The question is which limiting ζ functions, and therefore flows, are reached when several dynamic parameters assume special values.

The key parameter is the time scale ratio w_3 which determines the dissipative coupling between the two secondary densities. Its values may range from $w_3=0$ on the one hand, which is the case of a decoupling of the dissipative processes of the two secondary densities, and $w_3=\pm 1$ on the other hand, describing total coupling in the sense that the dissipation of two secondary densities is determined only by one density. In the second case a transformation to new secondary densities can be found where the first one satisfies a dynamic equation with a diffusion term while the dynamics of the second one is determined by a time reversible equation without any dissipation. The values of w_3^2 cannot exceed 1 because the matrix (7) of dynamic coefficients has to be positive definite.

1. Decoupled system at $w_3=0$

In this case we have $K_w=w_2-w_1$ from Eq. (120) and the components (123) of the dynamic transformation matrix reduce to $R_{11}=1$ and $R_{21}=0$. Inserting into Eqs. (119)–(122) leads to

$$\bar{\gamma}_1 = 0, \quad \bar{\gamma}_2=\gamma,$$
$$\bar{w}_1 = w_1, \quad \bar{w}_2 = w_2,$$
$$\bar{F}_1 = F_1, \quad \bar{F}_2 = F_2. \tag{135}$$

Inserting the above relations into Eq. (106) we obtain

$$\zeta_\Gamma(u,\gamma,w_1,w_2,w_3=0,F_1,F_2) = \zeta_\Gamma^{(F)}(u,\gamma,w_2,F_2) + \zeta_\Gamma^{(E^\star)}(u,w_1,F_1) - \zeta_\Gamma^{(A^\star)}(u,\Gamma,\Gamma^+) + \frac{F_1^2D_2}{w_1w_2(1+w_1)(1+w_2)}Y_0(\gamma,w_1,w_2,F_2). \tag{136}$$

From Eq. (129) it immediately follows that $\zeta_\Gamma^{(E^\star)}-\zeta_\Gamma^{(A^\star)}$ is proportional to F_1^2. Thus in the limit $F_1\to 0$ the ζ function of model F is obtained, i.e.,

$$\lim_{F_1\to 0}\lim_{w_3\to 0}\zeta_\Gamma(u,\gamma,w_1,w_2,w_3,F_1,F_2) = \zeta_\Gamma^{(F)}(u,\gamma,w_2,F_2). \tag{137}$$

The above relation describes the transition of the flow in ^{3}He-^{4}He mixtures to the flow in pure ^{4}He when the mole fraction X tends to zero. In fact the results for the flow presented in Ref. [27] revealed that the functions $w_3(l)$ and $f_1(l)$ [and therefore also $F_1(l)$], which have been found by fitting the experimental thermal conductivity and thermal diffusion ratio at several mole fractions, satisfy this property for small X. At $X=0$ the flow for w_2 and f_2 (and also F_2) turns into the flow of pure ^{4}He.

The limit $F_2\to 0$ alone does not lead to model F; one has in addition to perform the limit $\gamma\to 0$ in order to obtain model E.

2. Totally coupled system at $w_3=\pm 1$

In this case we have $K_w=w_2+w_1$ from Eq. (120) and the components (123) of the dynamic transformation matrix reduce to

$$R_{11}=\sqrt{\frac{w}{w_1}},\quad R_{21}=\pm\sqrt{\frac{w}{w_2}} \tag{138}$$

with

$$\frac{1}{w}\equiv\frac{1}{w_1}+\frac{1}{w_2}. \tag{139}$$

The sign of R_{21} is determined by the sign of w_3. Inserting into Eqs. (119) and (122) leads to

$$\bar{\gamma}_1=\pm\sqrt{\frac{w}{w_2}}\gamma,\quad \bar{\gamma}_2=\sqrt{\frac{w}{w_1}}\gamma,$$

$$\frac{1}{\bar{w}_1}=\frac{1}{w_1}+\frac{1}{w_2}=\frac{1}{w},\quad \frac{1}{\bar{w}_2}=0. \tag{140}$$

The diverging $\bar{w}_2$ would lead to a diverging mode coupling $\bar{F}_2$, and $\bar{f}_2$, respectively, except in one special case. Assuming that $w_3^2=1-\delta$ is in the vicinity of its limiting value 1, from Eq. (119), which is also valid for the real parts $\bar{w}_1'$ and $\bar{w}_2'$, it follows that

$$\bar{w}_2'=\frac{w_1'w_2'}{w'\delta}. \tag{141}$$

$\bar{w}_2'$ diverges when δ goes to zero. Using this result together with Eq. (121) we obtain for the coupling $\bar{f}_2=\bar{F}_2/\sqrt{\bar{w}_2'}$ the expression

$$\bar{f}_2=\frac{f_2\mp f_1}{\sqrt{\delta}}. \tag{142}$$

Again the sign in the above expression is connected to the sign of w_3. The mode coupling $\bar{f}_2$ stays finite only if we have

$$\frac{f_1}{f_2}=\pm(1-\sqrt{\delta}). \tag{143}$$

We obtain then for the mode couplings

$$\bar{f}_1=f_1=\pm\bar{f}_2=\pm f_2\equiv f. \tag{144}$$

When the two conditions $w_3^2=1$ and $f_1=\pm f_2$ are inserted into Eq. (106) it yields a function of the form

$$\zeta_\Gamma(u,\bar{\gamma}_i,\bar{w}_i,w_3=\pm1,f_1=\pm f_2)$$
$$=\zeta_\Gamma^{(F)}(u,\bar{\gamma}_1,w,f)+\bar{\gamma}_2^2Y_1(u,\bar{\gamma}_i,\bar{w}_2\to\infty,w,f), \tag{145}$$

where the function Y_1 diverges with $\bar{w}_2\to\infty$. A finite ζ function is obtained only in the case $\gamma=0$. Then we have $\bar{\gamma}_1=\bar{\gamma}_2=0$ and we obtain

$$\lim_{\gamma\to0}\lim_{\substack{w_3\to\pm1\\ f_1\to\pm f_2}}\zeta_\Gamma(u,\gamma,w_1,w_2,w_3,F_1,F_2)=\zeta_\Gamma^{(E^\star)}(u,w,f). \tag{146}$$

In the considered limit it is not possible to obtain a model F function and therefore the corresponding flow. The nonasymptotic behavior turns into the behavior of the model E*.

VIII. FIXED POINTS AND STABILITY

Within the GLW model (26) it is known that the fixed point $u^\star=u_H$ of the two-component Heisenberg model or *XY* model is the stable one. Its numerical value depends on the method of calculation, i.e., whether or not ϵ expansion is used. The value of a nonlinear solution of the equation $\beta_u(u^\star)=0$ with the β function (82) is $u_H/4!=0.0362$. For a two-component order parameter as in the current case the stable fixed point for the static coupling γ is $\gamma^\star=0$ (for a discussion of the static fixed points see, for instance, Ref. [23]). As a matter of fact this implies $\zeta_m(u_H,0)=0$ [see Eq. (99)]. The fixed points of the dynamic parameters are determined by the zeros of the right hand sides of Eqs. (94)–(98). Inserting the static fixed point $u^\star=u_H$, $\gamma^\star=0$ we obtain the equations

$$w_1^\star(\zeta_\Gamma^\star-\zeta_\lambda^{\star(d)})=0, \tag{147}$$

$$w_2^\star(\zeta_\Gamma^\star-\zeta_\mu^{\star(d)})=0, \tag{148}$$

$$w_3^\star\left(\zeta_L^{\star(d)}-\frac{1}{2}\zeta_\lambda^{\star(d)}-\frac{1}{2}\zeta_\mu^{\star(d)}\right)=0, \tag{149}$$

$$f_1^\star\left[\epsilon+\zeta_\lambda^{\star(d)}+\mathrm{Re}\left(\frac{\Gamma^\star}{\Gamma'^\star}\zeta_\Gamma^\star\right)\right]=0, \tag{150}$$

$$f_2^\star\left[\epsilon+\zeta_\mu^{\star(d)}+\mathrm{Re}\left(\frac{\Gamma^\star}{\Gamma'^\star}\zeta_\Gamma^\star\right)\right]=0. \tag{151}$$

From Eqs. (147) and (148) it follows that in any case $\zeta_\Gamma^\star=\zeta_\Gamma'^\star$ is at the fixed point real because the ζ functions corresponding to the secondary densities are all real functions. The origins of a nonzero ζ_Γ'' are (i) the imaginary part Γ'' of the OP kinetic coefficient, which is proportional to w_1'' and w_2'', and (ii) the product $ig_2\gamma$. At the static fixed point with $\gamma^\star=0$ the latter term does not contribute; thus the imaginary part of the OP relaxation coefficient $\Gamma''^\star$ could be the only origin of $\zeta_\Gamma''^\star$, but since $\zeta_\Gamma^\star$ is real $\Gamma''^\star$ has to be zero. At all fixed points with $\gamma^\star=0$ we have the condition $\Gamma''^\star=0$ in order to obtain

$$\zeta_\Gamma''^\star=0. \tag{152}$$

Therefore the fixed points of the imaginary parts of the time scale ratios are

$$w_1''^\star=w_2''^\star=0 \tag{153}$$

independent of the fixed point values of the other dynamic parameters. Then Eqs. (150) and (151) simplify to

$$f_1^\star(\epsilon+\zeta_\lambda^{\star(d)}+\zeta_\Gamma'^\star)=0, \tag{154}$$

$$f_2^\star(\epsilon+\zeta_\mu^{\star(d)}+\zeta_\Gamma'^\star)=0. \tag{155}$$

In order to obtain a complete set of fixed points it is convenient to introduce

TABLE I. Fixed points of model F′: $\gamma^{\star}=0$, $w_1''^{\star}=0$, and $w_2''^{\star}=0$. From the equations for the ζ functions at the fixed point one gets fixed point values and/or the correction exponents. The values f_{NSC}, f_∞, f_{SC}, and w_{SC} are those found in model F and E, respectively. The connection of the concentration dependent fixed points $w_1'^{\star}(X)$ and $w_2'^{\star}(X)$ and w_{SC} is given in Eq. (190).

$f_1^\star$	$f_2^\star$	$w_3^\star$	$w_1'^\star$	$w_2'^\star$	ζ-functions
0	0	$w_3^\star$	0, ∞	0, ∞	$\zeta_\Gamma'^\star=\zeta_\Gamma^{(A)}(u_H)$
f_{NSC}			0	0, ∞	$\zeta_\lambda^{\star(d)}+\zeta_\Gamma'^\star=-\epsilon$
f_∞	0	0	∞	0, ∞	
f_{SC}			w_{SC}	0, ∞	$\zeta_\Gamma'^\star=\zeta_\lambda^{\star(d)}=-\dfrac{\epsilon}{2}$
	f_{NSC}		0, ∞	0	$\zeta_\mu^{\star(d)}+\zeta_\Gamma'^\star=-\epsilon$
0	f_∞	0	0, ∞	∞	
	f_{SC}		0, ∞	w_{SC}	$\zeta_\Gamma'^\star=\zeta_\mu^{\star(d)}=-$
$\pm f_{NSC}$	f_{NSC}		0	0	$\zeta_\lambda^{\star(d)}+\zeta_\Gamma'^\star=-\epsilon$
$\pm f_\infty$	f_∞	±1	∞	∞	$\zeta_\lambda^{\star(d)}=\zeta_\mu^{\star(d)}=\zeta_L^{\star(d)}$
$\pm f_{SC}$	f_{SC}		$w_1'^\star(X)$	$w_2'^\star(X)$	$\zeta_\Gamma'^\star=\zeta_\lambda^{\star(d)}=\zeta_\mu^{\star(d)}=\zeta_L^{\star(d)}=-\epsilon/2$

$$\rho_i=\frac{w_i'}{1+w_i'},\quad i=1,2. \tag{156}$$

The values $0,\ldots,\infty$ adopted by w_i' are then mapped to the finite region of values $0,\ldots,1$ adopted by ρ_i. At the fixed points Eqs. (147) and (148) are replaced by

$$\rho_1^\star(1-\rho_1^\star)(\zeta_\Gamma'^\star-\zeta_\lambda^{\star(d)})=0, \tag{157}$$

$$\rho_2^\star(1-\rho_2^\star)(\zeta_\Gamma'^\star-\zeta_\mu^{\star(d)})=0. \tag{158}$$

The allowed fixed point values of ρ_1, ρ_2, and w_3 depend on the behavior of the $f_i^\star$ (see Table I). We have to consider four cases.

(i) $f_1^\star=0$, $f_2^\star=0$. In this case we simply have $\zeta_\lambda^{\star(d)}=\zeta_\mu^{\star(d)}=\zeta_L^{\star(d)}=0$ following from Eqs. (100)–(102). Equations (157) and (158) reduce to

$$\rho_1^\star(1-\rho_1^\star)\zeta_\Gamma'^\star=0,\quad \rho_2^\star(1-\rho_2^\star)\zeta_\Gamma'^\star=0 \tag{159}$$

while the expression in the parentheses of Eq. (149) is zero and $w_3^\star$ can be an arbitrary value because the equation is trivially fulfilled. $\zeta_\Gamma'^\star=\zeta_\Gamma'(u_H,0,\rho_1^\star,\rho_2^\star,w_3^\star,0,0)=\zeta_\Gamma^{(A)}(u_H)$ reduces to the model A function, which is finite at the fixed point. Thus each of the $\rho_i^\star$ has to be either 0 or 1, that is,

$$\rho_i^\star=0,1\quad (w_i'^\star=0,\infty),\quad i=1,2, \tag{160}$$

to satisfy the equations in (159).

(ii) $f_1^\star=0$, $f_2^\star\neq0$. In this case we have from Eqs. (100)–(102) $\zeta_\lambda^{\star(d)}=\zeta_L^{\star(d)}=0$. From Eq. (155) we obtain the condition

$$\zeta_\mu^{\star(d)}+\zeta_\Gamma'^\star=-\epsilon. \tag{161}$$

From Eq. (158) we obtain the nonscaling fixed point

$$\rho_2^\star=0\quad (w_2'^\star=0), \tag{162}$$

the infinite fixed point

$$\rho_2^\star=1\quad (w_2'^\star=\infty), \tag{163}$$

and the scaling fixed point

$$\rho_2^\star\neq0,1 \Rightarrow \zeta_\Gamma'^\star=\zeta_\mu^{\star(d)}=-\frac{\epsilon}{2} \tag{164}$$

where we have used Eq. (161) for the last equality in Eq. (164). Equations (157) and (149) reduce to

$$\rho_1^\star(1-\rho_1^\star)\zeta_\Gamma'^\star=0,\quad w_3^\star\zeta_\mu^{\star(d)}=0, \tag{165}$$

which can be solved only by

$$w_3^\star=0\quad\text{and}\quad \rho_1^\star=0,1. \tag{166}$$

In the case $w_3=0$ the dynamic transformation $\boldsymbol{R}$ reduces to an identity [see Eq. (135) and the text before]; thus it is not necessary to distinguish between the parameters in the dynamically diagonal and nondiagonal model.

At the nonscaling fixed point (162) the ζ functions (101) and (106) reduce to

$$\zeta_\mu^{(d)\star}=-\frac{f_2^{\star2}}{2}\left(1+\frac{f_2^{\star2}}{4}\right), \tag{167}$$

$$\zeta_\Gamma'^\star=-f_2^{\star2}+\frac{f_2^{\star4}}{2}\left(\frac{27}{2}\ln\frac{4}{3}-3\right)+\zeta_\Gamma^{(A)}(u_H) \tag{168}$$

with the model A contribution (real Γ)

$$\zeta_\Gamma^{(A)}(u_H)=\frac{u_H^2}{18}\left(6\ln\frac{4}{3}-1\right) \tag{169}$$

taken at the fixed point u_H. Equations (167)–(169) lead together with (161) to the equation

$$\left(a-\frac{1}{4}\right)f_{NSC}^4-3f_{NSC}^2+2\epsilon+\frac{u_H^2}{9}b=0 \tag{170}$$

for the nonscaling fixed point $f_2^{\star2}=f_{NSC}^2$. It is the same equation as in pure ^{4}He (model F). Thus we have the same nonscaling fixed point value f_{NSC}^2 as in pure ^{4}He. In Eq. (170) we have introduced the abbreviations $a\equiv 27/2\ln(4/3)-3$ and $b\equiv 6\ln(4/3)-1$. The solution of the equation is

$$f_{NSC}^2=\frac{3}{2\left(a-\frac{1}{4}\right)}\left[1\pm\sqrt{1-\frac{4}{9}\left(a-\frac{1}{4}\right)\left(2\epsilon+\frac{u_H^2}{9}b\right)}\right]. \tag{171}$$

Inserting the Borel resummed value $u_H/4!=0.0362$ into Eq. (171) we obtain a nonscaling fixed point value $f_{NSC}^2=0.8339$ at $d=3$ ($\epsilon=1$), which is in agreement with calculations in Ref. [8].

At the infinite fixed point (163) the ζ functions (101) and (106) reduce to

$$\zeta_\mu^{(d)\star}=-\frac{f_2^{\star 2}}{2}, \tag{172}$$

$$\zeta_\Gamma'^{\star}=\zeta_\Gamma^{(A)}(u_H). \tag{173}$$

From Eq. (161) we obtain immediately

$$f_\infty^2=2\left(\epsilon-\frac{u_H^2}{18}b\right) \tag{174}$$

which is $f_\infty^2=0.9696$ at $d=3$ ($\epsilon=1$).

At the scaling fixed point the ζ functions are equal to the corresponding functions in model E, which have been discussed extensively in Ref. [26]. We can write

$$\zeta_\mu^{(d)\star}=-\frac{f_2^{\star 2}}{2}[1+f_2^{\star 2}N(w_2'^{\star})], \tag{175}$$

$$\zeta_\Gamma'^{\star}=-\frac{f_2^{\star 2}}{1+w_2'^{\star}}[1-f_2^{\star 2}M(w_2'^{\star})]+\zeta_\Gamma^{(A)}(u_H) \tag{176}$$

with functions

$$N(w_2'^{\star})=\frac{1}{2(1+w_2'^{\star})}\left(\frac{1}{2}+w_2'^{\star}-w_2'^{\star 2}(2+w_2'^{\star})\times\ln\frac{(1+w_2'^{\star})^2}{w_2'^{\star}(2+w_2'^{\star})}\right) \tag{177}$$

and

$$M(w_2'^{\star})=\frac{1}{2(1+w_2'^{\star})}\left[\frac{27}{2}\ln\frac{4}{3}-3+\frac{1}{1+w_2'^{\star}}\left(w_2'^{\star}+(1+2w_2'^{\star})\ln\frac{(1+w_2'^{\star})^2}{1+2w_2'^{\star}}\right)\right]. \tag{178}$$

The above functions may easily be expressed as functions of $\rho_2^\star$ by inserting $w_2'^{\star}=\rho_2^\star/(1-\rho_2^\star)$. The two conditions in (161) and (164) determine the values w_{SC} and f_{SC} for the scaling fixed point. Inserting Eq. (175) and (176) into these conditions we obtain

$$f_{SC}^2[1+f_{SC}^2N(w_{SC})]-\epsilon=0, \tag{179}$$

$$\frac{f_{SC}^2}{2}[1+f_{SC}^2N(w_{SC})]-\frac{f_{SC}^2}{1+w_{SC}}[1-f_{SC}^2M(w_{SC})]+\zeta_\Gamma^{(A)}(u_H)=0. \tag{180}$$

The solution for w_{SC} and f_{SC} depends on whether the ϵ-expanded value or the Borel resummed value u_H is used. At $d=3$ ($\epsilon=1$) we obtain

$$w_{SC}=0.0185,\quad f_{SC}=0.8268 \tag{181}$$

with the ϵ-expanded value $u_H/4!=\epsilon/40$, which is in agreement with Refs. [26,28]. Using the value $u_H/4!=0.0362$ from Borel resummation no solution in $d=3$ exists since the borderline ϵ_b below which a finite solution for w_{SC} exists is smaller than 1.

(iii) $f_2^\star=0$, $f_1^\star\neq 0$. This case is quite analogous to case (ii). One has only to interchange the ζ functions $\zeta_\mu^{\star(d)}$ and $\zeta_\lambda^{\star(d)}$ as well as the fixed points $w_2'^{\star}$ and $w_1'^{\star}$ there.

(iv) $f_1^\star\neq 0, f_2^\star\neq 0$. Equations (154) and (155) imply the relations

$$\zeta_\lambda^{\star(d)}+\zeta_\Gamma'^{\star}=-\epsilon,\quad \zeta_\mu^{\star(d)}+\zeta_\Gamma'^{\star}=-\epsilon. \tag{182}$$

Inserting the general structure (100)–(102) of the dynamic secondary density ζ functions into Eq. (149) we obtain the condition

$$w_3^\star(f_1^{\star 2}+f_2^{\star 2})-2f_1^\star f_2^\star=0. \tag{183}$$

Assuming nonvanishing fixed point values for the mode couplings f_i the equation can only be satisfied by the two solutions

$$w_3^\star=1,\quad f_1^\star=f_2^\star, \tag{184}$$

$$w_3^\star=-1,\quad f_1^\star=-f_2^\star. \tag{185}$$

Because $w_3^\star$ is nonvanishing anyway in this case, the expression in parentheses in Eq. (149) has to be zero. Inserting Eq. (182) into it we obtain a third relation

$$\zeta_L^{\star(d)}+\zeta_\Gamma'^{\star}=-\epsilon. \tag{186}$$

From the three relations in (182) and (186) immediately follows $\zeta_\lambda^{\star(d)}=\zeta_\mu^{\star(d)}=\zeta_L^{\star(d)}$.

Relation (104) also has to be valid at the fixed point. In the current case we have $f_1^{\star 2}/f_2^{\star 2}=1$; thus at the fixed point Eq. (104) turns into

$$w_1'^{\star}=\frac{g_2^2}{g_1^2}w_2'^{\star}=\frac{Z_m^2\mathring{g}_2^2}{\mathring{g}_1^2}w_2'^{\star}. \tag{187}$$

The above relation restricts the allowed solutions for $w_1'^{\star}$ and $w_2'^{\star}$ and therefore also for $\rho_1^\star$ and $\rho_2^\star$, which follow from Eqs. (157) and (158). The mode coupling parameters g_i are different from zero in the considered case, therefore we have either a nonscaling fixed point with $w_1'^{\star}=w_2'^{\star}=0$, a fixed point with $w_1'^{\star}=w_2'^{\star}=\infty$, or a scaling fixed point where both time scale ratios are different from zero and infinity. At the scaling fixed point we additionally obtain from Eqs. (147) and (148) the conditions

$$\zeta_\Gamma'^\star = \zeta_\lambda^{\star(d)}, \quad \zeta_\Gamma'^\star = \zeta_\mu^{\star(d)}. \tag{188}$$

This leads together with Eqs. (182) and (186) to

$$\zeta_\Gamma'^\star = \zeta_\lambda^{\star(d)} = \zeta_\mu^{\star(d)} = \zeta_L^{\star(d)} = -\frac{\epsilon}{2}. \tag{189}$$

Nonscaling fixed point. In the case $w_1'^\star = w_2'^\star = 0$ we get from Eq. (122) the fixed point values in the dynamically diagonal model as $\bar{w}_1'^\star = 0$ and $\bar{w}_2'^\star = \infty$ ($w_3^{\star 2} = 1$ already causes the divergence of $\bar{w}_2'^\star$). Inserting into Eqs. (100) and (106) leads to the same functions (167) and (168) as in the previous case (ii) because an infinite time scale ratio has the same effect as a vanishing mode coupling parameter in the ζ functions when γ is equal to zero. The nonscaling fixed point value $f_1^{\star 2} = f_2^{\star 2} = f_{NSC}^2$ is therefore identical to Eq. (171).

Infinite fixed point. The same is true in the case $w_1'^\star = w_2'^\star = \infty$. The fixed point value $f_1^{\star 2} = f_2^{\star 2} = f_\infty^2$ is valid in this case.

Scaling fixed point. Similar to the situation at the nonscaling fixed point the infinite $\bar{w}_2'^\star$ leads to Eqs. (179) and (180). Consequently Eq. (181) is valid for w_{SC} and f_{SC} if the ϵ-expanded value of u_H is used.

From the previous discussion it is clear that in all cases the same nonscaling fixed point value f_{NSC}^2, the same infinite fixed point value f_∞^2, and the same scaling fixed point values w_{SC} and f_{SC}^2 appear. They are all concentration independent and identical to the corresponding values in pure ^{4}He (see the extensive discussion in Ref. [26]). A survey of all fixed points is given in Table I.

The concentration only enters the fixed point values if the concentration independent scaling fixed point $w^\star = w_{SC}$ in mixtures [case (iv)] is separated into $w_1'^\star$ and $w_2'^\star$ via relation (140). Using Eq. (187) we obtain the fixed points

$$w_1'^\star = w_{SC}[1 + B^{-1}(X)], \tag{190}$$

$$w_2'^\star = w_{SC}[1 + B(X)], \tag{191}$$

which depend on the mole fraction X of ^{3}He atoms in ^{4}He. $B(X)$ is

$$B(X) = \frac{g_1^2(X)}{g_2^2(X)}, \tag{192}$$

the ratio of the renormalized mode couplings g_i at the fixed point. It differs from the corresponding parameter in model E′ due to the renormalization of g_2 by a finite factor [see Eq. (187)]. The mode couplings g_i are related to thermodynamic derivatives along the λ line and background values of the specific heat. More details on the connection to experimental quantities have been presented elsewhere [13,23].

Because the ζ functions reduce always to model E functions at the fixed points, the transient exponents, and therefore the stability regions, also behave as in pure ^{4}He, which has been discussed extensively in the literature [6,8], and therefore will be not repeated here. We will give only a short summary of the results.

The stability boundary between the scaling fixed point $w_i'^\star \neq 0$ and the nonscaling fixed point $w_i'^\star = 0$ strongly depends on the ϵ dependence of $u_H(\epsilon)$. Using the loop expanded value $u_H/4! = \epsilon/40$, which is simply linear in ϵ, the scaling fixed point is stable and the nonscaling fixed point is unstable. In contrast using the Borel resummed value the ϵ dependence is nonlinear (see Fig. 2 in Ref. [8]) with $u_H(\epsilon=1)/4! = 0.0362$, and the stability boundary shifts to values of $\epsilon < 1$ so that the nonscaling fixed point is stable and the scaling fixed point is unstable. Because the scaling fixed point value is very close to zero [see Eq. (181)] and the corresponding transient exponent is also very small, there will be no noticeable difference in the critical behavior of the model between the scaling and nonscaling fixed point in any experimentally accessible temperature region.

Using Eqs. (82) and (83) from the Borel resummation procedure as the most convenient representation of the static functions, the following dynamic fixed points from Table I are stable:

(i) Mixture. At any finite mole fraction X the nonscaling fixed point

$$w_3^\star = \pm 1, \quad f_1^\star = \pm f_2^\star = \pm f_{NSC}, \quad w_1'^\star = w_2'^\star = 0 \tag{193}$$

determines the critical behavior of the system. The solutions of the flow equations will always end up in this fixed point when at least one of the initial values $w_3(l_0)$ and $f_1(l_0)$ is different from zero [we have always $w_i'(l_0) \neq 0$].

(ii) Pure ^{4}He. Model F′ reduces completely (also in the nonasymptotic region) to model F for $w_3 \equiv 0$, $f_1 \equiv 0$ (decoupled case) as discussed in Sec. VII B. Starting at initial values $w_3(l_0) = 0$ and $f_1(l_0) = 0$, the parameters remain at these values when the flow equations are solved. The flow of the remaining parameters is now identical to the corresponding flow in model F. The stable fixed point in this case is

$$w_3^\star = f_1^\star = 0, \quad f_2^\star = f_{NSC}, \quad w_1'^\star = \infty, \quad w_2'^\star = 0. \tag{194}$$

The fixed point values of the time scale ratios w_1' and w_2' in the above case correspond to the case where a mode coupling f_2 is present. In models without mode couplings (for instance model C′) the flow of the time scale ratios in the decoupled case ($w_3 = 0$) runs into the fixed point $w_1'^\star = w_2'^\star = 0$.

IX. NONASYMPTOTIC BEHAVIOR AND DECOUPLING POINT

An examination of the concentration dependence of the model parameters reveals the existence of two special mole fractions where one mode coupling, namely, $\mathring{f}_1$, which is proportional to (for the connection of the model parameters to experimental quantities see Ref. [13])

$$\mathring{g}_1 = a_1\left[\sigma + \left(\frac{\partial \Delta}{\partial T}\right)_{P\lambda} c\right], \tag{195}$$

goes to zero in the background. The first is $X = 0$, the case for pure ^{4}He, where a_1 goes to zero as $\sqrt{X}$. The second is the mole fraction $X_D = 0.37$, which is called the decoupling point, where the expression in square brackets goes to zero [4]. In both cases $\mathring{g}_1 = 0$ leads to an initial value $f_1(l_0) = 0$ for the flow which represents to some degree a decoupling of the secondary densities. The major difference between the two points is the behavior of $w_3(l_0)$.

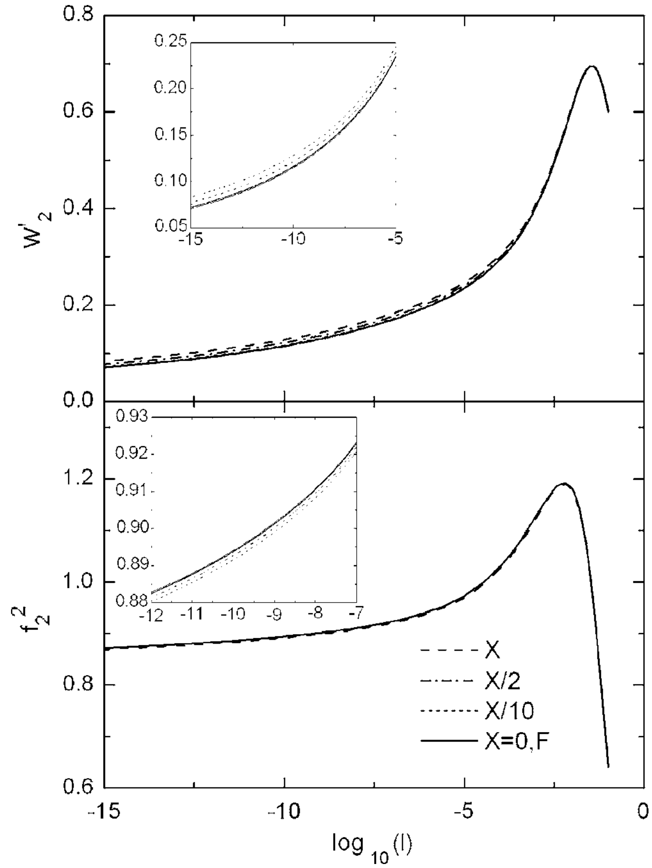

FIG. 1. Crossover to pure ^{4}He: Shown are the dynamic parameters, the real part of the time scale ratio w_2', and the mode coupling f_2, which cross over to the corresponding parameters of model F. The flow of model F′ at $X=0$ coincides with the flow of model F (solid curve).

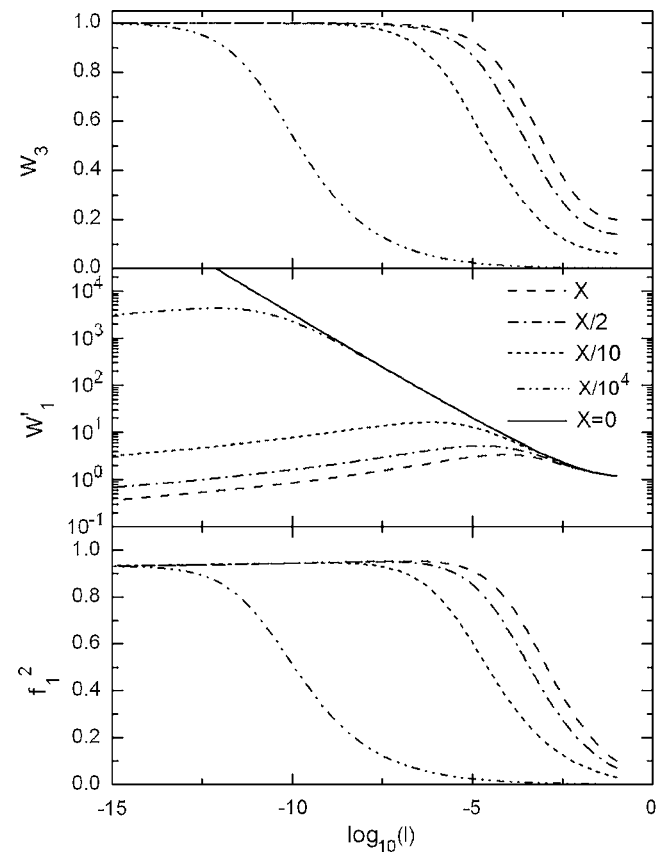

FIG. 2. Crossover to pure ^{4}He: The remaining dynamic parameters of model F′, the time scale ratios w_3, w_1' and the mode coupling f_1. These parameters have no counterpart in model F (see text).

In the case of $X=0$ (pure ^{4}He) one has to observe in addition that also $\mathring{L}$ goes to zero and thus also $w_3(l_0)$ (like $\sqrt{X}$). As mentioned in the previous section w_3 and f_1 stay identically zero for zero initial values. The flow reduces also in the nonasymptotic region to the flow of model F. Starting with finite initial values $w_3(l_0)$ and $f_1(l_0)$ the flow of the parameters $f_1(l)$ and $w_3(l)$ increases always to its finite "mixed" fixed point value $w_3^{\star 2}=1$, $f_1^{\star 2}=f_2^{\star 2}$ for small l independent of how small the initial values are. Thus although one has asymptotically the mixture behavior, the flow is almost model-F-like in the nonasymptotic region. The transition of the mixture flow into the flow of pure ^{4}He is demonstrated in Figs. 1 and 2. Starting with finite initial values for all parameters corresponding to a finite mole fraction X, the initial values for w_3 and f_1 are continuously lowered by the same factor. The parameters w_2' and f_2 turn systematically into the flow of model F, which is shown by the inset magnifications in more detail in Fig. 1. The solution calculated from the flow equations of model F′ at $X=0$ $[w_3(l_0)=f_1(l_0)=0]$ and the solution calculated from the model F flow equations are identical, which is indicated by $X=0,F$ in the figure. From Fig. 2 one can see that w_3 and f_1 begin to rise later the smaller the initial values are, but they always reach the mixture fixed point. In the same figure we have plotted w_1', which is increasing with smaller mole fractions, but always has to reach the fixed point value 0 as long as w_3 and f_1 are finite. For smaller X the curves join closely to the diverging curve of $X=0$ where w_1' has to reach the fixed point value ∞. The solution for mole fraction $X/10^4$ plotted in Fig. 2 has been omitted in Fig. 1 because the flow of the parameters plotted therein is not distinguishable from the $X=0$ case.

The other case $X=X_D=0.37$ differs from the previous one in the behavior of w_3. Although $f_1(l_0)=0$ at this mole fraction the initial value $w_3(l_0)$ stays finite. The finite initial value of w_3 causes a coupling of the two secondary densities by the kinetic coefficient L leading to a model which is not completely decoupled and therefore different from model F. As a matter of fact the flow of the parameters w_2' and f_2 is now different to the model F flow for $X\to X_D$. This is demonstrated in Figs. 3 and 4. Starting at a mole fraction $X\neq X_D$ the initial value $f_1(l_0)$ is systematically lowered while the initial value $w_3(l_0)$ stays fixed to simulate the approach to X_D. A smaller initial value of f_1 corresponds to a smaller distance $\Delta X=X_D-X$ to the decoupling point. In Fig. 3 the parameters w_2' and f_2 are plotted for different distances to X_D. From the magnified windows it is evident that the pure ^{4}He flow of model F represents no limiting case. The solutions for $\Delta X=0$ and for model F are different. While f_1 and w_1' behave similarly as in the $X\to 0$ limit as can be seen from Fig. 4, in w_3 occurs a nonasymptotic crossover effect. When f_1 starts at

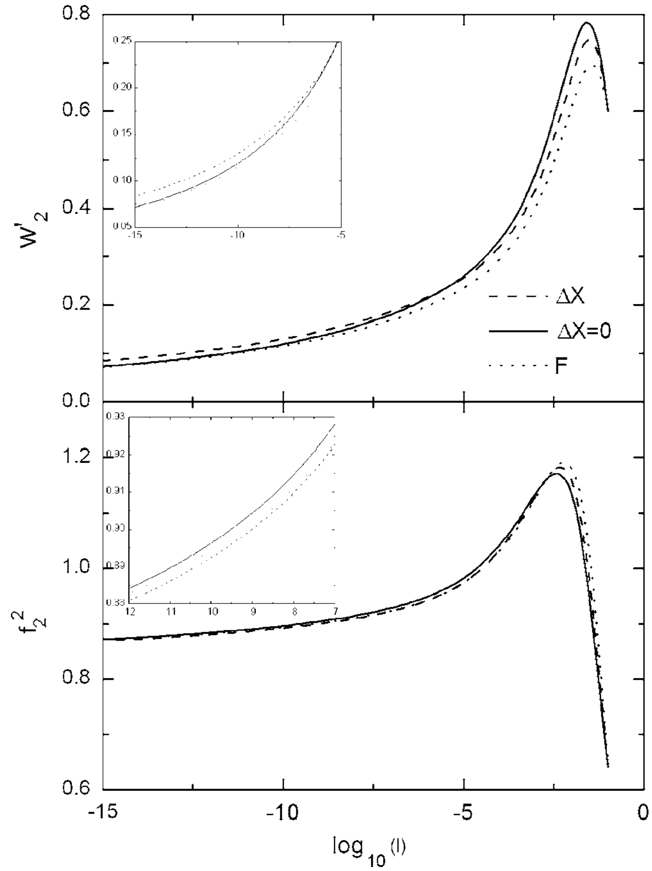

FIG. 3. Crossover to decoupling point: Shown are the dynamic parameters, the real part of the time scale ratio w_2', and the mode coupling f_2, which correspond to the parameters w' and f of model F. The flow at $\Delta X=0$ does not coincide with the model F flow (dashed curve) in the nonasymptotic region.

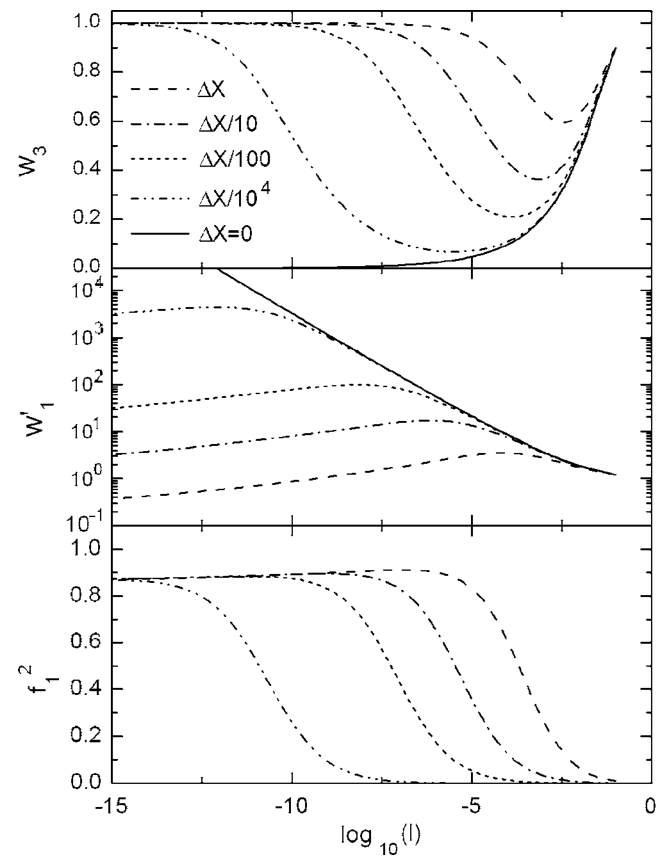

FIG. 4. Crossover to decoupling point: The remaining dynamic parameters of model F′, the time scale ratios w_3, w_1' and the mode coupling f_1. These parameters have no counterpart in model F (see text).

small values it stays at these small values for a while before it increases to its mixture fixed point value. As long as f_1 stays small the flow of w_3 drops down because at $f_1=0$ the fixed point value of w_3 would be 0. At increasing f_1 the flow of w_3 is turning around and increases also to the fixed point $w_3^\star=1$.

The limiting behavior of the flow at the other decoupling limit at $X_D=0.37$ is different and model F behavior is reached only in the asymptotic limit $l\to 0$. The reason is that although in this limit $f_1(l_0)=0$ and stays zero, the time scale parameter $w_3(l_0)$ in the background stays finite when approaching X_D. This time scale parameter is driven to its fixed point value zero, but for finite l it stays finite and determines the flow of the other dynamical parameters. In order to show this property (see Figs. 3 and 4) we start with an unusually large value of $w_3(0)$ in order to magnify the effect of the nonasymptotic dissipative coupling of the two densities m_1 and m_2.

In producing these figures we have chosen typical initial values for w_1, w_2, f_1, f_2, and w_3 corresponding to a fit of the transport coefficients at small mole fractions. The initial values of the parameters $w_2'(l_0)=0.6$, $w_2''(l_0)=0.3$, and $f_2(l_0)=0.8$, which turn in the limit $X\to 0$ into the parameters of pure model F are in a numerical range which can be found either by fits of the thermal conductivity with the corresponding model F functions in pure ^{4}He or by fits of the thermal conductivity and the thermal diffusion ratio with corresponding model F′ functions in mixtures at very low mole fractions [27]. The initial values of the remaining parameters $w_1'(l_0)=1.2$, $f_1(l_0)=0.1$, and $w_3(l_0)=0.2$ are in a numerical range of fits at finite mole fractions in mixtures. In order to simulate the approach to $X=0$ we reduce the initial values of f_1 and w_3 by the same factor as indicated in Figs. 1 and 2. A similar procedure has been used to simulate the approach to the decoupling point at X_D in Figs. 3 and 4. Starting from initial values corresponding to a fit of the transport coefficients apart from the value for w_3, which is chosen larger ($w_3=0.9$), now the initial value of f_1 is systematically decreased, while the initial value of w_3 stays constant.

X. CONCLUSION

The aim of the paper was to present the two-loop calculation of the model F′ field theoretic functions. The rearrangement of the loop expansion of the dynamic vertex functions into their general structure [see Eqs. (37) and (44)] leads to a considerable reduction of the number of contributions. Limiting behavior and general properties of the flow equations have been discussed. The theoretical results have already been used to analyze the critical transport properties of ^{4}He-^{4}He mixtures at the superfluid transition at mole fractions $0 \leqslant X \leqslant 0.366$ [27]. The dynamic functions can also be used to analyze higher mole fractions and the tricritical behavior at $X=0.67$ in two loop order. This may clarify the problems mentioned in Ref. [29].

ACKNOWLEDGMENT

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APPENDIX A: CALCULATION OF THE DYNAMIC ORDER PARAMETER VERTEX FUNCTION

1. Perturbation expansion in two-loop order

In perturbation expansion up to two-loop order the functions $\mathring{\Omega}_{\psi\tilde{\psi}^+}$ and $\mathring{\Gamma}^{(d)}_{\psi\tilde{\psi}^+}$, which appear in Eq. (37), can be written as

$$\mathring{\Omega}_{\psi\tilde{\psi}^+}(\xi,k,\omega) = 1 + \mathring{\Omega}^{(1L)}_{\psi\tilde{\psi}^+}(\xi,k,\omega) + \mathring{\Omega}^{(2L)}_{\psi\tilde{\psi}^+}(\xi,k,\omega), \quad \text{(A1)}$$

$$\mathring{\Gamma}^{(d)}_{\psi\tilde{\psi}^+}(\xi,k,\omega) = 2[\mathring{\Gamma} + \mathring{G}^{(1L)}_{\psi\tilde{\psi}^+}(\xi,k,\omega) + \mathring{G}^{(2L)}_{\psi\tilde{\psi}^+}(\xi,k,\omega)]. \quad \text{(A2)}$$

The superscript (iL) indicates the loop order. Of course all functions considered depend on all model parameters (couplings and kinetic coefficients), but only the independent lengths ξ, k, and ω will be mentioned explictly in the following. The contributions from perturbation expansion are calculated within the dynamically diagonal model introduced in Eqs. (13)–(25). The one-loop contributions are

$$\mathring{\Omega}^{(1L)}_{\psi\tilde{\psi}^+}(\xi,k,\omega) = \sum_i (\mathring{\Gamma}\mathring{\bar{\gamma}}_i - i\mathring{\bar{g}}_i)\mathring{\bar{\gamma}}_i I_i^{(F)}(\xi,k,\omega), \quad \text{(A3)}$$

$$\mathring{G}^{(1L)}_{\psi\tilde{\psi}^+}(\xi,k,\omega) = \sum_i (\mathring{\Gamma}\mathring{\bar{\gamma}}_i - i\mathring{\bar{g}}_i) i\mathring{\bar{g}}_i I_i^{(F)}(\xi,k,\omega). \quad \text{(A4)}$$

The sum in the above expressions runs over the number of secondary densities. The two-loop contributions have the structure

$$\mathring{\Omega}^{(2L)}_{\psi\tilde{\psi}^+}(\xi,k,\omega) = \frac{2}{9}\mathring{\Gamma}\mathring{u}^2 \mathring{W}^{(A)}_{\psi\tilde{\psi}^+}(\xi,k,\omega) - \frac{2}{3}\sum_i [\mathring{\Gamma}\mathring{u}\mathring{\bar{\bar{\gamma}}}_i + (\mathring{\Gamma}\mathring{\bar{\bar{\gamma}}}_i - i\mathring{\bar{g}}_i)\mathring{u}]\mathring{F}^{(T3)}_{i,i,\psi\tilde{\psi}^+}(\xi,k,\omega) + \sum_{i,j} \mathring{\bar{\bar{\gamma}}}_i \mathring{F}_{i,j,\psi\tilde{\psi}^+}(\xi,k,\omega) \quad \text{(A5)}$$

and

$$\mathring{G}^{(2L)}_{\psi\tilde{\psi}^+}(\xi,k,\omega) = -\frac{2}{3}\sum_i \mathring{\Gamma}\mathring{u} i\mathring{\bar{g}}_i \mathring{F}^{(T3)}_{i,i,\psi\tilde{\psi}^+}(\xi,k,\omega) + \sum_{i,j} i\mathring{\bar{g}}_i \mathring{F}_{i,j,\psi\tilde{\psi}^+}(\xi,k,\omega). \quad \text{(A6)}$$

Note that both two-loop functions differ only in terms containing the static fourth-order coupling $\mathring{u}$. The remaining contributions are the same in both functions apart from a factor $\mathring{\bar{\bar{\gamma}}}_i$ and $i\mathring{\bar{g}}_i$, respectively,

$$\mathring{F}_{i,j,\psi\tilde{\psi}^+}(\xi,k,\omega) = (\mathring{\Gamma}\mathring{\bar{\gamma}}_j - i\mathring{\bar{g}}_j)[(\mathring{\Gamma}\mathring{\bar{\gamma}}_i - i\mathring{\bar{g}}_i)(\mathring{\Gamma}\mathring{\bar{\gamma}}_j - i\mathring{\bar{g}}_j) \times \mathring{F}^{(T4)}_{i,j,\psi\tilde{\psi}^+}(\xi,k,\omega) + \mathring{F}^{(T5)}_{i,j,\psi\tilde{\psi}^+}(\xi,k,\omega) + \mathring{F}^{(T6)}_{i,j,\psi\tilde{\psi}^+} \times (\xi,k,\omega) - 2\mathring{\bar{\gamma}}_j \mathring{F}^{(T3)}_{i,i,\psi\tilde{\psi}^+}(\xi,k,\omega)]. \quad \text{(A7)}$$

The one-loop integral $I_i^{(F)}$ in Eqs. (A1) and (A2) reads

$$I_i^{(F)}(\xi,k,\omega) = \int_{k'} \frac{1}{[\xi^{-2} + (k+k')^2](-i\omega + \alpha_i')}. \quad \text{(A8)}$$

The dynamic propagator α_i' is defined by

$$\alpha_i' = \mathring{\Gamma}[\xi^{-2} + (k+k')^2] + \mathring{\lambda}_i k'^2. \quad \text{(A9)}$$

The first two-loop contribution in Eq. (A5) comes from the well known model A. $\mathring{W}^{(A)}_{\psi\tilde{\psi}^+}$ is defined by

$$\mathring{W}^{(A)}_{\psi\tilde{\psi}^+}(\xi,k,\omega) = \int_{k'}\int_{k''} \frac{1}{(\xi^{-2}+k'^2)(\xi^{-2}+k''^2)[\xi^{-2}+(k+k'+k'')^2](-i\omega + A)} \quad \text{(A10)}$$

with

$$A = \mathring{\Gamma}(\xi^{-2}+k'^2) + \mathring{\Gamma}^+(\xi^{-2}+k''^2) + \mathring{\Gamma}[\xi^{-2}+(k+k'+k'')^2]. \quad \text{(A11)}$$

The further two-loop contributions in Eqs. (A5)–(A7) are marked with superscripts (Ti), which indicate the different graph topologies. The explicit expressions are

$$\mathring{F}^{(T3)}_{i,i,\psi\tilde{\psi}^+}(\xi,k,\omega) = \int_{k'}\int_{k''} \frac{1}{[\xi^{-2}+(k+k')^2](-i\omega+\alpha_i')(-i\omega+A)}\left(\frac{\mathring{\Gamma}\mathring{\bar{\gamma}}_i - i\mathring{\bar{g}}_i}{\xi^{-2}+k''^2} + \frac{\mathring{\Gamma}^+\mathring{\bar{\gamma}}_i + i\mathring{\bar{g}}_i}{\xi^{-2}+(k'+k'')^2}\right), \quad \text{(A12)}$$

$$\mathring{F}^{(T4)}_{i,j,\psi\tilde{\psi}^+}(\xi,k,\omega)=\int_{k'}\int_{k''}\frac{1}{[\xi^{-2}+(k+k'+k'')^2](-i\omega+\alpha_i')^2(-i\omega+\beta_{ij})},\tag{A13}$$

$$\mathring{F}^{(T5)}_{i,j,\psi\tilde{\psi}^+}(\xi,k,\omega)=\int_{k'}\int_{k''}\frac{\mathring{\lambda}_j\mathring{\bar{\gamma}}_jk'^2-i\mathring{\bar{g}}_j[(k'+k'')^2-k''^2]}{[\xi^{-2}+(k+k')^2](-i\omega+\alpha_i')(-i\omega+\alpha_j^i)(-i\omega+A)}\left(\frac{\mathring{\Gamma}\mathring{\bar{\gamma}}_i-i\mathring{\bar{g}}_i}{\xi^{-2}+k''^2}+\frac{\mathring{\Gamma}^+\mathring{\bar{\gamma}}_i+i\mathring{\bar{g}}_i}{\xi^{-2}+(k'+k'')^2}\right),\tag{A14}$$

$$\begin{aligned}\mathring{F}^{(T6)}_{i,j,\psi\tilde{\psi}^+}(\xi,k,\omega)=&\int_{k'}\int_{k''}\frac{\mathring{\lambda}_j\mathring{\bar{\gamma}}_jk''^2+i\mathring{\bar{g}}_j[(k+k'+k'')^2-\ (k+k')^2)]}{[\xi^{-2}+(k+k')^2](-i\omega+\alpha_i')(-i\omega+\alpha_j'')(-i\omega+A')}\left(\frac{\mathring{\Gamma}\mathring{\bar{\gamma}}_i-i\mathring{\bar{g}}_i}{\xi^{-2}+(k+k'+k'')^2}+\frac{\mathring{\Gamma}^+\mathring{\bar{\gamma}}_i+i\mathring{\bar{g}}_i}{\xi^{-2}+(k+k'')^2}\right)\\&+\int_{k'}\int_{k''}\frac{\mathring{\Gamma}\mathring{\bar{\gamma}}_i-i\mathring{\bar{g}}_i}{[\xi^{-2}+(k+k')^2](-i\omega+\alpha_j'')(-i\omega+\beta_{ij})}\left(\frac{\mathring{\Gamma}\mathring{\bar{\gamma}}_j-i\mathring{\bar{g}}_j}{-i\omega+\alpha_i'}+\frac{\mathring{\bar{\gamma}}_j}{\xi^{-2}+(k+k'+k'')^2}\right.\\&\left.+\frac{\mathring{\lambda}_j\mathring{\bar{\gamma}}_jk''^2+i\mathring{\bar{g}}_j[(k+k'+k'')^2-\ (k+k')^2)]}{[\xi^{-2}+(k+k'+k'')^2](-i\omega+\alpha_i')}\right)\end{aligned}\tag{A15}$$

with the dynamic propagators

$$\beta_{ij}=\mathring{\Gamma}[\xi^{-2}+(k+k'+k'')^2]+\mathring{\lambda}_ik'^2+\mathring{\lambda}_jk''^2,\tag{A16}$$

$$\begin{aligned}A'=&\mathring{\Gamma}[\xi^{-2}+(k+k')^2]\\&+\mathring{\Gamma}^+[\xi^{-2}+(k+k'+k'')^2]+\mathring{\Gamma}[\xi^{-2}+(k+k'')^2]\end{aligned}\tag{A17}$$

which are both invariant under an interchange of k' and k''.

2. General integral types

In the above expressions the same integral may appear several times with different parameters inserted. To identify the independent basic integral types it is convenient to express them in a generalized form.

The one-loop contributions contain only one integral (A8) which is at vanishing external frequency ω and external wave vector k of the form

$$I_F^{(1L)}=\int_{k'}\frac{1}{(a+k'^2)(b+k'^2)}.\tag{A18}$$

A closer examination of the two-loop expressions (A10)–(A15) reveals that they contain eight independent integrals of the following types:

$$I_{F_1}=\int_{k'}\int_{k''}\frac{1}{(a+k'^2)(A+k''^2)[B+(k'+k'')^2][e+\mu k'^2+\nu k''^2+(k'+k'')^2]},\tag{A19}$$

$$I_{F_2}=\int_{k'}\int_{k''}\frac{1}{(a+k'^2)(b+k'^2)(A+k''^2)[e+\mu k'^2+\nu k''^2+(k'+k'')^2]},\tag{A20}$$

$$I_{F_3}=\int_{k'}\int_{k''}\frac{k'^2}{(a+k'^2)(b+k'^2)(c+k'^2)(A+k''^2)[e+\mu k'^2+\nu k''^2+(k'+k'')^2]},\tag{A21}$$

$$I_{F_4}=\int_{k'}\int_{k''}\frac{(k'+k'')^2-k''^2}{(a+k'^2)(b+k'^2)(c+k'^2)(A+k''^2)[e+\mu k'^2+\nu k''^2+(k'+k'')^2]},\tag{A22}$$

$$I_{F_5}=\int_{k'}\int_{k''}\frac{k''^2}{(a+k'^2)(b+k'^2)(A+k''^2)(B+k''^2)[e+\mu k'^2+\nu k''^2+(k'+k'')^2]},\tag{A23}$$

$$I_{F_6}=\int_{k'}\int_{k''}\frac{k''^2}{(a+k'^2)(b+k'^2)(A+k''^2)[B+(k'+k'')^2][e+\mu k'^2+\nu k''^2+(k'+k'')^2]},\tag{A24}$$

$$I_{F_7}=\int_{k'}\int_{k''}\frac{(k'+k'')^2-k''^2}{(a+k'^2)(b+k'^2)(A+k''^2)(B+k''^2)[e+\mu k'^2+\nu k''^2+(k'+k'')^2]}, \tag{A25}$$

$$I_{F_8}=\int_{k'}\int_{k''}\frac{(k'+k'')^2-k''^2}{(a+k'^2)(A+k''^2)(B+k''^2)[C+(k'+k'')^2][e+\mu k'^2+\nu k''^2+(k'+k'')^2]}. \tag{A26}$$

APPENDIX B: SECONDARY DENSITY VERTEX FUNCTIONS

1. Perturbation expansion

In perturbation expansion up to two-loop order the dynamic functions appearing in Eq. (44) are

$$\mathring{\Omega}_{m_i\tilde{m}_j}(\xi,k,\omega)=\delta_{ij}+\mathring{\Omega}^{(1L)}_{m_i\tilde{m}_j}(\xi,k,\omega)+\mathring{\Omega}^{(2L)}_{m_i\tilde{m}_j}(\xi,k,\omega), \tag{B1}$$

$$\mathring{\Gamma}^{(d)}_{m_i\tilde{m}_j}(\xi,k,\omega)=\mathring{\lambda}_j k^2\delta_{ij}+\mathring{G}^{(1L)}_{m_i\tilde{m}_j}(\xi,k,\omega)+\mathring{G}^{(2L)}_{m_i\tilde{m}_j}(\xi,k,\omega). \tag{B2}$$

The one-loop contributions to the dynamic vertex functions of the secondary densities are

$$\mathring{\Omega}^{(1L)}_{m_i\tilde{m}_j}(\xi,k,\omega)=\frac{1}{2}\mathring{\bar{\gamma}}_i I_{1j}(\xi,k,\omega)+\overline{cc}, \tag{B3}$$

$$\mathring{G}^{(1L)}_{m_i\tilde{m}_j}(\xi,k,\omega)=-\frac{1}{2}i\mathring{\bar{g}}_i I^{(P)}_{1j}(\xi,k,\omega)+\overline{cc}, \tag{B4}$$

where the expression $\overline{cc}$ denotes the same terms as explicitly given but with all complex quantities replaced by their complex conjugated counterparts except the $-i\omega$ terms in the integrals. Thus the above expressions are not real quantities. The one-loop integrals I_{1j} and $I_{1j}^{(P)}$ in Eqs. (B3) and (B4) read

$$I_{1j}(\xi,k,\omega)=\int_{k'}\frac{M_j(k,k')}{(\xi^{-2}+k'^2)[\xi^{-2}+(k+k')^2](-i\omega+\alpha'_{(m)})}, \tag{B5}$$

$$I^{(P)}_{1j}(\xi,k,\omega)=\int_{k'}\frac{[(k+k')^2-k'^2]M_j(k,k')}{(\xi^{-2}+k'^2)[\xi^{-2}+(k+k')^2](-i\omega+\alpha'_{(m)})}. \tag{B6}$$

The dynamic propagator $\alpha'_{(m)}$ is defined by

$$\alpha'_{(m)}=\mathring{\Gamma}(\xi^{-2}+k'^2)+\mathring{\Gamma}^{+}[\xi^{-2}+(k+k')^2] \tag{B7}$$

and $M_j(k,k')$ is given by

$$M_j(k,k')=\mathring{\lambda}_j\mathring{\bar{\gamma}}_j k^2+i\mathring{\bar{g}}_j[(k+k')^2-k'^2]. \tag{B8}$$

The two-loop contributions in Eqs. (B1) and (B2) are of the form

$$\mathring{\Omega}^{(2L)}_{m_i\tilde{m}_j}(\xi,k,\omega)=-\mathring{\bar{\gamma}}_i\Bigg[\Bigg(\frac{2}{3}\mathring{u}+\sum_l\mathring{\bar{\gamma}}_l^2\Bigg)\Bigg(\frac{1}{2}I(\xi,k)I_{1j}(\xi,k,\omega)+J_1(\xi,k,\omega)W_{1j}(\xi,k,\omega)\Bigg)+\frac{1}{2}\sum_l i\mathring{\bar{g}}_l\mathring{\bar{\gamma}}_l J_1(\xi,k,\omega)I^{(P)}_{1j}(\xi,k,\omega)-W_{3j}(\xi,k,\omega)-W_{4j}(\xi,k,\omega)+\overline{cc}\Bigg] \tag{B9}$$

and

$$\mathring{G}^{(2L)}_{m_i\tilde{m}_j}(\xi,k,\omega)=i\mathring{\bar{g}}_i\Bigg[\Bigg(\frac{2}{3}\mathring{u}+\sum_l\mathring{\bar{\gamma}}_l^2\Bigg)J^{(P)}_1(\xi,k,\omega)W_{1j}(\xi,k,\omega)+\frac{1}{2}\sum_l i\mathring{\bar{g}}_l\mathring{\bar{\gamma}}_l J^{(P)}_1(\xi,k,\omega)I^{(P)}_{1j}(\xi,k,\omega)-W^{(P)}_{3j}(\xi,k,\omega)+W^{(P)}_{4j}(\xi,k,\omega)\Bigg]+\overline{cc}. \tag{B10}$$

The integral expressions appearing in the above two relations are defined as

$$J^{(P)}_1(\xi,k,\omega)=\int_{k'}\frac{[(k+k')^2-k'^2]}{(\xi^{-2}+k'^2)[\xi^{-2}+(k+k')^2](-i\omega+\alpha'_{(m)})}, \tag{B11}$$

$$W_{1j}(\xi,k,\omega)=\int_{k'}\frac{\mathring{\Gamma}M_j(k,k')}{[\xi^{-2}+(k+k')^2](-i\omega+\alpha'_{(m)})}, \tag{B12}$$

$$W^{(P)}_{3j}(\xi,k,\omega)=\sum_l\int_{k'}\int_{k''}\frac{(\mathring{\Gamma}\mathring{\bar{\gamma}}_l-i\mathring{\bar{g}}_l)^2}{[\xi^{-2}+(k+k')^2][\xi^{-2}+(k'+k'')^2]}\times\frac{[(k+k')^2-k'^2]M_j(k,k')}{(-i\omega+\alpha'_{(m)})^2(-i\omega+\beta^{(m)}_l)}, \tag{B13}$$

$$W_{4j}^{(P)}(\xi,k,\omega)=\sum_l \int_{k'}\int_{k''}\frac{(\mathring{\Gamma}\mathring{\bar{\gamma}}_l-i\mathring{\bar{g}}_l)(\mathring{\Gamma}^{+}\mathring{\bar{\gamma}}_l+i\mathring{\bar{g}}_l)}{[\xi^{-2}+(k+k')^2][\xi^{-2}+(k+k'')^2]}\times\frac{[(k+k'')^2-k''^2]M_j(k,k')}{(-i\omega+\alpha'_{(m)})(-i\omega+\alpha''^{+}_{(m)})(-i\omega+\delta_l^{(m)+})},\qquad\text{(B14)}$$

where we have introduced the expressions

$$\beta_l^{(m)}=\mathring{\Gamma}^{+}[\xi^{-2}+(k+k')^2]+\mathring{\Gamma}[\xi^{-2}+(k'+k'')^2]+\mathring{\lambda}_l k''^2,\qquad\text{(B15)}$$

$$\delta_l^{(m)}=\mathring{\Gamma}[\xi^{-2}+(k+k')^2]+\mathring{\Gamma}^{+}[\xi^{-2}+(k+k'')^2]+\mathring{\lambda}_l(k+k'+k'')^2.\qquad\text{(B16)}$$

The corresponding integral contributions without the superscript (P) in Eqs. (B9) and (B10) are like Eqs. (B11)–(B14) but without the term $[(k+k')^2-k'^2]$ or $[(k+k'')^2-k''^2]$ in the numerator [compare Eqs. (B5) and (B6)].

2. k^2 derivatives of the integrals

The renormalization of the kinetic coefficients of the secondary densities are determined from the k^2 derivative of $\mathring{\Gamma}^{(d)}_{m_l\tilde{m}_j}(\xi,k,\omega)$ at $\omega=0$ and $k=0$. Therefore we need the corresponding derivatives of the integrals appearing in Eq. (B2). For the one-loop contribution (B6) we obtain

$$\left.\frac{\partial}{\partial k^2}I_{1j}^{(P)}\right|_{\substack{\omega=0\\k=0}}=2\frac{i\mathring{\bar{g}}_j}{\mathring{\Gamma}'}\int_{k'}\frac{k'^2\cos^2\theta'}{(\xi^{-2}+k'^2)^3}\qquad\text{(B17)}$$

with θ' as the polar angle to the z axis of a d-dimensional unit sphere in k' space. The derivatives of the two-loop contributions in Eq. (B10) are

$$\left.\frac{\partial}{\partial k^2}J_1^{(P)}W_{1j}\right|_{\substack{\omega=0\\k=0}}=0,\qquad\text{(B18)}$$

$$\left.\frac{\partial}{\partial k^2}J_1^{(P)}I_{1j}^{(P)}\right|_{\substack{\omega=0\\k=0}}=0,\qquad\text{(B19)}$$

$$\left.\frac{\partial}{\partial k^2}W_{3j}^{(P)}\right|_{\substack{\omega=0\\k=0}}=\frac{i\mathring{\bar{g}}_j}{4\mathring{\Gamma}'^2}\sum_l(\mathring{\Gamma}\mathring{\bar{\gamma}}_l-i\mathring{\bar{g}}_l)^2\int_{k'}\frac{(\partial/\partial k^2)\,[(k+k')^2-k'^2]^2|_{k=0}}{(\xi^{-2}+k'^2)^3}\times\int_{k''}\frac{1}{[\xi^{-2}+(k'+k'')^2]\{\mathring{\Gamma}^{+}(\xi^{-2}+k'^2)+\mathring{\Gamma}[\xi^{-2}+(k'+k'')^2]+\mathring{\lambda}_l k''^2\}}.\qquad\text{(B20)}$$

The derivative in the numerator of the first integral in Eq. (B20) is not affected by shifts of the second integration variable k''. Thus we can insert immediately

$$\frac{\partial}{\partial k^2}[(k+k')^2-k'^2]^2|_{k=0}=4k'^2\cos^2\theta'\qquad\text{(B21)}$$

in the above equation. The last two-loop contribution is

$$\left.\frac{\partial}{\partial k^2}W_{4j}^{(P)}\right|_{\substack{\omega=0\\k=0}}=\frac{i\mathring{\bar{g}}_j}{4\mathring{\Gamma}'^2}\sum_l(\mathring{\Gamma}\mathring{\bar{\gamma}}_l-i\mathring{\bar{g}}_l)(\mathring{\Gamma}^{+}\mathring{\bar{\gamma}}_l+i\mathring{\bar{g}}_l)\int_{k'}\frac{1}{(\xi^{-2}+k'^2)^2}\times\int_{k''}\frac{(\partial/\partial k^2)\,\{[(k+k')^2-k'^2][(k+k'')^2-k''^2]\}|_{k=0}}{(\xi^{-2}+k''^2)^2[\mathring{\Gamma}^{+}(\xi^{-2}+k'^2)+\mathring{\Gamma}(\xi^{-2}+k''^2)+\mathring{\lambda}_l(k'+k'')^2]}.\qquad\text{(B22)}$$

Because the second integral in Eq. (B22) needs to be shifted in the further calculation, it is not convenient to introduce polar coordinates for the derivative at this stage.

3. General integral types

The one-loop integral (B17) has the structure

$$I_m^{(1L)}=\int_{k'}\frac{k'^2\cos^2\theta'}{(a+k'^2)^3}.\qquad\text{(B23)}$$

In two-loop order we need the singular parts of the contributions (B18)–(B22). A closer examination of Eqs. (B18) and (B19) reveals that their singular parts are vanishing, i.e.,

$$\left[\left.\frac{\partial}{\partial k^2}J_1^{(P)}W_{1j}\right|_{\substack{\omega=0\\k=0}}\right]_S=\left[\left.\frac{\partial}{\partial k^2}J_1^{(P)}I_{1j}^{(P)}\right|_{\substack{\omega=0\\k=0}}\right]_S=0. \tag{B24}$$

Thus we only have to consider the contributions (B20) and (B22). Their general form is

$$I_{m_1}=\int_{k'}\int_{k''}\frac{(\partial/\partial k^2)\,[(k+k')^2-k'^2]^2|_{k=0}}{(a+k'^2)^3[A+(k'+k'')^2][e+\mu k'^2+\nu k''^2+(k'+k'')^2]} \tag{B25}$$

$$I_{m_2}=\int_{k'}\int_{k''}\frac{(\partial/\partial k^2)\,\{[(k+k')^2-k'^2][(k+k'')^2-k''^2]\}|_{k=0}}{(a+k'^2)^2(A+k''^2)^2[e+\mu k'^2+\nu k''^2+(k'+k'')^2]}. \tag{B26}$$

APPENDIX C: ϵ EXPANSION OF THE INTEGRALS

1. Order parameter vertex functions

The general integrals in Appendix A 2 can be calculated in ϵ expansion to identify the pole terms.

The one-loop integral is needed up to order ϵ^0 in a two-loop calculation. For Eq. (A18) we obtain

$$I_F^{(1L)}=\frac{A_d}{\epsilon}\left(1-\frac{\epsilon}{2}\frac{a\ln a-b\ln b}{a-b}\right)+O(\epsilon). \tag{C1}$$

In order to calculate the ζ functions only the singular part $[I_{F_i}]_S$ of the corresponding two-loop integral, which contains the ϵ pole terms, is necessary. For I_{F_1} we obtain

$$[I_{F_1}]_S=\frac{A_d^2}{4\epsilon}\left[\frac{1}{\mu}\ln\left(1+\frac{\mu^2}{\sigma}\right)+\frac{1}{\nu}\ln\left(1+\frac{\nu^2}{\sigma}\right)+\ln\frac{1+\sigma}{\sigma}\right] \tag{C2}$$

where we have introduced

$$\sigma=\mu+\nu+\mu\nu. \tag{C3}$$

Note that in the case $\mu=\nu=1$ and $a=A=B=e=\xi^{-2}/\kappa^2$ the above integral reduces to the integral appearing in the well known model A (pure relaxation model without mode coupling terms and without secondary densities). Inserting into the above result we obtain $[I_{F_1}]_S(\mu=\nu=1)=(3A_d^2/4\epsilon)\ln\frac{4}{3}$, which is consistent with previous calculations [25].

The pole terms of the second integral (A20) and the fifth integral (A23) are equal. They read

$$[I_{F_2}]_S=[I_{F_5}]_S=\frac{A_d^2}{2\epsilon^2(1+\nu)}\left[1+\frac{\epsilon}{2}\left(1+\ln\frac{1+\nu}{1+\mu}-\sigma\ln\frac{1+\sigma}{\sigma}\right)-\epsilon\frac{a\ln a-b\ln b}{a-b}\right]. \tag{C4}$$

The integrals (A21), (A23), and (A24) appear in models C and C′ (pure relaxation model without mode coupling terms) for special values of their parameters. The pole terms of these integrals $\{[I_{F_5}]_S$; see Eq. (C4)$\}$ read

$$[I_{F_3}]_S=\frac{A_d^2}{2\epsilon^2(1+\nu)}\left[1+\frac{\epsilon}{2}\left(1+\ln\frac{1+\nu}{1+\mu}-\sigma\ln\frac{1+\sigma}{\sigma}\right)-\epsilon\left(\frac{a^2\ln a}{(b-a)(c-a)}+\frac{b^2\ln b}{(a-b)(c-b)}+\frac{c^2\ln c}{(a-c)(b-c)}\right)\right], \tag{C5}$$

$$[I_{F_6}]_S=\frac{A_d^2}{2\epsilon^2(1+\nu)}\left[1+\frac{\epsilon}{2}\left(1+\ln\frac{1+\nu}{\mu+\nu}+\frac{\sigma}{\nu^2}\ln\frac{\sigma}{(1+\nu)(\mu+\nu)}\right)-\epsilon\frac{a\ln a-b\ln b}{a-b}\right]. \tag{C6}$$

The remaining integrals (A22), (A25), and (A26) appear in models E and E′ (no static coupling $\bar{\gamma}$) for special values of its parameters. The pole terms are

$$[I_{F_4}]_S=\frac{A_d^2}{2\epsilon^2}\frac{\nu}{(1+\nu)^2}\left[1-\frac{\epsilon}{2\nu}\left((1+\nu)\mu-\nu\ln\frac{1+\nu}{1+\mu}-(1+\mu+\mu\nu)\sigma\ln\frac{1+\sigma}{\sigma}\right)-\epsilon\left(\frac{a^2\ln a}{(b-a)(c-a)}+\frac{b^2\ln b}{(a-b)(c-b)}+\frac{c^2\ln c}{(a-c)(b-c)}\right)\right], \tag{C7}$$

$$[I_{F_7}]_S=\frac{A_d^2}{2\epsilon^2(1+\mu)}\left[1-\frac{\epsilon}{2}\left(1+2\mu+\ln\frac{1+\nu}{1+\mu}-(1+2\mu)\sigma\ln\frac{1+\sigma}{\sigma}\right)-\epsilon\frac{A\ln A-B\ln B}{A-B}\right], \tag{C8}$$

$$[I_{F_8}]_S = \frac{A_d^2}{2\epsilon^2(1+\mu)}\Bigg[1+\frac{\epsilon}{2}\Bigg(1+\ln\frac{1+\mu}{\mu+\nu} + \frac{1}{\mu}\ln\frac{\sigma}{(\mu+\nu)(1+\mu)} + \frac{1+\mu}{\nu}\ln\frac{\sigma}{(\mu+\nu)(1+\nu)} - (2+\mu+\sigma)\ln\frac{1+\sigma}{\sigma}\Bigg) - \epsilon\frac{A\ln A - B\ln B}{A-B}\Bigg]. \quad \text{(C9)}$$

2. Secondary density vertex functions

The pole terms of the two-loop integrals (B25) and (B26) read

$$[I_{m_1}]_S = \frac{A_d^2}{2\epsilon^2(1+\nu)}\Bigg[1-\frac{\epsilon}{2}\Bigg(\frac{3}{2}-\ln\frac{1+\nu}{\mu+\nu} - \frac{\sigma}{\nu^2}\ln\frac{\sigma}{(1+\nu)(\mu+\nu)}\Bigg) - \epsilon\ln a\Bigg], \quad \text{(C10)}$$

$$[I_{m_2}]_S = -\frac{A_d^2}{4\epsilon}\Bigg(1-\sigma\ln\frac{1+\sigma}{\sigma}\Bigg) \quad \text{(C11)}$$

with σ introduced in Eq. (C3).

APPENDIX D: RENORMALIZATION FACTORS IN TWO-LOOP ORDER

The renormalization factors are calculated within the dynamically diagonal model introduced in Eqs. (18)–(20). The definition of the Z factors can be found in Secs. IV A 3 and IV B 2.

1. Static Z factors

The static Z factors of the GLW model are well known for general OP component number n. In two-loop order they read for $n=2$ in our notation

$$Z_\psi = 1 - \frac{1}{\epsilon}\frac{u^2}{36}, \quad \text{(D1)}$$

$$Z_u = 1 + \frac{1}{\epsilon}\frac{5u}{3} + \frac{1}{\epsilon}\frac{u^2}{9}\left(\frac{25}{\epsilon}-8\right), \quad \text{(D2)}$$

$$Z_{\psi^2} = 1 + \frac{1}{\epsilon}\frac{2u}{3} + \frac{1}{\epsilon}\frac{u^2}{9}\left(\frac{7}{\epsilon}-\frac{5}{4}\right). \quad \text{(D3)}$$

The components of the static renormalization matrix $\mathbf{Z}_{\bar{m}}$ introduced in Eq. (55) are

$$[\mathbf{Z}_{\bar{m}}]_{ij} = \delta_{ij} + \bar{\gamma}_i\bar{\gamma}_j\left[\frac{1}{2\epsilon}+\frac{1}{\epsilon^2}\left(\frac{u}{3}+\frac{3}{8}\sum_k\bar{\gamma}_k^2\right)\right]. \quad \text{(D4)}$$

2. Dynamic Z factors

From the pole terms of $\mathring{\Omega}_{\psi\tilde{\psi}^+}$ and $\mathring{\Gamma}^{(d)}_{\psi\tilde{\psi}^+}$ in Eq. (37) we obtain

$$Z_{\tilde{\psi}^\star}^{1/2} = 1 - \frac{1}{\epsilon}\sum_i\frac{\bar{\gamma}_i\bar{D}_i}{1+\bar{w}_i} - \frac{1}{\epsilon}\frac{u^2}{18}\left(L_0+x_1L_1-\frac{1}{4}\right) + \frac{1}{4\epsilon}\left(\frac{2}{3}\sum_i\frac{u(\bar{w}_i\bar{\gamma}_i+\bar{D}_i)}{w_i(1+\bar{w}_i)}A_i + \sum_{i,j}\frac{\bar{\gamma}_i\bar{D}_j}{\bar{w}_j(1+\bar{w}_i)(1+\bar{w}_j)}B_{ij}\right) + \frac{1}{2\epsilon^2}\Bigg\{-\frac{2}{3}\sum_i\frac{u\bar{\gamma}_i(\bar{w}_i\bar{\gamma}_i+\bar{D}_i)}{1+\bar{w}_i} + \sum_{i,j}\frac{\bar{\gamma}_i\bar{D}_j}{\bar{w}_j(1+\bar{w}_i)(1+\bar{w}_j)}\Bigg[\frac{\bar{w}_i\bar{D}_i\bar{D}_j}{1+\bar{w}_i} - \bar{w}_i\bar{\gamma}_i\bar{D}_j - i\bar{F}_i\Bigg(\bar{w}_j\bar{\gamma}_j - \frac{i\bar{F}_j}{1+\Gamma^\star/\Gamma}\Bigg)\Bigg]\Bigg\}, \quad \text{(D5)}$$

$$Z_\Gamma^{(d)} = 1 - \frac{1}{\epsilon}\sum_i\frac{i\bar{F}_i\bar{D}_i}{\bar{w}_i(1+\bar{w}_i)} + \frac{1}{4\epsilon}\left(\frac{2}{3}\sum_i\frac{ui\bar{F}_i}{w_i(1+\bar{w}_i)}A_i + \sum_{i,j}\frac{i\bar{F}_i\bar{D}_j}{\bar{w}_i\bar{w}_j(1+\bar{w}_i)(1+\bar{w}_j)}B_{ij}\right) + \frac{1}{2\epsilon^2}\Bigg\{-\frac{2}{3}\sum_i\frac{u\bar{\gamma}_i i\bar{F}_i}{1+\bar{w}_i} + \sum_{i,j}\frac{i\bar{F}_i\bar{D}_j}{\bar{w}_i\bar{w}_j(1+\bar{w}_i)(1+\bar{w}_j)}\Bigg[\frac{\bar{w}_i\bar{D}_i\bar{D}_j}{1+\bar{w}_i} - \bar{w}_i\bar{\gamma}_i\bar{D}_j - i\bar{F}_i\Bigg(\bar{w}_j\bar{\gamma}_j - \frac{i\bar{F}_j}{1+\Gamma^\star/\Gamma}\Bigg)\Bigg]\Bigg\} \quad \text{(D6)}$$

by using the results from Appendix C 1. The coupling $\bar{D}_i$ and the functions A_i and B_{ij} are defined in Eqs. (107)–(109). The components of the Z matrix $\mathbf{Z}_\Lambda^{(d)}$ introduced in Eq. (70) are

$$[\mathbf{Z}_\Lambda^{(d)}]_{ij} = \delta_{ij} + \frac{1}{\epsilon}\frac{i\bar{F}_i i\bar{F}_j}{2\bar{w}'_j}\left[1+\frac{\mathcal{Q}}{2}-\frac{1}{4\epsilon}\sum_k\frac{1}{\bar{w}'_k}\left(\frac{\bar{D}_k^2}{1+\bar{w}_k}+\frac{\bar{D}_k^{+2}}{1+\bar{w}_k^+}\right)\right] \quad \text{(D7)}$$

where we have used the pole terms from Appendix C 2.

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