Exact site percolation thresholds using a site-to-bond transformation and the star-triangle transformation

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I construct a two-dimensional lattice on which the inhomogeneous site percolation threshold is exactly calculable and use this result to find two more lattices on which the site thresholds can be determined. The primary lattice studied here, the "martini lattice," is a hexagonal lattice with every second site transformed into a triangle. The site threshold of this lattice is found to be 0.764826…, i.e., the solution to $p^4-3p^3+1=0$, while the others have $(\sqrt{5}-1)/2$ (the inverse of the golden ratio) and $1/\sqrt{2}$. This last solution suggests a possible approach to establishing the bound for the hexagonal site threshold, $p_c<1/\sqrt{2}$. To derive these results, I solve a correlated bond problem on the hexagonal lattice by use of the star-triangle transformation and then, by a particular choice of correlations derived from a site-to-bond transformation, solve the site problem on the martini lattice.

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I. INTRODUCTION

Percolation theory [1,2] concerns the appearance of infinitely connected components in randomly populated lattices. The problem was originally stated by Broadbent and Hammersley [3] and is one of the simplest examples of a process that exhibits a phase transition. Given a lattice, such as the one shown in Fig. 2, and referring to the line segments as bonds and the vertices between bonds as sites, we consider site percolation by declaring each site to be occupied with probability p and unoccupied with probability $1-p$. Two occupied sites that are connected by a bond are said to be part of the same cluster and it is clear that as p is increased, clusters of ever increasing size will appear. As shown by Broadbent and Hammersley, there is a clearly defined threshold, p_c, above which there is, with probability 1, a cluster containing an infinite number of sites, and below which there is no such cluster. This number is called the percolation threshold and is unique to each lattice. We can also have bond percolation, in which bonds are declared open or closed with probability p and $1-p$. A bond problem can be transformed into a site problem by covering every bond with a site, and then connecting two sites if their underlying bonds were adjacent, thus forming the "covering lattice." For example, the kagomé lattice (Fig. 6) is the covering lattice of the hexagonal lattice, and so the former's site threshold is equal to the latter's bond threshold. Not every lattice is a covering lattice, however, so in this sense site percolation is more general.

Despite the problem's innocuous appearance, the exact calculation of thresholds on non-trivial lattices is very difficult. Many problems are still outstanding, such as site percolation on the square and hexagonal lattices and bond percolation on the kagomé lattice [4]. There are a few two-dimensional (2D) lattices that have been solved, however. For example, 2D lattices with a certain property, self-duality for bond problems and self-matching for site problems, are known to all have $p_c=\frac{1}{2}$ [1]. In a pioneering paper, Sykes and Essam [5] used a nonrigorous method called the star-triangle transformation to find the exact values of the bond thresholds for the triangular and hexagonal lattices. With some ingenuity, Wierman [6] adapted the star-triangle transformation to find the bond threshold on the bowtie lattice. The star-triangle transformation has clearly been a very useful tool; every exact result in two dimensions is either $\frac{1}{2}$, meaning it is self-matching or self-dual, or relies on the star-triangle transformation in some way. In the present work, I extend this method to problems with some limited correlation structure.

It is necessary to make a distinction between exact solutions and proofs. As mentioned before, the star-triangle transformation is a nonrigorous method that nonetheless predicts the correct bond thresholds for the triangular, hexagonal, and square lattices. Although it was already widely accepted, it was not until 1980 that the value of $\frac{1}{2}$ for the bond threshold of the square lattice was eventually proved by Kesten [7]. That the predictions for the other two lattices were also correct was proved by Wierman [8] in 1981, many years after the conjecture of Sykes and Essam. The present paper presents nonrigorous but nevertheless exact results. However, it is hoped that it is a straightforward exercise to put the arguments on a rigorous footing.

The primary lattice studied here, which I call the "martini lattice" due to the shape of the basic cell, is the one shown in Figs. 1 and 2. Each site has three nearest neighbors, but the lattice is nonuniform because some sites are $(3,9^2)$ while others are (9^3) in the notation of Grünbaum and Shephard [9], in which one lists the number of sides of each polygon that surround a site. This lattice is mentioned on p. 186 of that book as an example of what they call a 2-homeohedral tiling of valence 3.

II. STAR-TRIANGLE TRANSFORMATION

The star-triangle transformation exploits the fact that if the bonds of a unit cell of the triangular lattice (**T**) are re-

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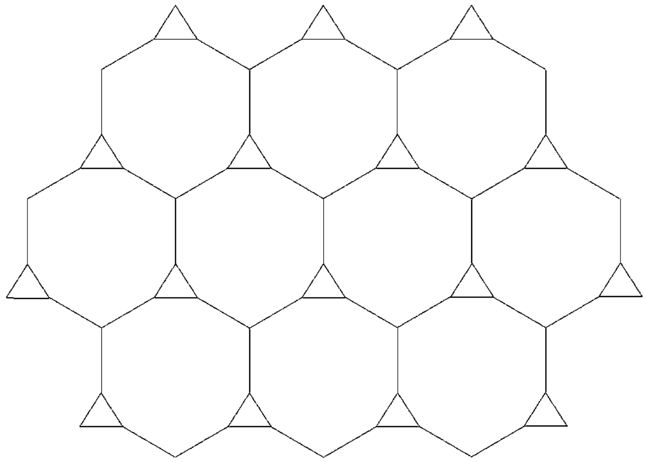

FIG. 1. The martini lattice, drawn to emphasize its origin as a hexagonal lattice with every second site transformed into a triangle.

placed by a corresponding star, as illustrated in Fig. 3, the result is the hexagonal lattice (**H**), which is the triangular lattice's dual (Fig. 4). The dual, L_d, of a graph L is formed by placing sites in the faces of L and connecting them with bonds that cross the bonds of L. The bond percolation threshold of a lattice and its dual are related, in 2D, by the well-known formula [1]

$$p_c^{\text{bond}}(L) = 1 - p_c^{\text{bond}}(L_d). \tag{1}$$

This means that the appearance of the infinite open cluster on L coincides with the disappearance of the infinite closed cluster on L_d. The star-triangle transformation leads to another relationship between the critical probabilities of **H** and **T** besides $p_c(\mathbf{T}) = 1 - p_c(\mathbf{H})$, which allows both to be determined exactly. The method even works for the inhomogeneous case, where the probabilities of each bond being open on the base triangle are different, resulting in a critical surface rather than a critical point.

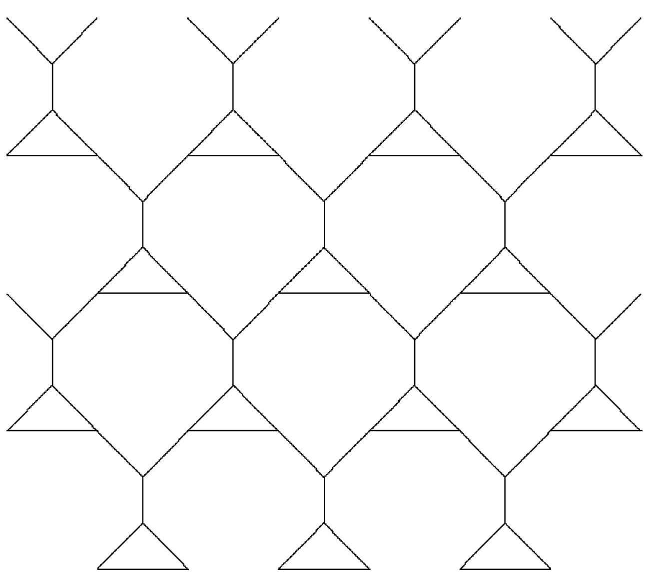

FIG. 2. Another representation of the martini lattice.

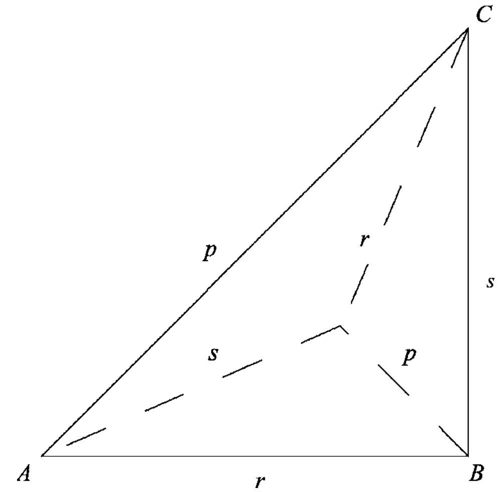

FIG. 3. The star-triangle transformation. p,r,s denote the probabilities of their corresponding bonds on the triangle and star, and A,B,C label the sites.

The argument proceeds as follows. Consider bond percolation on the triangle and superimposed star shown in Fig. 3. The probabilities p,s,r refer to the probabilities that their corresponding bonds are open on either the star or triangle. We can ask several questions about the connectedness of the sites A,B,C. For example, what is the probability that A is connected to both B and C, an event we will denote $P(A \rightarrow B, A \rightarrow C)$, through open bonds on the triangle? This is easily found to be

$$P(A \rightarrow B, A \rightarrow C) = ps + pr(1-s) + sr(1-p). \tag{2}$$

Next, we want the probability that A,B,C are connected through *closed* bonds on the star. We denote this event $Q^*(A \rightarrow B, A \rightarrow C)$. Q will hereafter denote the probability of events that happen in closed bonds and $*$ will indicate that the event happens by traversing the star rather than the tri-

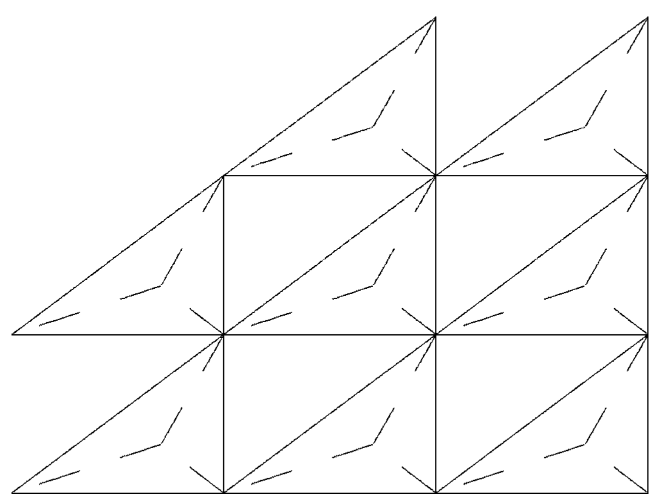

FIG. 4. The star-triangle transformation. Replacing each triangle with a dashed star transforms the triangular lattice into the hexagonal lattice.

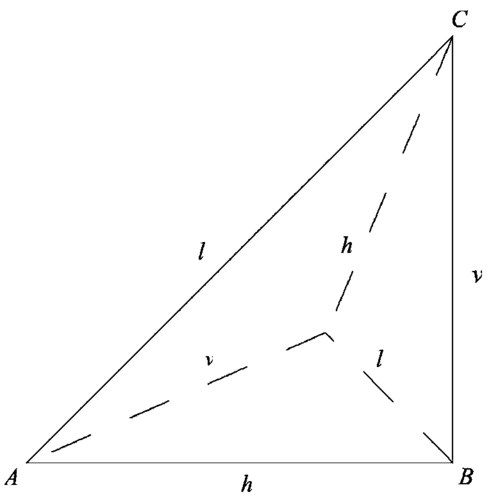

FIG. 5. Labels used in treating the correlated triangle. A,B,C label sites and v,h,l label bonds on the triangle and the corresponding bonds on the star—note that these are now names, not probabilities.

angle. Since this event only happens if all three bonds are closed, we have

$$Q^*(A \rightarrow B, A \rightarrow C) = (1-p)(1-r)(1-s). \tag{3}$$

If we now consider the entire lattice, we can see that the replacement of triangles by stars turns the triangular lattice into the hexagonal lattice, i.e., the dual (Fig. 4). As we will see, the condition

$$P(A \rightarrow B, A \rightarrow C) = Q^*(A \rightarrow B, A \rightarrow C) \tag{4}$$

defines the critical surface. Substituting our previous results into Eq. (4) and simplifying, we get

$$psr - p - s - r + 1 = 0. \tag{5}$$

Setting $s=r=p$ leads to the critical point for bond percolation on the triangular lattice, $p_c = 2\sin \pi/18$. To convince ourselves that Eq. (4) defines the critical surface, we can examine the other ways of connecting the sites. The probability that A connects B, but not C, denoted $P(A \rightarrow B, A \nrightarrow C)$, is given by

$$P(A \rightarrow B, A \nrightarrow C) = r(1-p)(1-s). \tag{6}$$

Also,

$$Q^*(A \rightarrow B, A \nrightarrow C) = r(1-p)(1-s) \tag{7}$$

so that $P(A \rightarrow B, A \nrightarrow C) = Q^*(A \rightarrow B, A \nrightarrow C)$ for all (p,r,s) and similarly for the events $(A \nrightarrow B, A \rightarrow C)$ and $(B \rightarrow C, B \nrightarrow A)$. The condition

$$P(A \nrightarrow B, A \nrightarrow C) = Q^*(A \nrightarrow B, A \nrightarrow C) \tag{8}$$

also leads to Eq. (5). Because all open connectivities internal to a triangle are equivalent to the closed connectivities on the star when this condition is satisfied, the connectivity of open bonds on **T** is exactly the same as that of the closed bonds on **H**. This means that if there is an infinite open cluster on **T**, then there is an infinite closed cluster on **H**. This leads, by our previous discussion of duality, to the conclusion that there is neither an infinite open cluster on **T** nor an infinite closed cluster on **H**, i.e., we are on the critical surface.

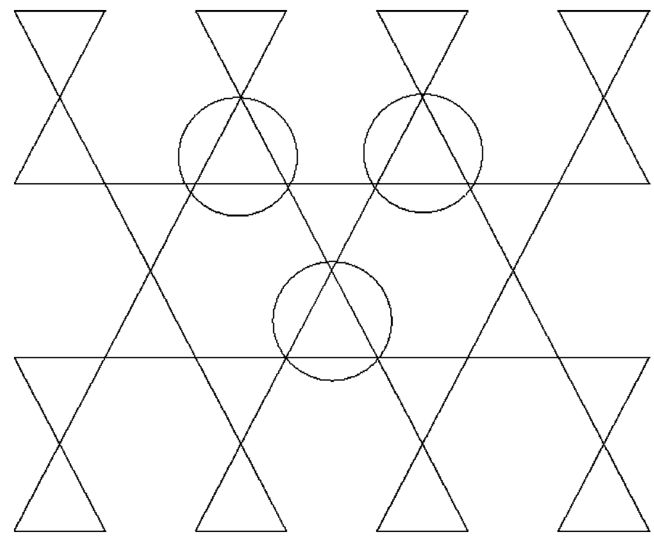

FIG. 6. The kagomé lattice. The circled triangles are the ones on which we apply the site-to-bond transformation.

III. CORRELATED BOND PERCOLATION ON THE TRIANGULAR LATTICE

The method of Sykes and Essam can be extended to the case in which the bonds of the triangle are not independent. Our critical surface will now appear as a constraint between the one-, two-, and three-point joint probabilities. It is important to note that although all bonds in a triangle are correlated, there are no correlations between neighboring triangles, so a given bond is only correlated to two of its neighbors. The dual lattice is constructed analogously to the uncorrelated case, with each bond in the dual inheriting the probabilities and now the correlations of the original lattice. Labeling the bonds v,h,l as shown in Fig. 5, we will deal with the quantities $P(v)$, $P(h)$, $P(l)$, $P(h,l)$, $P(v,h)$, $P(v,l)$, and $P(h,v,l)$, which are the set of one-, two-, and three-point joint probabilities of the indicated bonds being open. Probabilities of bonds being closed will be denoted with a bar over the bond name, e.g., $P(\bar{v})$. We can now repeat the procedure outlined above but with our joint probabilities,

$$P(A \rightarrow B, A \rightarrow C) = P(v,h) + P(v,l,\bar{h}) + P(h,l,\bar{v}) \tag{9}$$

and

$$Q^*(A \rightarrow B, A \rightarrow C) = P(\bar{v},\bar{l},\bar{h}). \tag{10}$$

Equating these gives our critical surface,

$$P(v,h) + P(v,l,\bar{h}) + P(h,l,\bar{v}) - P(\bar{v},\bar{l},\bar{h}) = 0. \tag{11}$$

There are many equivalent ways this can be expressed. For example, if we use the condition

$$P(A \nrightarrow B, A \nrightarrow C) = Q^*(A \nrightarrow B, A \nrightarrow C),$$

we obtain the more compact

$$P(v) + P(\bar{v},h,l) - P(\bar{h},\bar{l}) = 0. \qquad (12)$$

Although they look dissimilar, Eqs. (11) and (12) are in fact the same constraint and it is a simple matter to relate them using identities. The conditions $P(A \to B, A \nrightarrow C) = Q^*(A \to B, A \nrightarrow C)$, etc., are also satisfied for all choices of the probabilities, as before.

To compare with our earlier results, it is easy to see that setting $P(\bar{h},\bar{l}) = (1-r)(1-s)$, $P(\bar{v},h,l) = (1-p)rs$, $P(v) = p$ leads to Eq. (5).

IV. SITE-TO-BOND TRANSFORMATION

The process of forming the covering lattice of a given bond problem, by covering each bond with a site, is commonly referred to as the bond-to-site transformation. As previously mentioned, not every site problem is a covering lattice, so there is no inverse "site-to-bond" transformation that works in general. I will therefore use this name to describe a different transformation by which it is always possible to transform from a site problem into a bond problem. If we consider a realization of site percolation on a given lattice, we can transform it into a bond process by declaring a bond to be open if both its bounding sites are occupied, and closed otherwise. By doing this, we introduce correlations between neighboring bonds; the probability that a given bond is open is p^2, but the probability that a bond is open given that one of its neighbors is open is p. Furthermore, there are three-point correlations; the probability that a bond is open is 1 if two of its neighbors on opposite ends of the bond are open. It is clear that the existence or lack of an infinite open cluster are properties that will be shared by both the site and transformed bond problems. If we now consider sites on a triangle, we can use these rules to derive joint probabilities for the bonds, which we can then use in the criticality condition (12). It is easy to see that if the sites are occupied with probability p,

$$P(v) = p^2, \qquad (13)$$

$$P(\bar{h},\bar{l}) = (1-p)^3 + 3p(1-p)^2 + p^2(1-p), \qquad (14)$$

$$P(h,l,\bar{v}) = 0. \qquad (15)$$

If we use these in Eq. (12), we cannot expect to have solved the site problem on the triangular lattice. The critical surface is not appropriate to that problem because we have not included correlations between triangles. I suggest that the threshold we will discover is that of the site problem on the kagomé lattice (Fig. 6). To see this, consider the triangles outlined in Fig. 6. Clearly they are not correlated to each other if we use the site-to-bond transformation. But percolation of bonds on these triangles implies percolation of the lattice since, due to three-point correlations, a connecting bond on the separating triangles will be open with probability 1 if two bonds on either side of it are open. More specifically, consider the bonds on the two top circled triangles. Under the site-to-bond transformation, if the horizontal bond on each is open, then both bounding sites on each of the bonds are occupied. But if this is true, then it follows that the horizontal bond on the separating triangle is also open, thus our two original bonds are connected to each other by virtue of being open themselves, and they are independent. Thus, by inserting the separating triangles, we have preserved the way in which the circled triangles were connected to each other on the original triangular lattice and ensured that neighboring triangles are independent, thus enabling us to use our star-triangle result. Plugging expressions (13)–(15) into Eq. (12), we obtain the polynomial

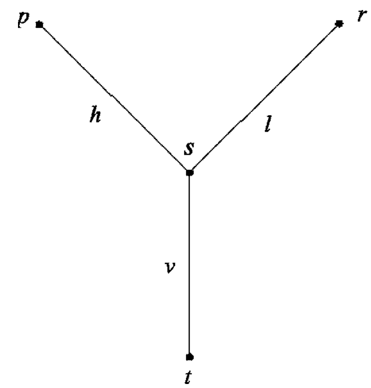

FIG. 7. The assignment of probabilities to the sites on the martini lattice.

$$1 - 3p_c^2 + p_c^3 = 0 \qquad (16)$$

with solution $p_c = 1 - 2\sin\pi/18$, which is indeed the critical threshold of the kagomé lattice. However, since this is just the covering lattice of **H**, its threshold has long been known through more elementary means.

We can obtain our new results by considering percolation on the star. The critical surface in this case will be given by the complement of Eq. (12),

$$P(\bar{v}) + P(v,\bar{h},\bar{l}) - P(h,l) = 0. \qquad (17)$$

In fact, we will consider the inhomogeneous site problem, and assign probabilities p,r,s,t, to the sites as shown in Fig. 7,

$$P(\bar{v}) = 1 - st, \qquad (18)$$

$$P(h,l) = prs, \qquad (19)$$

$$P(v,\bar{h},\bar{l}) = st(1-p)(1-r). \qquad (20)$$

This leads to the critical surface

$$1 - rst - prs - pst + stpr = 0, \qquad (21)$$

which is the central result of this work. But to what lattice does it correspond? In the previous example, where we obtained the kagomé lattice, we needed to insert extra triangles to separate our correlated triangles. Inserting these separating triangles in between the correlated stars, we obtain the martini lattice shown in Fig. 2. The critical threshold for site percolation is obtained by setting $r=s=t=p$,

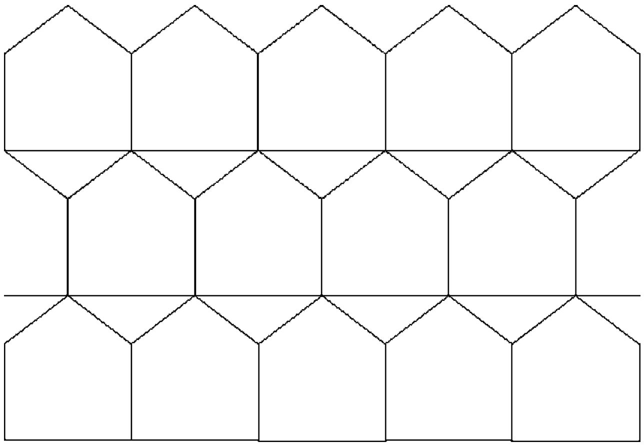

FIG. 8. The lattice obtained from the martini lattice by setting $t=r=1$. On this lattice, $p_c^{\text{site}}=(\sqrt{5}-1)/2=0.618034\ldots$ and $p_c^{\text{bond}}=\frac{1}{2}$.

$$1-3p_c^3+p_c^4=0, \tag{22}$$

which has a solution on [0,1],

$$p_c=0.764826\ldots\ . \tag{23}$$

This number has an exact representation in terms of radicals, but it is complicated and not very illuminating. We can obtain further results by making different choices for the probabilities. For example, by setting $s=1$, $r=t=p$, we expect the kagomé lattice to reappear, since $s=1$ effectively turns the star back into a triangle. Plugging these into Eq. (21), we indeed get Eq. (16). Other choices are possible that lead to a variety of results.

A. $t=r=1$

The corresponding lattice is the one shown in Fig. 8 and it resembles a stack of houses or a neighborhood. It is nonuniform, with some sites $(5^2,3,5,3)$ with five nearest neighbors and others $(3,5^2)$ with three nearest neighbors, and falls somewhere between the hexagonal lattice and the $(3^3,4^2)$ lattice. It is also self-dual, meaning we can immediately locate its bond threshold at $\frac{1}{2}$. If the $(5^2,3,5,3)$ sites have probability s and the $(3,5^2)$ sites p, the critical locus for site percolation is

$$1-s-ps=0. \tag{24}$$

Setting $s=p$ leads to the critical threshold

$$1-p_c-p_c^2=0 \tag{25}$$

or $p_c=(\sqrt{5}-1)/2=1/\phi=0.618034\ldots$, where $\phi=(\sqrt{5}+1)/2=1.618034\ldots$ is the golden ratio [10].

This site threshold is an unexpected place for the golden ratio to appear, and it is interesting to note that the bond threshold of the square lattice with every horizontal bond doubled also satisfies Eq. (25), even though the two problems are not related by any obvious transformation.

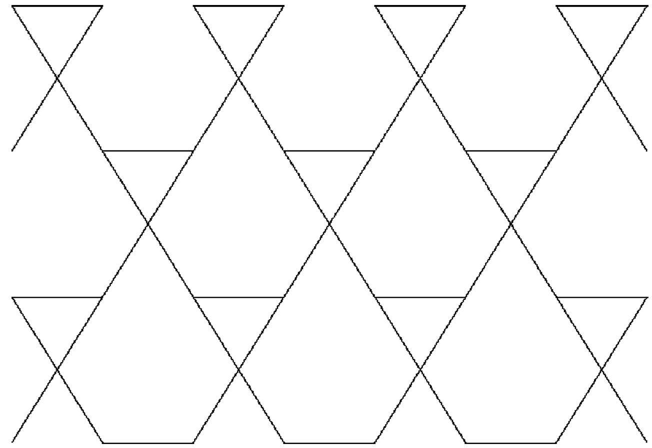

FIG. 9. The lattice obtained by setting $r=1$. The site threshold is $p_c=1/\sqrt{2}$.

B. $s=t=1$

Setting $s=t=1$ and $r=p$, we get the covering lattice of the square bond problem, leading to $p_c=\frac{1}{2}$, which is a very roundabout way of solving that problem.

C. $r=1$

Again, a star is turned into a triangle, but a different one from that which produced the kagomé lattice earlier. The lattice that results here is shown in Fig. 9. The critical surface is

$$1-st-ps=0. \tag{26}$$

Setting $s=t=p$ yields

$$1-2p_c^2=0, \tag{27}$$

which means $p_c=1/\sqrt{2}=0.707107\ldots$. This value has been verified numerically by Robert Ziff [11] to within ± 0.000001. This is an interesting result for several reasons. For one, some sites have three nearest neighbors while others have four. Thus, if we were to make a guess strictly on the basis of nearest neighbors, we might be led to believe that

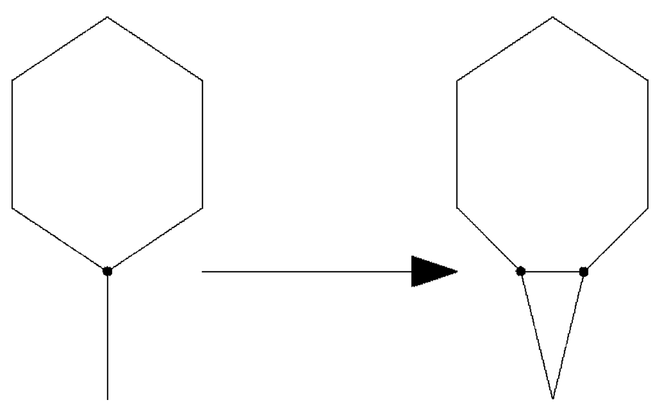

FIG. 10. The transformation that takes the hexagonal lattice to the one shown in Fig. 9. The site at the bottom of every hexagon is divided into two sites, and a bond is inserted between them.

this lattice's site threshold is smaller than that of the hexagonal lattice, where every site has three nearest neighbors. The contrary is true, however, as we know from numerical results [12]. Actually, the value $1/\sqrt{2}$ is noteworthy in itself. Usually, simple values like this indicate an elementary transformation from a known lattice with $p_c=\frac{1}{2}$. If such a transformation exists in this case, it does not appear to be obvious. Interestingly, $p_c=1/\sqrt{2}$ was once conjectured to be the exact site threshold for the hexagonal lattice [13] but was shortly thereafter judged unlikely from numerical considerations [14].

This result suggests an approach to finding an upper bound for $p_c^{\text{site}}(\mathbf{H})$. Consider the following procedure for producing this lattice. Starting with the usual hexagonal lattice, take the site at the bottom of each hexagon, split it into two sites, and connect them together with a bond, as shown in Fig. 10. All that is required is to show that this procedure always increases the critical probability and one will have shown that

$$p_c^{\text{site}}(\mathbf{H}) < \frac{1}{\sqrt{2}}. \tag{28}$$

It remains to be seen whether this suggestion actually simplifies the problem.

V. CONCLUDING REMARKS

I have found three lattices whose site percolation thresholds can be calculated exactly, and shown that one of these solutions might lead to an upper bound for the hexagonal lattice.

It should be noted that a special case of Eq. (21) has previously appeared in the literature, though in a slightly different context. In considering a mixed site/bond problem, Kondor [13] derived an expression that matches Eq. (21) for $r=t=p$. The situation he considered was bond percolation on the triangular lattice in which there is a "three-site" bond in the middle of every second triangle. The dual is a hexagonal lattice with a site in the middle of every second star. This problem is evidently isomorphic to site percolation on the martini lattice.

ACKNOWLEDGMENTS

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