Specific heat of random fractal energy spectra

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The specific heat corresponding to systems with deterministic fractal energy spectra is known to present logarithmic-periodic oscillations as a function of the temperature T in the low T region around a mean value given by a characteristic dimension of the energy spectrum. In general, it is considered that the presence of disorder does not affect strongly these results, and that the fractal structure of the energy spectrum dominates. In this paper, we study the properties of the specific heat derived from random fractal energy spectra as a function of the degree of disorder present in the spectra. To study the influence of the disorder, we analyze the specific heat using three different properties: the specific heat mean value and the periods and amplitudes of the oscillations of the specific heat around its mean value. By studying the distributions and the mean values of these three properties, we obtain that the disorder does not influence very much the mean value of the specific heat. However, concerning the behavior of periods and amplitudes, we obtain a critical value of the disorder present in the energy spectra. Below this critical value, we find a low effect of the disorder and quasideterministic behavior indicating that the fractal structure is the dominant effect, but above the critical value, the disorder dominates and the behavior of the specific heat is practically chaotic.

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I. INTRODUCTION

The study of the thermodynamical properties derived from systems with hierarchical or fractal structure started more than a decade ago. The first results [1] showed that the specific heat of quasiperiodic spin chains presents logarithmic-periodic oscillations in the low temperature region. Similar results were found in the specific heat properties associated to hierarchical structures [2] or to the specific heat corresponding to the Heisenberg model with quasiperiodic exchange couplings at some circumstances [3]. All these examples share in common that the corresponding energy spectra present fractal properties. This was the motivation for the work by Tsallis *et al.* [4], where they studied theoretically the main properties that can be expected for the specific heat of a system with a fractal energy spectrum. Specifically, they studied the properties of the specific heat derived from a fractal energy spectrum with the structure of the triadic Cantor set. The most relevant results found in Ref. [4] were that in the region of low temperature T the specific heat presents logarithmic-periodic oscillations around the fractal dimension of the energy spectrum ($\log_{10} 2/\log_{10} 3$), and that the logarithmic period of the oscillations is given by the logarithm of the inverse of the scale factor used to construct the fractal spectrum ($\log_{10} 3$). These results were generalized to more complicated fractal spectra, either derived from generalized Cantor sets [5–7] or from energy spectra obtained from iterated maps [8,9]. In all cases, the specific heat presents logarithmic-periodic oscillations around a mean value given by a characteristic dimension of the energy spectrum. The energy spectra with fractal structure present an additional interest: quasiperiodic sequences, often used to model quasicrystals, are known to have energy spectra with fractal properties [10], similar in structure to fractal sets of Cantor type. This is the reason why the results obtained from studies performed on energy spectra of Cantor type have been used to explain the properties of the specific heat of Fibonacci sequences, either modeled as one-dimensional (1D) tight-binding Hamiltonians [7,11] or as superlattices [12]. Also, the properties of the specific heat associated to fractal spectra presents similar properties when quantum, fermionic, or bosonic statistics are considered [13,14].

Nevertheless, all the results described above have been obtained by considering *deterministic* fractal energy spectra, and the presence of disorder in the fractal structure has not been systematically studied; although, in general, it is considered that the presence of some degree of disorder does not influence very much the hierarchical structure of the underlying fractal energy spectra (see Ref. [6], and also Refs. [15,16]). The aim of this paper is to study the effect produced by the presence of disorder in fractal energy spectra of Cantor type in the corresponding specific heat properties. When disorder is introduced in the fractal spectra, there is a competition between the fractal properties and the disorder: We show below that there exists a threshold value in the degree of disorder introduced in the fractal spectra below which the specific heat resembles statistically the behavior found for deterministic fractal spectra, i.e., the fractal structure dominates. Above this threshold, we find a completely irregular behavior of the specific heat, indicating that the disorder overcomes the fractal structure.

To study the properties of the specific heat, we first generate discrete random fractal energy spectra $\{\varepsilon_i\}$ with different degrees of disorder. Once the spectra are generated (see below for the generation procedure), in all our calculations, we consider Boltzmann's statistics to calculate the corresponding specific heat $C(T)$. In this way, the partition function is obtained as $Z(T)=\Sigma_i \exp(-\beta\varepsilon_i)$, with $\beta=1/k_BT$ and k_B is the Boltzmann's constant, which is set to unity in the calculations. The internal energy is obtained as $U(T)=T^2 d\ln Z/dT$, and finally $C(T)=dU/dT$.

We characterize $C(T)$ using three different properties: the mean value of $C(T)$ and the behavior of the periods and amplitudes of the oscillations of the specific heat around its mean value. We study the dependence of these three properties on the degree the disorder introduced in the fractal energy spectrum. The paper is organized as follows. In Sec. II, we discuss the effect in the specific heat of introducing disorder in the classical (triadic) Cantor set. In Sec. III, we study the specific heat properties of more general disordered fractal spectra. In Sec. IV, we discuss the possibility of constructing a deterministic fractal spectrum for which the corresponding specific heat presents on average the same properties as the specific heats derived from a set of random energy spectra. Finally, we present our conclusions.

II. DISORDER IN THE CANTOR SET

We start by studying the effect of disorder in one of the simplest fractal energy spectra, the one based on the triadic Cantor set. This set is constructed in the following way: Let us consider our energy interval in the range $[0,\Delta]$, where from now on we take $\Delta=1$. In the first step of the generation process, the unity interval is divided into three parts, each one of size 1/3. Then, the central part is neglected, and in the two remaining segments, we apply the same procedure with the same scale factor 1/3. If this procedure is iterated n times, we say that we have the Cantor set in the nth step of the generation process. The whole Cantor set is obtained in the $n\to\infty$ limit. In this paper, we consider only *discrete* energy spectrum, i.e., for any n, we take as our energy levels the right and left borders of the segments obtained. Thus, the energy spectrum is formed by 2^{n+1} energy levels.

The disorder is introduced in a simple way: At any step of the generation process, instead of applying the constant scale factor $1/3$, we select randomly the scale factor in the interval $[1/3-\epsilon,1/3+\epsilon]$. The parameter ϵ quantifies the amount of disorder introduced in the spectrum, and the case $\epsilon=0$ corresponds to the ordered case. It is well known [4] that for $\epsilon=0$, the specific heat $C(T)$ of the spectrum presents logarithmic-periodic oscillations as a function of the temperature T in the low T region ($T<1$), and with a period in logarithmic-scale given by $\log_{10}3$ [i.e., $C(3T)=C(T)$]. The oscillations take place around the fractal dimension of the set, $d_{\rm box}=\log_{10}2/\log_{10}3$, which gives the specific heat mean value $\langle C\rangle$ in the oscillatory regime. Note that the scale factor $1/3$ controls the period of the oscillations (given by the logarithm of the inverse of $1/3$) and also the average value of $C(T)$, because the factor $\log_{10}3$ in $d_{\rm box}$ comes also from the scale factor.

In Fig. 1, we show several specific heats $C_n(T)$ obtained for different values of the disorder parameter ϵ. In all cases, we consider energy spectra obtained for $n=15$. In general, we see that when ϵ increases, the oscillations become more irregular, as expected. Let us define the ith period [17] of the oscillations as

$$\tau_i \equiv \log_{10}T_{i+1}-\log_{10}T_i, \qquad (1)$$

where T_i stands for the T value for which $C(T)$ reaches its ith local maxima (minima). For $\epsilon=0$, the period is constant and is given by $\log_{10}3$. For increasing ϵ, the period starts to be a nonconstant value and depends on the particular oscillation considered. Similarly, let us define the amplitude of the oscillations as

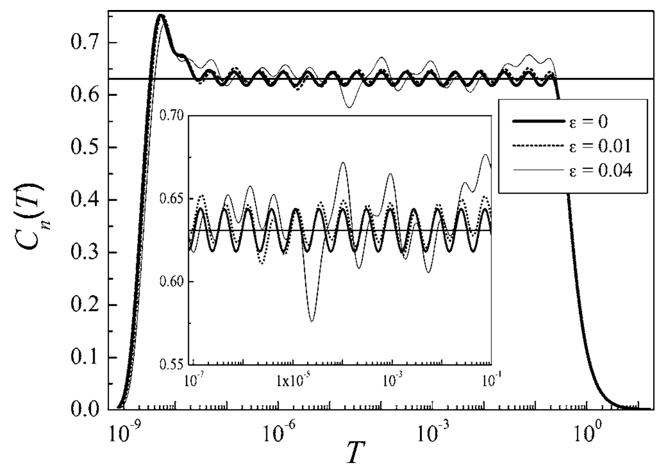

FIG. 1. Specific heats as a function of T in logarithmic scale corresponding to the disordered Cantor set for three different values of ϵ. All cases represent a particular realization of the disorder and $n=20$. The horizontal line corresponds to the fractal dimension of the Cantor set $d_{\rm box}=\log_{10}2/\log_{10}3=0.6309\ldots$. Inset: A zoom of the oscillatory part of the specific heats.

$$A_+ \equiv C(T_+)-\langle C\rangle \qquad (2)$$

when $C(T_+)$ stands for a local maximum of $C(T)$ and

$$A_- \equiv \langle C\rangle - C(T_-) \qquad (3)$$

when $C(T_-)$ stands for a local minimum of $C(T)$. For $\epsilon=0$, the amplitude is constant. For increasing ϵ, the amplitudes start to be nonconstant and depend on the particular oscillation considered.

These results suggest that when $\epsilon\neq0$, instead of single values of period and amplitude, we have *distributions* of these values. Thus, we study the distributions of these magnitudes as a function of the disorder parameter ϵ. In Fig. 2, we show the normalized distributions of periods for several values of ϵ. For each ϵ value, we have generated 10^4 random spectra up to $n=20$, calculated the corresponding specific heats and obtained the individual periods as defined in (1). For $\epsilon=0$, the distribution is a δ function centered at $\log_{10}3$, corresponding to the ordered Cantor set. When ϵ increases, the distribution becomes lower and wider, reflecting the disorder effect and the loss of regularity. If ϵ increases more, the distribution can become bimodal and even trimodal for very high ϵ, the new peaks centered at $\log_{10}9$ and $\log_{10}27$. The reason for these new peaks is explained below.

In Fig. 3, we show the normalized distribution of the amplitude values for several values of ϵ. As before, we have generated 10^4 energy spectra ($n=20$) for any ϵ value, calculated the corresponding specific heats, and obtained the individual amplitudes as defined in (2) and (3). As expected, for increasing ϵ, we obtain wider distributions, i.e., a greater diversity of amplitudes, showing again the irregularity introduced by the disorder. Note also that the distributions of amplitudes extend toward negative values as ϵ increases.

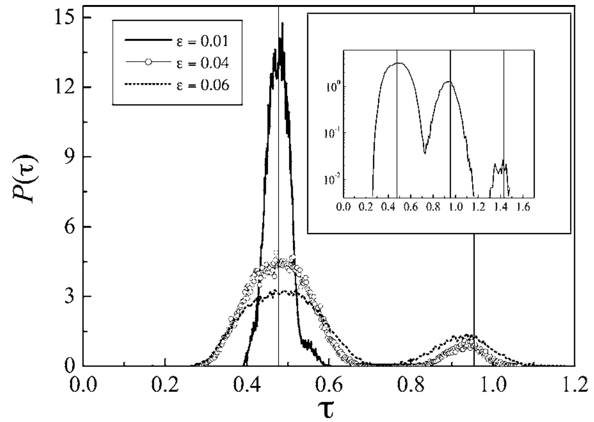

FIG. 2. Probability distributions of the periods of the oscillations of the specific heat obtained for different values of ϵ. The vertical lines corresponds to $\log_{10} 3$ and $\log_{10} 9$. Inset: The same with the vertical axis in logarithmic scale for the case $\epsilon=0.06$ to better observe the trimodal structure. Now the vertical lines correspond to $\log_{10} 3$, $\log_{10} 9$, and $\log_{10} 27$.

The reason for the multimodal behavior of the distributions of periods and for the existence of negative amplitudes can be explained observing Fig. 4, where we show the oscillations of the specific heat obtained for a realization of the disordered Cantor set with $\epsilon=0.05$ and $n=20$. On the one hand, note that the disorder stretches and compresses the oscillations, and even can make some local maxima or minima disappear, thus producing lost oscillations (as the ones marked with an arrow in Fig. 4). Thus, the new period for the corresponding oscillation will be on average twice the previous period, i.e., $2 \log_{10} 3=\log_{10} 9$. This double period is also shown in Fig. 4, where a normal period is also shown for comparison. This effect produces a peak in the distributions of periods around $\log_{10} 9$, transforming the distribution into a bimodal one, as shown in Fig. 2. For higher values of ϵ, even two consecutive oscillations can disappear, thus leading to periods three times larger on average than the period without disorder, i.e., $3 \log_{10} 3=\log_{10} 27$, producing a third peak in the distributions of periods, which now would become trimodal as the distribution shown in the inset of Fig. 2.

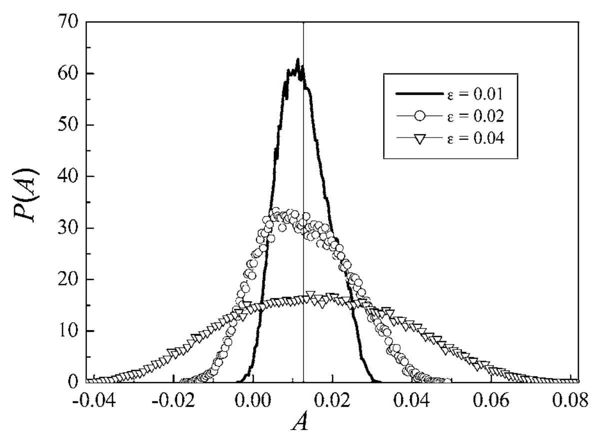

FIG. 3. Normalized probability distributions of the amplitudes of the oscillations of the specific heat for several values of ϵ. The vertical line corresponds to the only amplitude value of the ordered Cantor spectrum.

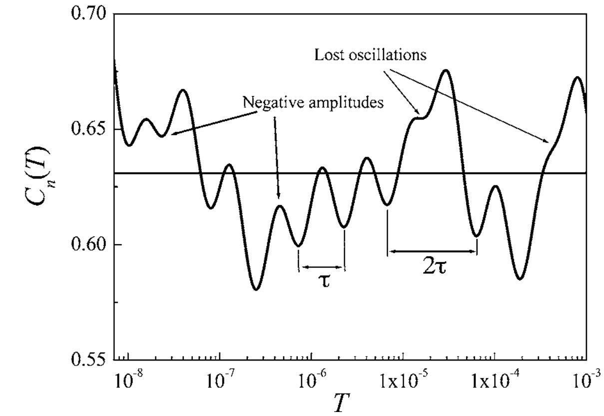

FIG. 4. The oscillatory part of the specific heat obtained for $\epsilon=0.05$. We mark with arrows negative amplitudes and lost oscillations.

Concerning the existence of negative amplitudes, note that due to the irregularity introduced by the disorder, one can find local maxima of $C(T)$ smaller than the average value, and local minima of $C(T)$ larger than the average value, leading directly to negative amplitudes. Both cases are marked with arrows in Fig. 4. As the oscillations become more irregular, i.e., for increasing ϵ, this effect can appear more often, as reflected in Fig. 3, where negative amplitudes are more likely to appear for larger ϵ.

In general, we can conclude that when the triadic Cantor energy spectrum is considered, the main effects produced by the disorder in the oscillations of the specific heat are such that, as the oscillations become more irregular, on the one hand, one can obtain more likely negative amplitudes for higher disorder, and on the other hand, larger periods than expected are also obtained because some oscillations can disappear. We use these results below to explain the behavior found in the specific heat of more general random fractal spectra.

III. GENERAL RANDOM CANTOR SPECTRA

In Sec. III, we have considered the effect of the disorder on the classical or triadic Cantor energy spectra. Now, we study the properties of the specific heat of more general random fractal spectra. The generation of these spectra is as follows. As usual, our starting point is the unity interval $[0,1]$. In the first step of the generation process, we select two segments of equal size at the left and right sides of the unity interval and neglect the central part. The size p_1 of the two selected segments is drawn at random from the uniform probability distribution

$$P(p) = \frac{1}{p_{\max}} \quad \text{if } 0 < p \leq p_{\max} \tag{4}$$

and $P(p)=0$ otherwise. The parameter $p_{\max}$ controls the amount of disorder present in the spectrum. In the second step of the generation process, a second scale factor p_2 is

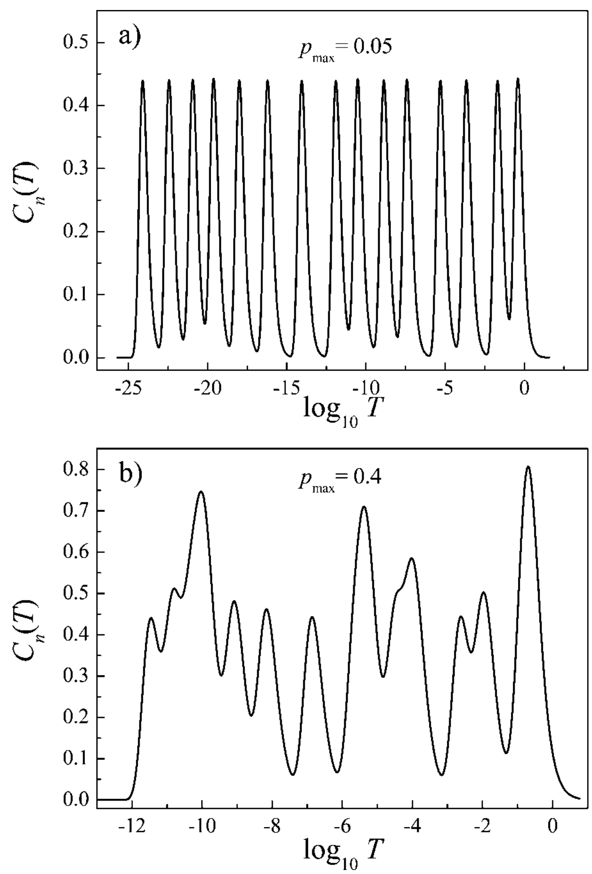

FIG. 5. (a) Specific heat as a function of $\log_{10} T$ obtained for a random spectrum generated with $p_{\max}=0.05$ and $n=15$. (b) The same, but for a spectrum generated with $p_{\max}=0.4$.

extracted randomly from (4). This scale factor is applied on the two segments of size p_1 in a similar way as before. Each one is divided in three parts: the leftmost and rightmost ones of size p_1p_2, and the central part is neglected. If this procedure is iterated n times, we say we have a disordered spectrum at the nth step of generation. The resulting structure after n steps of generation consists of 2^{n-1} segments, each one of size S_n given by

$$S_n = \prod_{i=1}^{n} p_i. \tag{5}$$

As we consider discrete spectrum, our energy levels at the nth step of the generation are given by the borders of the segments above referred, and therefore, the spectrum is formed by 2^{n+1} energy levels with the corresponding structure.

Intuitively, the larger $p_{\max}$, the wider the distribution of scales present in the energy spectrum, and therefore, one should expect a higher irregularity in the corresponding specific heat by increasing $p_{\max}$. We show the specific heats obtained for $p_{\max}=0.05$ and $p_{\max}=0.4$ in Figs. 5(a) and 5(b), respectively. In both cases, we show a particular realization of the disorder for $n=15$. As expected, for low $p_{\max}$, we observe a fairly regular oscillatory behavior of $C(T)$, while for high $p_{\max}$ the oscillations are mostly chaotic.

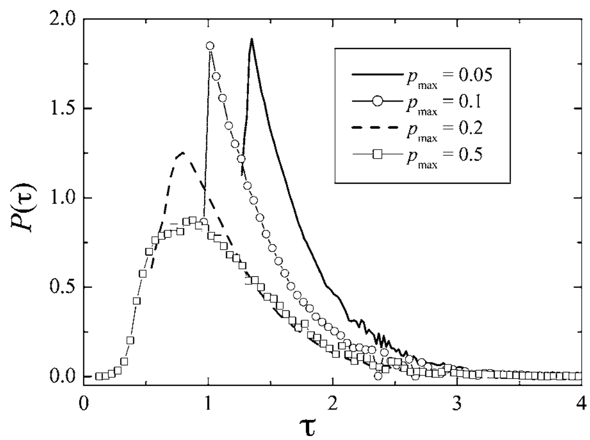

FIG. 6. Behavior of the normalized distributions of periods of the specific heat for different degrees of disorder in the random spectra.

In general, the presence of disorder produces a variety of periods and amplitudes, both defined in (1)–(3). Thus, instead of a single period and a single amplitude, we obtain now distributions of these magnitudes. Nevertheless, not only periods and amplitudes are affected by the disorder, but also the proper mean value of $C(T)$, which can be different for different disorder realizations. As the disorder present in the spectrum is quantified by $p_{\max}$, we study the distributions of periods, amplitudes, and mean values of $C(T)$ as a function of $p_{\max}$ in Secs. III A–III C.

A. Periods

The effect of increasing $p_{\max}$ should lead, in principle, to a wider distribution of periods, because of the presence of a larger diversity of scales in the fractal spectrum. To show this behavior, in Fig. 6, we plot the normalized distribution of periods for different values of $p_{\max}$. In order to obtain the distributions, for any $p_{\max}$ value, we have generated 10^4 random fractal energy spectra up to $n=20$, calculated the corresponding specific heats, and measured the individual periods according to (1). For low $p_{\max}$ (see the cases $p_{\max}=0.05$ and 0.1 in Fig. 6), we obtain a sharp peak in the region of low periods from where the distribution decreases monotonically as the period value increases. As $p_{\max}$ increases, on the one hand, the peak of the distribution is shifted to the left, and on the other hand, the peak looses gradually its sharp profile and the distribution becomes wider.

It is possible to explain the behavior of the distribution of periods as a function of $p_{\max}$ if we interpret the results (Fig. 6) using the known results of deterministic fractal energy spectra. For a deterministic fractal spectrum, it is known that by increasing in a unity the step of generation n of the fractal, we obtain an additional oscillation in the specific heat. In addition, as the scale factor p is constant (for example, $p=1/3$ for the triadic Cantor set), any new step in the generation process produces an oscillation with a period given by $\tau=\log_{10}(1/p)=-\log_{10} p$, which is also constant ($\tau=\log_{10} 3$ in the triadic Cantor set). Thus, in principle, we could hypothesize that any scale factor p_i introduced in the generation of

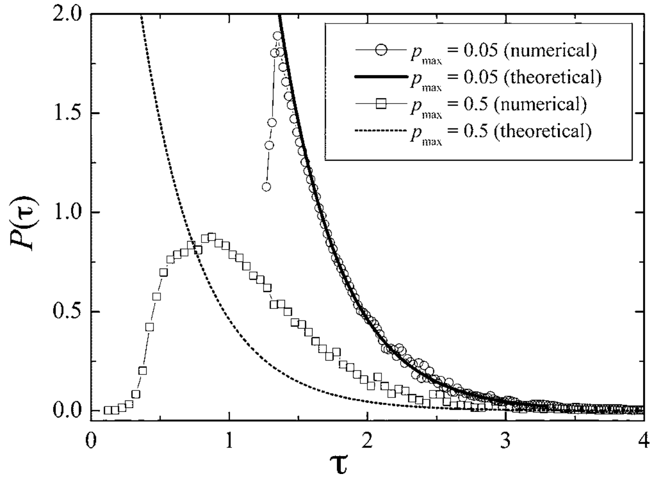

FIG. 7. Behavior of the distributions of periods for two extreme values of the disorder. The solid lines correspond to the analytical predictions obtained by considering that the specific heat behaves quasideterministically.

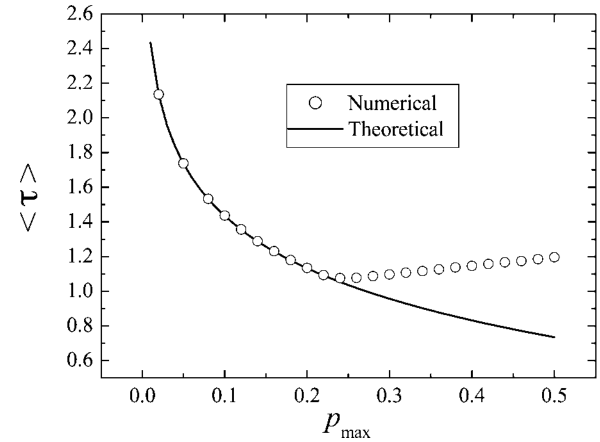

FIG. 8. Behavior of the mean value of the periods of the oscillations of the specific heat as a function of p_{max}. The solid line corresponds to the analytical result obtained by considering quasideterministic behavior.

the random fractal spectrum could lead to a period given by

$$\tau_i = -\log_{10} p_i. \tag{6}$$

Taking into account that our scale factors p_i are uniformly distributed in the interval $[0, p_{max}]$ according to (4), it is straightforward to obtain that the normalized distribution of periods τ_i as defined in (6) is given by

$$P(\tau) = \ln 10 \frac{10^{-\tau}}{p_{max}}, \tag{7}$$

where the range of variation of τ is $[-\log_{10} p_{max}, \infty)$, because the smallest period is obtained for the larger p_i value (p_{max}), and the largest one corresponds to the case $p_i \to 0$. In Fig. 7, we show the distribution of periods obtained numerically for two extreme values of p_{max} together with the corresponding theoretical distributions given by (7).

We observe that for low p_{max} (see the case p_{max}=0.05 in Fig. 7), our hypothetical distribution of periods (7) is essentially correct, because it agrees fairly well with the numerical result. Nevertheless, for high p_{max}, the theoretical distribution does not reproduce at all the numerical result. On the one hand, we find a smaller probability of small periods than expected, and on the other hand, a much larger number of large periods than expected. Thus, we can conclude that for low p_{max} (low disorder), our working hypothesis is basically correct: Each scale factor p_i produces a period τ_i given by (6), in the same way as in the case of deterministic fractal spectra. But when a large value of p_{max} is considered (large disorder), this is not true. The reason for this behavior is essentially similar to the one we have discussed before in Sec. II. When the disorder is so large, the oscillations become very irregular [see Fig. 5(b)] and some oscillations can even disappear (as we shown before in Fig. 4), thus biasing the distribution of periods to larger values. We can say that while for low p_{max} values, we observe a quasideterministic behavior, for high p_{max} values, the disorder dominates.

We next study quantitatively the effect of the disorder on the distribution of periods to find where the disorder starts to be more important than the deterministic behavior. To perform the analysis, we use the mean value of the distribution of periods, $\langle \tau \rangle$, and we study this mean value as a function of p_{max}. Note that the theoretical value of $\langle \tau \rangle$ can be obtained directly from (7)

$$\langle \tau \rangle = \int_{-\log_{10} p_{max}}^{\infty} \tau P(\tau) d\tau = \frac{1 - \ln p_{max}}{\ln 10}. \tag{8}$$

In Fig. 8, we show the behavior of $\langle \tau \rangle$ as a function of p_{max}. The solid line represents the theoretical value (8) and the symbols correspond to the numerical result.

We find that the quasideterministic behavior is the dominant effect up to approximately $p_{max} \simeq 0.25$, because the numerical result coincides perfectly with the theoretical one when $p_{max} < 0.25$. When $p_{max} > 0.25$, both curves separate from each other, and even the numerical result becomes an increasing function of p_{max}, while the theoretical one continues decreasing as p_{max} increases. In the value $p_{max} \simeq 0.25$, we find a transition point (a critical value of the disorder) below which the behavior is similar to the one obtained in deterministic fractal spectra and above which the disorder is dominant and the specific heat presents mostly chaotic oscillations.

B. Amplitudes

Now we study the behavior of the amplitudes of the oscillations in a similar way as we have just done with the periods. In Fig. 9, we show the normalized distributions of amplitudes $P(A)$ for three different values of the disorder parameter p_{max}. For any p_{max} value, we have generated 10^4 random energy spectra up to n=20 and calculated any individual amplitude as defined in (2) and (3).

For low p_{max} values (low disorder), we observe a distribution of bimodal nature (see the case p_{max}=0.05 in Fig. 9). This finding is a reminiscence of the deterministic behavior.

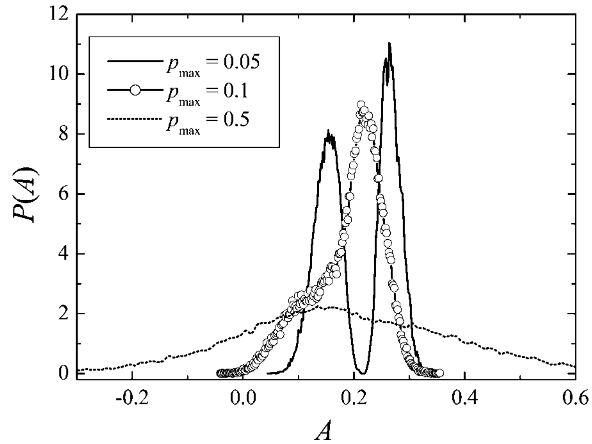

FIG. 9. Normalized distributions of amplitudes for several degrees of disorder.

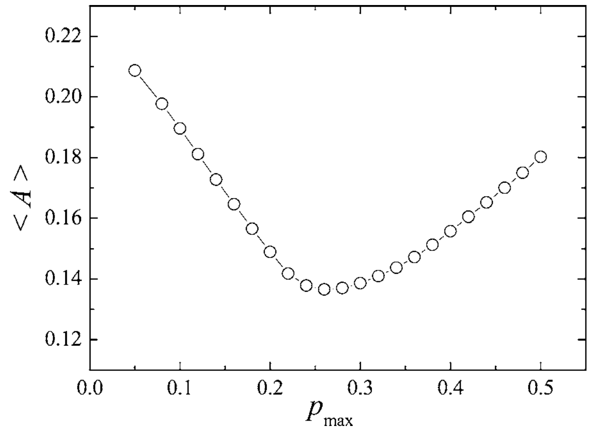

FIG. 10. Behavior of the mean value of the amplitudes as a function of p_{max}.

It is known [18] that when deterministic fractal spectra are considered, when the spectrum is generated with a small and constant scale factor p, the oscillations of the specific heat, although perfectly regular and periodic, are nonharmonic. This nonharmonicity is reflected in the amplitudes. Although the amplitudes are constant, it can be distinguished two different values: a large value corresponding to the amplitude of the oscillations above the mean value of $C(T)$ [which is defined in (2)] and a small value corresponding to the amplitude of the oscillations below the mean value of $C(T)$ [defined in (3)]. When small disorder is introduced in the spectrum (low p_{max}), the amplitudes are nonconstant, and thus, we obtain distributions; but still the distributions reflect the behavior of the deterministic case, and this is the reason of the bimodal structure observed in Fig. 9 for $p_{max}=0.05$. If p_{max} increases, we observe that the distribution is wider and also that the two peaks approach to each other and eventually collapse and give rise to a single peak (as the case $p_{max}=0.1$ in Fig. 9). The collapse of the two peaks into a single one also appears for deterministic fractal spectra if the scale factor p increases: In this case, the oscillations of the specific heat become harmonic [18], and therefore, the two amplitude values collapse into a single one. If p_{max} increases more, the distributions of amplitudes become wider and lower, just reflecting the increasing irregularity of the oscillations. We note that for intermediate and high values of p_{max}, the probability of negative amplitudes is not zero and increases with p_{max}. The reason is that for high disorder, the oscillations are very irregular, and one can find either local maxima of $C(T)$ below the mean value $\langle C\rangle$ or local minima of $C(T)$ above $\langle C\rangle$ (see Fig. 4), both cases producing negative amplitudes which are more likely to appear for higher p_{max} value.

We next study the behavior of the mean value of the amplitudes, $\langle A\rangle$ as a function of the disorder parameter p_{max}, which is shown in circles in Fig. 10. For low p_{max}, we obtain a large value of $\langle A\rangle$. As p_{max} increases, $\langle A\rangle$ decreases monotonically up to a certain value of p_{max} (around $p_{max}\simeq 0.25$), for which the trend of $\langle A\rangle$ changes and it becomes an increasing function of p_{max}. Strikingly, the p_{max} value at which the behavior of p_{max} changes coincides with the point from where the periods of the oscillations are controlled by the disorder. Nevertheless, in the amplitude case, we have not an analytical way to predict $\langle A\rangle$ in the same way we have done before with $\langle\tau\rangle$, which was useful to detect the point where the disorder starts to dominate. In spite of the lack of this analytical way to determine $\langle A\rangle$, it is known that for deterministic fractal spectra the amplitude of the oscillations is a decreasing function of the fixed scale factor p in the whole range of p. Thus, according to Fig. 10, we can conclude that when $p_{max}<0.25$, the amplitude of the oscillations behave similarly to the deterministic case. Note that this result is in agreement with the behavior of the distribution of amplitudes for low p_{max} (see Fig. 9) where the bimodal structure of $P(A)$ was a consequence of the quasideterministic behavior. When $p_{max}>0.25$, the disorder dominates, the quasideterministic behavior disappears, and the amplitudes reflect the chaotic behavior of the specific heat. Note that this transition point in the disorder parameter p_{max} around 0.25 coincides with our previous results of the behavior of the periods of the oscillations.

C. Mean value of *C*(*T*)

As we referred above, the presence of disorder in the fractal spectrum not only affects to the amplitudes and periods of the oscillations of the specific heat, but also to the corresponding mean value of the oscillations $\langle C\rangle$, which is different for different realizations of the disorder. Thus, for any p_{max} value, we obtain distributions of $\langle C\rangle$ values (one value for each energy spectrum). The probability distributions $P(\langle C\rangle)$ of the $\langle C\rangle$ values for different values of p_{max} are shown in Fig. 11. For any p_{max} value, we have generated 5×10^4 random energy spectra up to $n=20$, calculated the corresponding specific heat for each spectrum and obtained the individual average value.

In general, the distributions $P(\langle C\rangle)$ are centered distributions very symmetric around the single peak, and they look approximately like Gaussian distributions. For low p_{max} values, corresponding to low disorder, the distribution is narrow and the peak is placed at low values of $\langle C\rangle$ (see Fig. 11). For increasing p_{max}, the peak is shifted to higher values of $\langle C\rangle$, and the distribution becomes lower and wider, simply reflect-

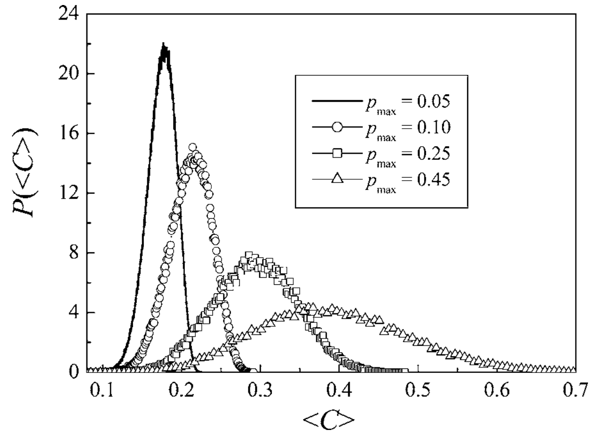

FIG. 11. Normalized distributions of the specific heat mean value for different degrees of disorder.

ing the presence of large disorder in the spectrum.

To better quantify the effect of the disorder over $\langle C\rangle$, we next study the behavior of the average value of the distributions shown in Fig. 11 as a function of p_{max}. Actually, this quantity would be *the average value of the mean values*, and we term it from now on as $\overline{\langle C\rangle}$. The behavior of $\overline{\langle C\rangle}$ as a function of p_{max} is shown in circles in Fig. 12. We observe that $\overline{\langle C\rangle}$ increases almost linearly as a function of p_{max}.

We note that when deterministic fractal spectra are considered, the average value $\langle C\rangle$ of the specific heat is given by the fractal dimension d_{box} of the spectrum, which for a generic spectrum generated with a scale factor p is given by

$$d_{box} = -\frac{\log_{10} 2}{\log_{10} p} = \langle C\rangle. \tag{9}$$

This equation shows that in deterministic fractals, when p increases, $\langle C\rangle$ also increases, so in some sense, we can say that our results for random fractals shown in Fig. 12 seem to indicate a quasideterministic behavior for $\overline{\langle C\rangle}$. But this conjecture can be studied on an analytical basis: If we assume quasideterministic behavior, Eq. (9), valid for deterministic spectra, can be easily generalized to random spectra in the following way:

$$\overline{\langle C\rangle} = -\frac{\log_{10} 2}{\langle \log_{10} p\rangle}, \tag{10}$$

where $\langle \log_{10} p\rangle$ corresponds to the average value the logarithms of our scale factors p_i, which are uniformly distributed in the interval $[0, p_{max}]$. Thus, according to (4), this average value can be easily determined

$$\langle \log_{10} p\rangle = \int_0^{p_{max}} P(p)\log_{10} p dp = \frac{\ln p_{max} - 1}{\ln 10}, \tag{11}$$

and therefore, introducing (11) into (10) and simplifying, we get

$$\overline{\langle C\rangle} = \frac{\ln 2}{\ln p_{max} - 1}. \tag{12}$$

This theoretical behavior of $\overline{\langle C\rangle}$ is represented in Fig. 12 in a solid line and coincides almost perfectly with the numerical behavior. The coincidence is better in the low p_{max} region (in the low disorder region), but the deviation is not significant in the large p_{max} region.

Thus, we can conclude that, concerning the average value of the specific heat, the disorder does not introduce any significant influence, since the mean value of the average specific heat behaves similarly to the deterministic case. In other words, it behaves in a quasideterministic way in the whole range of p_{max}, in contrast to the behavior of the periods and of the amplitudes, for which we found a different behavior in the low disorder region and in the high disorder region.

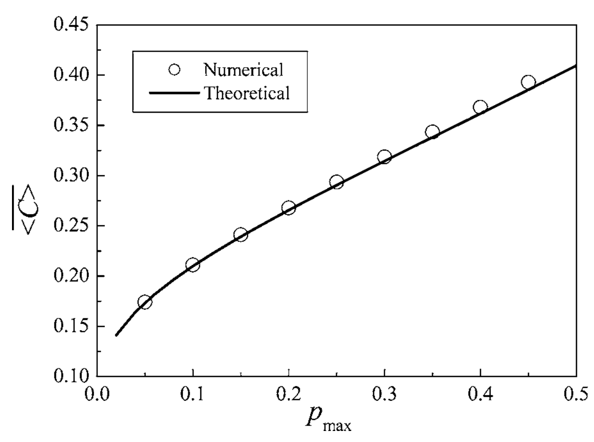

FIG. 12. Behavior of $\overline{\langle C\rangle}$ as a function of p_{max}. The solid line correspond to the theoretical prediction by considering quasideterministic behavior of the specific heat.

IV. AVERAGE DETERMINISTIC FRACTAL SPECTRUM

In this section, we consider the following question: Given a set of random fractal energy spectra generated with the same value of p_{max} is it possible to construct a deterministic fractal energy spectra for which the properties of the specific heat coincide with the corresponding average values of the set of random spectra? If this is the case, the deterministic spectrum can represent the set of random spectra, and we could say that the effect of the disorder is not very important.

Using the results of Sec. III, we show that indeed it is possible to construct this deterministic fractal spectrum under certain conditions. First, we study how to construct a deterministic fractal, i.e., generated with a single scale factor p, for which the average value of the specific heat $\langle C\rangle$ coincides with the value of $\overline{\langle C\rangle}$ of the set of random spectra generated with the same p_{max}. According to Eqs. (9) and (12), the scale factor p must satisfy the following equation:

$$-\frac{\log_{10} 2}{\log_{10} p} = \frac{\ln 2}{\ln p_{max} - 1} \tag{13}$$

that can be solved easily to give

$$p = \frac{p_{\max}}{e}, \tag{14}$$

i.e., a deterministic fractal generated with this p value presents the same value of $\langle C \rangle$ as the set of random spectra generated with $p_{\max}$. In addition, according to Fig. 12, the deterministic fractal provides the correct result in the whole range of $p_{\max}$, i.e., independently on the amount of disorder present in the energy spectrum.

We next study the deterministic fractal for which the period of the oscillations τ coincides with the average period value $\langle \tau \rangle$ of the set of random spectra generated with the same $p_{\max}$. Taking into account Eq. (8) and also that for a deterministic fractal the period is $\tau = -\log_{10} p$, the scale factor p must satisfy now

$$-\log_{10} p = \frac{1 - \ln p_{\max}}{\ln 10}, \tag{15}$$

which can be solved to give exactly the same p value as in (14). This means that if we construct a deterministic fractal with the scale factor p in (14), not only the average value of the specific heat agrees with the average value of the random spectra, but also the period of the oscillations agrees with the average period $\langle \tau \rangle$ of the random spectra. Nevertheless, we point out that this latter affirmation is only true when $p_{\max} < 0.25$, i.e., when we obtained quasideterministic behavior for the periods (see Fig. 8). When $p_{\max} > 0.25$, Eq. (8) does not provide the correct $\langle \tau \rangle$ value, and it is not possible to construct the correct deterministic fractal spectrum. As we discussed before, the range $p_{\max} > 0.25$ can be identified with the region where the disorder dominates the behavior of the specific heat.

Now we study the possibility of generating a deterministic fractal spectrum with an amplitude of the specific heat oscillations identical to the average value $\langle A \rangle$ of a set of random fractal spectra generated with the same value of $p_{\max}$. The problem in this case is that, in contrast to the period of the oscillations and the average value of the specific heat, there are no analytical results for the amplitudes. In addition, while $\langle C \rangle$ and τ depend *logarithmically* on the scale factors [see Eqs. (9)–(13)], the amplitudes of the oscillations depend on the scale factors themselves [18]. This is the reason why we hypothesize that the average value of the amplitudes of the oscillations $\langle A \rangle$ of the specific heat for a set of random spectra generated with the same $p_{\max}$ (i.e., with scale factors uniformly distributed in $(0, p_{\max}]$) should be similar to the amplitude of the oscillation of the deterministic spectrum generated with the *average scale factor* $\langle p \rangle$. According to (4), we have simply

$$\langle p \rangle = \frac{1}{p_{\max}} \int_0^{p_{\max}} p\,dp = \frac{p_{\max}}{2}. \tag{16}$$

In Fig. 13, we show the behavior of $\langle A \rangle$ for random and deterministic fractal energy spectra. In the random case, the data corresponds to the ones shown in Fig. 9, simply rescaled in the horizontal axis, because they are shown as a function of $p_{\max}/2$. Concerning the curve for deterministic fractals, it has been obtained by generating 100 deterministic fractal spectra with $n=20$ using 100 $\langle p \rangle$ values uniformly distributed in the interval $(0, 1/2)$, calculating for each spectrum the corresponding specific heat and averaging for each one the amplitude of the oscillations. We observe that the deterministic and the random results coincide almost exactly for $\langle p \rangle \lesssim 0.13$. At $\langle p \rangle \simeq 0.13$, they start to separate from each other. For $\langle p \rangle > 0.13$, the deterministic $\langle A \rangle$ value continues decreasing as $\langle p \rangle$increases, while the random value changes its previous decreasing trend and increases as $\langle p \rangle$ increases. These results are not surprising, because we have obtained quasideterministic behavior of $\langle A \rangle$ when $p_{\max} < 0.25$ in Sec. III, which corresponds exactly to the region of Fig. 13 where the two curves overlap, since $p_{\max} < 0.25$ corresponds to $\langle p \rangle \lesssim 0.13$. Thus, we can conclude that it is possible to construct a deterministic fractal spectrum for which the amplitude of the oscillations of the specific heat coincides with the corresponding average value of the random fractals provided that $p_{\max} < 0.25$.

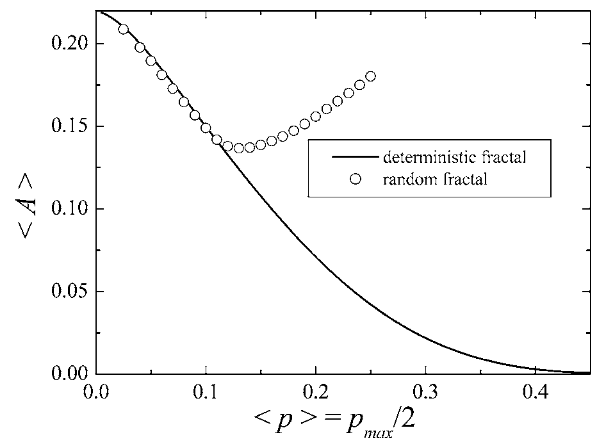

FIG. 13. The average value of the amplitude of the oscillations of the specific heat of random fractal spectra as a function of the average scale factor $\langle p \rangle$. The solid line corresponds to deterministic fractal spectra generated with a constant scale factor given by $\langle p \rangle = p_{\max}/2$.

To summarize, we have shown that when $p_{\max} < 0.25$, we can construct *two* different deterministic fractal spectra with specific heat properties identical to the average values of the specific heat obtained for the random spectra. The first one, generated with a scale factor $p = p_{\max}/e$ presents an average value of $C(T)$ and a period of the oscillations identical to the corresponding average values of the set of random spectra. The second one, generated with a scale factor $p = p_{\max}/2$, presents a value of the amplitude of the oscillations of the specific heat identical to the corresponding average value of the sets of random spectra. For $p_{\max} > 0.25$, the disorder dominates and the oscillatory regime becomes completely irregular.

V. CONCLUSIONS

In this paper, we have studied the properties of the specific heat obtained from random fractal energy spectra, which we have characterized using three properties: specific heat mean value, period of the oscillations around this mean value, and amplitude of the oscillations. We have shown that when disorder is introduced in the deterministic fractal spectra, distributions of these magnitudes are obtained, which we have studied as a function of the degree of disorder present in the spectra. In general, we have found a critical value of the disorder ($p_{\max}=0.25$) present in the random fractal spectra below which we obtain a quasideterministic behavior of the specific heat (reflected also in the possibility of constructing deterministic fractal spectra with the same properties as the random spectra). In this region, the effects of disorder are not so important and the fractal properties dominate, as it is usually considered in the bibliography. Nevertheless, for degrees of disorder above the critical value, the disorder becomes more important than the fractal properties: It controls the properties of the specific heat, which behaves chaotically.

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