Scale invariant forces in one-dimensional shuffled lattices

Andrea Gabrielli
Istituto dei Sistemi Complessi, CNR, Via dei Taurini 19, 00185-Rome (Italy) and SMC-INFM, Department of Physics, University "La Sapienza" of Rome, P.le Aldo Moro 2, 00185-Rome, Italy
(Received 15 June 2005; published 13 December 2005)

We present a detailed and exact study of the probability density function $P(F)$ of the total force F acting on a point particle belonging to a perturbed lattice of identical point sources of a power-law pair interaction. The main results concern the large-F tail of $P(F)$ for which two cases are mainly distinguished: (i) Gaussian-like fast decreasing $P(F)$ for lattice with perturbations forbidding any pair of particles to be found arbitrarily close to one each other and (ii) Lévy-like power-law decreasing $P(F)$ when this possibility is instead permitted. It is important to note that in the second case the exponent of the power-law tail of $P(F)$ is the same for all perturbations (apart from very singular cases) and is in a one-to-one correspondence with the exponent characterizing the behavior of the pair interaction with the distance between the two particles.

DOI: 10.1103/PhysRevE.72.066113 PACS number(s): 02.50.−r, 05.40.−a, 61.43.−j

I. INTRODUCTION

Knowledge of the statistical properties of the force acting on a particle belonging to a gas and exerted by all the other particles provides important information in many physical contexts and applications. Typical examples are the (i) distribution of the gravitational force in a gas of masses in cosmological and stellar astrophysical applications [1–3], (ii) distribution of molecular and dipolar interactions [4] in a gas of particles, (iii) theory of defects in condensed matter physics [5], and (iv) granular materials [6,7]. The first seminal work in this field was due to Chandrasekhar [1], which deals, among many subjects, with the gravitational force probability distribution in a homogeneous Poisson spatial distribution of identical particles. By studying the characteristic function of the sum of the stochastic forces due to single particles, the probability distribution of this total force is exactly found to be given by the so-called Holtzmark distribution, which is a three-dimensional analog of the one-dimensional fat-tailed stable Lévy distribution. In [2,8,4,9] approximated extensions to different branches of physics of this approach can be found for more complex particle distributions (i.e., point processes) obtained by perturbing a homogeneous Poisson point process. In this paper we present a study of the total force probability distribution for a very different class of spatial particle distributions (i.e., point processes), the perturbed lattices of point particles, in the case in which the pair interaction decays spatially as a general power law. We think that this study can be very useful for applications in both solid-state physics (e.g., in the case of Coulomb or dipolar pair interaction) [4] and cosmology where n-body gravitational simulations (introduced to study the problem of "structure formation" due to gravitational collapse from primordial cosmological mass density fluctuations) are performed usually starting from suitable perturbed lattice initial conditions [10]. We limit the study to the one-dimensional (1D) case in order to avoid difficulties related to the anisotropies of higher-dimensional lattices. However, the exact results we present in 1D are suggestive of the behavior of the same quantities in higher dimensions. In fact one can see [11] that the change of spatial dimension only renders calculations not explicitly performable, keeping qualitatively the behavior we present below.

II. DEFINITIONS AND FORMALISM

In order to approach in the proper way the problem of the global force probability distribution in a perturbed lattice of particles interacting via a power-law spatially decreasing pair interaction, let us consider first a gas of such identical particles with microscopic density

$$n(x) = \sum_i \delta(x - x_i),$$

where x_i is the position of the ith particle. Let us assume that the average number density $n_0=\langle n(x)\rangle > 0$ (where the average $\langle\cdots\rangle$ is to be intended as an ensemble average) is well defined (i.e., the particle distribution is uniform on sufficiently large scales [9]). We then suppose that particles interact via a pair force $f(x)$ depending on the mutual pair distance x as

$$f(x) = -C\frac{x}{|x|^{\alpha+1}}.$$

This means that $f(x)$ gives the force exerted by a particle in the origin on another particle in x (the force is attractive if $C>0$ and repulsive if $C<0$). Therefore the force field in the point x of the space will be

$$\mathcal{F}(x) = C\sum_i \frac{x_i - x}{|x_i - x|^{\alpha+1}} \equiv C\int dy n(y)\frac{y-x}{|y-x|^{\alpha+1}}, \tag{1}$$

where the last integral is over all the space. However, from the last expression of Eq. (1), being $n_0>0$, we have that for a given realization of the stochastic density field $n(x)$, the infinite volume limit of $\mathcal{F}(x)$ is not univocally defined for $\alpha \leqslant 1$ (i.e., the integral is *absolutely diverging*, and its value depends on how this limit is taken). The same feature is present in higher spatial dimensions. For example in $d=3$ the same problem is present for $\alpha \leqslant 3$ and in general $\alpha \leqslant d$ in d

dimensions. For instance this is the case of the gravitational force in a self-gravitating homogeneous gas of identical masses [1] and of a Coulomb interaction in the one-component plasma (OCP) [12] of identical electrical charges in both the disordered and ordered (i.e., the Coulomb lattice [13]) phases. This problem is well known in condensed matter physics about the OCP. However, in this case, the problem is automatically solved by the presence in the physical system of a uniform background charge density $n_b(x)=-n_0$ with opposite sign with respect to the identical charged particles and such that to conserve global charge neutrality in the system. Once the attractive force of the background is considered on a charged particle together with the repulsive forces exerted by the other particles, the problem of an infinite-volume limit of $\mathcal{F}$ is solved and its value unique. For what concerns the self-gravitating systems, in Newtonian gravitation an analog of the uniform background of the OCP [i.e., a negative uniform mass density $n_b(x)=-n_0$ such as generate a repulsive force on the particles] does not exist, and it has to be introduced in the system artificially to regularize the problem (an approach usually called *Jeans' swindle* [14]). However this negative background comes out naturally as an effect of space expansion when the gravitational motion of particles is described, starting from the equations of general relativity, in comoving coordinates in a quasiuniform expanding Einstein–de Sitter universe [15] which is the main model of universe used in cosmology. In practice considering the presence of such a balancing background will give for $\mathcal{F}$ the following expression:

$$\mathcal{F}(x) = C\int dy\,\delta n(y)\frac{y-x}{|y-x|^{\alpha+1}}, \tag{2}$$

where $\delta n(x)=n(x)-n_0$. This makes $\mathcal{F}$ to be defined also for smaller α depending on the small-k behavior of the power spectrum $S(k)\sim\langle|\widetilde{\delta n}(k)|^2\rangle$ of the density field where $\widetilde{\delta n}(k)$ is the Fourier transform of $\delta n(x)$. By studying the large-distance scaling behavior of the integrated fluctuations of $n(x)$ [9], it is simple to show that, assuming $S(k)\sim k^\beta$ at small k, in d dimension $\mathcal{F}$ is a well-defined statistical quantity (i.e., its value does not depend on the way in which the infinite-volume limit is taken) for $\alpha>(d-\beta)/2$ if $\beta<1$ and $\alpha>(d-1)/2$ if $\beta\geq 1$ (see also the discussion in Appendix B on the definiteness of the force $\mathcal{F}$, respectively, in the shuffled lattice and the homogeneous Poisson particle distributions). Note finally that taking the infinite-volume limit symmetrically with respect to the point x on which the force is calculated, the background gives a zero net force on the point x. Therefore the value of $\mathcal{F}$ obtained calculating Eq. (1) taking the infinite-volume limit symmetrically with respect to the point x gives automatically the well-defined value obtained by Eq. (2) (i.e., subtracting the effect of the background). This preliminary discussion of the statistical definiteness of the force $\mathcal{F}$ is useful to justify the symmetrical way in which we take the infinite-volume limit in the one-dimensional shuffled lattice case we analyze in the rest of the paper. The statistical properties of $\mathcal{F}$ we willfind in this peculiar way (no background and symmetrical limit) coincide with those of the case in which the effect of a negative background is considered independently of the way in which the infinite-volume limit is taken, which therefore can be considered the *real* physical case. Moreover, from the above considerations we deduce that, as in the shuffled lattice $S(k)\sim k^2$ at small k (see [16]), the results are valid for all values $\alpha>0$ in $d=1$.

Let us take, therefore, a 1D regular chain of $2N+1$ unitary mass particles with a lattice spacing $a>0$ (we will take eventually the limit $N\to+\infty$); i.e., the position of the nth particle is $X_n=na$. Therefore the microscopic density can be written as

$$n_{\text{in}}(x) = \sum_{n=-N}^{N}\delta(x-na).$$

Clearly the average density of particles in the system is $n_0=1/a$. We now apply an uncorrelated displacement field (i.e., a random shuffling) to this system; i.e., a random displacement U_n is applied to the generic nth particle independently of the other particles. This displacement field is completely characterized by the one-displacement probability density function (PDF) $p(u)$ [i.e., $\text{Prob}(u\leq U_n<u+du)=p(u)du$]. After the application of the displacements the new microscopic density will be

$$n(x) = \sum_{n=-N}^{N}\delta(x-na-u_n), \tag{3}$$

the u_n's being realizations of the random variables U_n all extracted from $p(u)$ independently of each other. We consider the case $p(u)=p(-u)$ for simplicity. Equation (3) says that the particle originally in $X_n=na$, after the displacement, will be in $X_n=na+U_n$. For the analysis of spatial density correlations in such a system see [16]. Let us call $q_n(x)$ the PDF of the position of the nth particle [i.e., $\text{Prob}(x\leq x_n<x+dx)=q_n(x)dx$]. Clearly it is given by

$$q_n(x) \equiv p(x-na).$$

Let us now assume, as above, that the nth particle creates a force field in the point x of the type

$$f_n(x) = C\frac{X_n-x}{|X_n-x|^{\alpha+1}},$$

with $\alpha>0$ and C a constant. Therefore the total stochastic field $\mathcal{F}$ generated at a generic point x of the space by all the system particles is

$$\mathcal{F}(x) = C\sum_{n=-N}^{N}\frac{X_n-x}{|X_n-x|^{\alpha+1}}. \tag{4}$$

Note that it is a sum of random variables. Let us call $W_0(F)$ its PDF; it will be given by

$$W_0(F) = \int \int_{-\infty}^{\infty} \left[\prod_{n=-N}^{N} dx_n p(x_n - na) \right] \times \delta\left(F - C \sum_{n\neq 0}^{-N,N} \frac{x_n - x}{|x_n - x|^{\alpha+1}} \right).$$

It is immediate to see that $\mathcal{F}(x)$ is the sum of independent random variables $C(X_n - x)/|X_n - x|^{\alpha+1}$. However, as the PDF's $q_n(x)$ change with n, these variables are not identically distributed. As shown below, this and the fact that $\mathcal{F}$ needs not a normalization in N to be well defined in the large-N limit are the reasons why we do not obtain in general an exact Gaussian or Lévy limit [17,18] for the $W_0(F)$. In order to study the asymptotic behavior in F and N, it is usual to introduce the so-called *characteristic function* of $\mathcal{F}$—i.e., the Fourier transform (FT) of $W_0(F)$:

$$\hat{W}_0(k) \equiv \int_{-\infty}^{\infty} dF W_0(F) e^{ikF} = \prod_{n=-N}^{N} \int_{-\infty}^{\infty} dy p(y - na) \exp\left(iCk \frac{y - x}{|y - x|^{\alpha+1}} \right). \quad (5)$$

By studying the small-k behavior of the single integrals in Eq. (5) and taking appropriately the limit $N \to \infty$ we can deduce the moments and the large-F behavior of $W_0(F)$.

However, we will study a slightly different and more difficult problem, which is of particular interest if we want to study the dynamics of the system particles under the effect of only this mutual force. We study directly the statistical properties of the stochastic force acting on one generic system particle. In particular we calculate the total force acting on the particle initially located at the origin of the space and displaced to $X_0 = U_0$:

$$\mathcal{F} = \sum_{n\neq 0}^{-N,N} f_n(X_0) = C \sum_{n\neq 0}^{-N,N} \frac{X_n - X_0}{|X_n - X_0|^{\alpha+1}}. \quad (6)$$

In this way, taking the limit $N \to \infty$, we get the symmetric infinite-volume limit with no negative background of Eq. (1) with $x = X_0$, which is, as explained in the first paragraph of this section, equal to Eq. (2) with an arbitrary way of taking the infinite-volume limit, and therefore giving the force on any particle belonging to the system with the uniform balancing background. Note that now, because of the presence of the variable X_0 in each term of the sum (6), $\mathcal{F}$ is no longer a sum of independent terms. However, we show below how to reduce the problem to that of a sum of independent stochastic terms by introducing the concept of a *conditional probability density function*. The solution of this *conditional* problem will give also a way to approach the study of the first *unconditional* case given by Eq. (4).

About $\mathcal{F}$ in Eq. (6), we want again to find the PDF $W(F)$ of the value F of this force. As before, since the displacements applied to the particles are independent of each other, we have the exact relation

$$W(F) = \int \int_{-\infty}^{+\infty} \left[\prod_{n=-N}^{N} dx_n p(x_n - na) \right] \times \delta\left(F - C \sum_{n\neq 0}^{-N,N} \frac{x_n - x_0}{|x_n - x_0|^{\alpha+1}} \right),$$

which, through a simple change of variables $\Delta_n = x_n - x_0$ or $n \neq 0$, can be rewritten as

$$W(F) = \int_{-\infty}^{+\infty} dx_0 p(x_0) \int \int_{-\infty}^{+\infty} \left[\prod_{n\neq 0}^{-N,N} d\Delta_n p(\Delta_n + x_0 - na) \right] \times \delta\left(F - C \sum_{n\neq 0}^{-N,N} \frac{\Delta_n}{|\Delta_n|^{\alpha+1}} \right).$$

Let us analyze the behavior of the conditional PDF $P(F;x_0)$, conditioned to the fact that the particle on which the force is evaluated is at $X_0 = x_0$:

$$P(F;x_0) = \int \int_{-\infty}^{+\infty} \left[\prod_{n\neq 0}^{-N,N} d\Delta_n p(\Delta_n + x_0 - na) \right] \times \delta\left(F - C \sum_{n\neq 0}^{-N,N} \frac{\Delta_n}{|\Delta_n|^{\alpha+1}} \right). \quad (7)$$

In this way, once x_0 is fixed, the total force $\mathcal{F}$ is the sum of independent contributions

$$f_n = C \frac{\Delta_n}{|\Delta_n|^{\alpha+1}}. \quad (8)$$

It is interesting to see in which cases $\mathcal{F}$ satisfies the central-limit theorem. We will see that it never satisfies this theorem even when its PDF is rapidly decreasing at large values. More precisely, we will see that even in this last case its PDF is dependent on the details of $p(u)$.

For the sake of simplicity of notation let us assume in the rest of the paper that $C=1$ (the repulsive case $C=-1$ will be trivially deduced from this). By performing the simple change of variables given by Eq. (8), it is possible to find the conditional (i.e., conditioned to $X_0 = x_0$) PDF $g_n(f;x_0)$ of the single stochastic force generated by the particle in X_n on the particle fixed in $X_0 = x_0$:

$$g_n(f;x_0) = \frac{|f|^{-1-1/\alpha}}{\alpha} p\left(\frac{f}{|f|^{1+1/\alpha}} + x_0 - na \right). \quad (9)$$

Clearly also the forces f_n so distributed are independent of each other. The support of $g_n(f;x_0)$ can be simply deduced from the one of $p(u)$.

III. LARGE-F ANALYSIS AND LIMIT THEOREMS

Let us take the FT of Eq. (7) in order to evaluate the conditional *characteristic function* of the stochastic variable $\mathcal{F}$:

$$\hat{P}(k;x_0) \equiv \int_{-\infty}^{+\infty} dF P(F;x_0) e^{ikF} = \prod_{n\neq 0}^{-N,N} \int_{-\infty}^{+\infty} dx p(x+x_0-na) \exp\left(ik\frac{x}{|x|^{\alpha+1}}\right). \tag{10}$$

Note that the quantity

$$\int_{-\infty}^{+\infty} dx p(x+x_0-na) \exp\left(ik\frac{x}{|x|^{\alpha+1}}\right) = \int_{-\infty}^{+\infty} df g_n(f;x_0) e^{ikf} \equiv \{e^{ikf}\}_{n;x_0}, \tag{11}$$

where $g_n(f;x_0)$, given by Eq. (9), is the conditional PDF of the field f felt by the particle in x_0 due to the particle in x_n, is the conditional characteristic function of this field f. Moreover, $\{a(f)\}_{n;x_0} = \int_{-\infty}^{+\infty} df g_n(f;x_0) a(f)$. It is important to note that, if $x_0=0$ (i.e., the particle on which we calculate the force is stuck at the origin), the condition $p(u)=p(-u)$ on the displacements of any particle would imply

$$\int_{-\infty}^{+\infty} dx p(x+na) \exp\left(ik\frac{x}{|x|^{\alpha+1}}\right) = \int_{-\infty}^{+\infty} dx p(x-na) \exp\left(-ik\frac{x}{|x|^{\alpha+1}}\right).$$

Consequently, if one fixes $x_0=0$, Eq. (10) becomes

$$\hat{W}(k) = \prod_{n=1}^{N} \left| \int_{-\infty}^{+\infty} dx p(x-na) \exp\left(ik\frac{x}{|x|^{\alpha+1}}\right) \right|^2.$$

However, the shift $x_0 \neq 0$ of the particle initially at the origin, and on which we calculate the force, breaks this symmetry, which is anyway recovered when the average over $p(x_0)$ is performed to proceed from $P(F;x_0)$ to $W(F)$ (however, we will see that this is a further source of noise when we calculate explicitly the variance of the force $\mathcal{F}$).

In order to proceed into the analysis of the PDF of $\mathcal{F}$, we have to distinguish two basically different cases.

(i) *Nonoverlapping condition* (NOC). No particle can be found arbitrarily close to any other particle; i.e., the supports,[1] respectively, of $p(u)$ and of $p(u-na)$, for all integer $n \neq 0$, have an empty overlap. The main case of physical interest in this class of displacement fields is when $\exists\, 0 < u_0 < a/2$ such that $p(u)=0$ for $|u| > u_0$.

(ii) *Overlapping condition* (OC). Particles can cross one each other and at least one pair of particles can be found arbitrarily close to one each other; i.e., the supports, respectively, of $p(u)$ and of $p(u-na)$, for at least an integer $n \neq 0$, have a nonzero overlap. The main case of physical interest in which this happens is when $\exists\, \epsilon > 0$ such that $p(u) > 0$ for all $|u| < a/2 + \epsilon$.

We will see that in the first case we obtain a rapidly decreasing $W(F)$ even though there is no constraint toward Gaussianity in the large-N limit, while in the second case we have a power-law-tailed $W(F)$ similarly to that of the three-dimensional Holtzmark distribution [1].

IV. DETAILED ANALYSIS OF Eq. (10)

Let us analyze the single factor of Eq. (10) which, as aforementioned, is the FT of the conditional PDF $g_n(f;x_0)$ of the force felt by the particle in $X_0=x_0$ due to only the particle in X_n:

$$\int_{-\infty}^{+\infty} dx p(x+x_0-na) \exp\left(ik\frac{x}{|x|^{\alpha+1}}\right) = \left\langle \exp\left(ik\frac{x}{|x|^{\alpha+1}}\right) \right\rangle_{n,x_0}, \tag{12}$$

where $\langle s(x) \rangle_{n,x_0}$ denotes the average of the function $s(x)$ over the shifted PDF $h_{n,x_0}(x) = p(x+x_0-na)$. In practice, if we indicate with simply $\langle s(u) \rangle = \int_{-\infty}^{+\infty} du p(u) s(u)$, then we can say that

$$\langle s(x) \rangle_{n,x_0} = \langle s(u+na-x_0) \rangle.$$

We want to study Eq. (11) in the limit of small k. Similarly to what pointed out in the previous section, the small-k behavior of $\langle \exp(ikx/|x|^{\alpha+1}) \rangle_{n,x_0}$ is different in the two cases in which, as a consequence of displacements, the pair of particles initially in $x=0$ and $x=na$ (i) cannot or (ii) can be found arbitrarily close to one each other—i.e., respectively, if the supports of $p(u)$ and $p(u-na)$ have an empty or a nonzero overlap.

Let us start with the case (i). If $\exists\, 0 < u^* < |n|a/2$ such that $p(u)=0$ for $|u| \geq u^*$, the exponent of $\exp(ikx/|x|^{\alpha+1})$ can take only limited values in the integral (12) [i.e., the support of $g_n(f;x_0)$ is restricted to only a finite interval of values of f]. Note that in the given hypothesis (i), if $n>0$, the quantity x can take only strictly positive values, while if $n<0$, it takes only strictly negative values. In this case, if $n>0$, we can write

$$\left\langle \exp\left(ik\frac{x}{|x|^{\alpha+1}}\right) \right\rangle_{n,x_0} = \int_{-\infty}^{+\infty} dx p(x+x_0-na) \sum_{m=0}^{+\infty} \frac{(ikx^{-\alpha})^m}{m!} = \sum_{m=0}^{+\infty} \frac{(ik)^m}{m!} \langle x^{-\alpha m} \rangle_{n,x_0} = \sum_{m=0}^{+\infty} \frac{(ik)^m}{m!} \langle (u+na-x_0)^{-\alpha m} \rangle, \tag{13}$$

where

$$\langle (u+na-x_0)^{-\alpha m} \rangle = \int_{-u^*}^{u^*} du p(u) (u-x_0+na)^{-\alpha m}.$$

If instead $n<0$, Eq. (13) becomes

[1] We call *support* of $p(u)$ simply the set of real values of u such that $p(u)>0$.

$$\left\langle \exp\left(ik\frac{x}{|x|^{\alpha+1}}\right)\right\rangle_{n,x_0} = \sum_{m=0}^{+\infty}\frac{(-ik)^m}{m!}\langle(-x)^{-\alpha m}\rangle_{n,x_0} = \sum_{m=0}^{+\infty}\frac{(-ik)^m}{m!}\langle(-u-na+x_0)^{-\alpha m}\rangle. \tag{14}$$

Note that, as $p(u)=p(-u)$, we have $\langle(-u-na+x_0)^{-\alpha m}\rangle=\langle(u-na+x_0)^{-\alpha m}\rangle$. In both cases,

$$\left|\left\langle\left(\frac{x}{|x|^{\alpha+1}}\right)^m\right\rangle_{n,x_0}\right| < (|n|a-2u^*)^{-\alpha m}$$

for any $m\geqslant 0$, and therefore the series in Eq. (13) absolutely converges. It is very important to note that, if $u^*<a/2$ (i.e., no pair of particles can be found arbitrarily close to one each other), all the factors in Eq. (10) can be represented as a Taylor series (13) to all orders m. As shown below in more detail, this implies that when $u^*<a/2$, $W(F)$ has all finite moments and therefore is rapidly decreasing at large F.

In the second case (ii) instead $\exists\ 0<\epsilon<a/2$ such that, at least $\forall\ u$ satisfying $|n|a/2-\epsilon<|u|<|n|a/2+\epsilon$, we have $p(u)>0$. In this case the particle initially at $\pm na$ and the particle initially at the origin can be found arbitrarily close. This implies that the quantity x (i.e., $u-x_0+na$) in Eq. (12) is permitted to take arbitrary small values up to zero, and therefore in the last expression of Eq. (13) there would be an infinite number of diverging terms of the last series. In other words, the Taylor-series sum in the second expression of Eq. (13) cannot be exchanged with the average operation $\langle\cdots\rangle_{n,x_0}$, and we expect a singular part in the small-k expansion of the average $\langle\exp(ikx/|x|^{\alpha+1})\rangle_{n,x_0}$. In order to find it, we rewrite it as in Eq. (11):

$$\left\langle \exp\left(ik\frac{x}{|x|^{\alpha+1}}\right)\right\rangle_{n;x_0} \equiv \int_{-\infty}^{+\infty} df g_n(f;x_0)e^{ikf} = \int_{-\infty}^{+\infty} dx p(x+x_0-na)\exp\left(ik\frac{x}{|x|^{\alpha+1}}\right).$$

Note that in this case, differently from the previous one, the support of $g_n(f;x_0)$ includes arbitrarily large values of $|f|$ for which, using Eq. (9), we have

$$g_n(f;x_0)=\frac{p(x_0-na)}{\alpha}|f|^{-(\alpha+1)/\alpha}+o(|f|^{-(\alpha+1)/\alpha}).$$

Let us call $M=[\alpha^{-1}]$ the integer part of α^{-1}. By using the results presented in Appendix A we can conclude that

$$\left\langle \exp\left(ik\frac{x}{|x|^{\alpha+1}}\right)\right\rangle_{n;x_0} = \sum_{m=0}^{M}\frac{(ik)^m}{m!}\langle(u+na-x_0)^{-\alpha m}\rangle + S_n(k;x_0), \tag{15}$$

where $S_n(k;x_0)$ contains all the terms of order higher than M, including the singular part of the small-k expansion of $\langle\exp(ikx/|x|^{\alpha+1})\rangle_{n;x_0}$ which is of order $1/\alpha$ in k. By using Eq. (A7), we can finally write

$$S_n(k;x_0)=\begin{cases}\dfrac{(-1)^{(M+1)/2}\pi p(x_0-na)}{\alpha\Gamma[(\alpha+1)/\alpha]\cos\left(\dfrac{\alpha^{-1}-M}{2}\pi\right)}k^{1/\alpha}+o(k^{1/\alpha}) & \text{for odd } M,\\[2ex] \dfrac{(-1)^{M/2}\pi p(x_0-na)}{\alpha\Gamma[(\alpha+1)/\alpha]\sin\left(\dfrac{\alpha^{-1}-M}{2}\pi\right)}k^{1/\alpha}+o(k^{1/\alpha}) & \text{for even } M.\end{cases} \tag{16}$$

Here, for simplicity, we have excluded the case in which exactly $M=1/\alpha$ for which we have logarithmic corrections in k to the above equations.

V. FINDING $W(F)$

At this point we can go further and classify the possible behaviors of $P(F;x_0)$ and $W(F)$. Basically we again distinguish the following two cases.

(i) $\exists\ \epsilon>0$ such that $\forall\ |u|>a/2-\epsilon$, one has $p(u)=0$.

(ii) $\exists\ 0<\epsilon<a/2$ such that, at least, $\forall\ u$ satisfying $a/2-\epsilon<|u|\leqslant a/2+\epsilon$, $p(u)>0$.

A. Case (i): Fast decreasing $W(F)$

In this case the system satisfies the NOC and all the factors in Eq. (10) can be expanded in the Taylor series (13) and (14) for all different n.

We can then write

$$\hat{P}(k;x) \equiv \int_{-\infty}^{+\infty} dF P(F;x) e^{ikF}$$
$$= \prod_{n=1}^{N} \langle e^{ik/(u-x+na)^{\alpha}} \rangle \langle e^{-ik/(u+x+na)^{\alpha}} \rangle$$
$$= \prod_{n=1}^{N} \left[\sum_{m=0}^{+\infty} \frac{(ik)^m}{m!} \langle (u-x+na)^{-\alpha m} \rangle \times \sum_{l=0}^{+\infty} \frac{(-ik)^l}{l!} \langle (u+x+na)^{-\alpha l} \rangle \right], \qquad (17)$$

where in the average $\langle\cdots\rangle$ over the displacement u we have used the symmetry property $p(u)=p(-u)$. Note that from Eq. (17), we have $\hat{P}(k;-x)=\hat{P}^{\dagger}(k;x)$, where $A^{\dagger}$ indicates the complex conjugate of A.

By calling again $u^* < a/2$ the maximal permitted displacement for each particle (i.e., the support of $p(u)$ is $[-u^*, u^*]$), we can find $\hat{W}(k)=\mathcal{F}[W(F)]$ by simply calculating the average

$$\hat{W}(k) = \int_{-u^*}^{u^*} dx p(x) \hat{P}(k;x). \qquad (18)$$

It is simple to verify that, as $\hat{P}(k;-x)=\hat{P}^{\dagger}(k;x)$ and $p(u)=p(-u)$, the function $\hat{W}(k)$ is real and $\hat{W}(k)=\hat{W}(-k)$. The Taylor expansion in k of $\hat{W}(k)$ is obtainable from Eqs. (17) and (18). Since it is a real function and is the characteristic function, only even powers of k are present. In particular the coefficient of the k^2 term is $-\overline{\mathcal{F}^2}/2$ where

$$\overline{h(\mathcal{F})} = \int_{-\infty}^{+\infty} dF W(F) h(F).$$

Actually, rigorously speaking, we should show that all the coefficients of the Taylor expansion of $\hat{W}(k)$ are convergent to finite values in the limit $N\to\infty$. It is simple to show it by expanding the terms $\langle (u\pm x_0+na)^{-\alpha m}\rangle$ of Eq. (17) in Taylor series of $(u\pm x_0)/na$ for $n\geqslant 1$ which is justified by the fact that in the given hypothesis $|u|+|x_0|\leqslant 2u^* < a$ and considering that

$$\sum_{m=0}^{+\infty} \frac{(ik)^m}{m!} (na)^{-\alpha m} \sum_{l=0}^{+\infty} \frac{(-ik)^l}{l!} (na)^{-\alpha l} = 1, \quad \forall \ n \geqslant 1.$$

Therefore we conclude that we can write $\hat{W}(k)$ in the following form:

$$\hat{W}(k) = \sum_{n=0}^{+\infty} (-1)^n \frac{\overline{\mathcal{F}^{2n}}}{(2n)!} k^{2n}.$$

It is simple to see that

$$\frac{\overline{\mathcal{F}^2}}{2} = \sum_{n=1}^{N} \{ \langle \langle (u-x+na)^{-2\alpha} \rangle_u \rangle_x - \langle \langle (u-x+na)^{-\alpha} \rangle_u \langle (u+x+na)^{-\alpha} \rangle_u \rangle_x \} + \sum_{n<l}^{1,N} \langle [\langle (u-x+na)^{-\alpha} \rangle_u - \langle (u+x+na)^{-\alpha} \rangle_u][\langle (u-x+la)^{-\alpha} \rangle_u - \langle (u+x+la)^{-\alpha} \rangle_u] \rangle_x, \qquad (19)$$

where for clarity we have redefined

$$\langle a(u) \rangle_u = \int_{-u^*}^{u^*} du a(u) p(u),$$

$$\langle a(x) \rangle_x = \int_{-u^*}^{u^*} dx a(x) p(x),$$

$$\langle \langle b(u,x) \rangle_u \rangle_x = \int_{-u^*}^{u^*} \int_{-u^*}^{u^*} du dx b(u,x) p(u) p(x).$$

It is matter of simple algebra to show that, for $p(u)=p(-u)$, Eq. (19) can be rewritten as

$$\frac{\overline{\mathcal{F}^2}}{2} = \sum_{n=1}^{N} \langle \langle (u-x+na)^{-2\alpha} \rangle_u - \langle (u-x+na)^{-\alpha} \rangle_u^2 \rangle_x + \sum_{n,l}^{1,N} \langle \langle (u-x+na)^{-\alpha} \rangle_u [\langle (u-x+la)^{-\alpha} \rangle_u - \langle (u+x+la)^{-\alpha} \rangle_u] \rangle_x. \qquad (20)$$

We see that the force variance is composed of two different contributions: the former, given by the first sum in Eq. (20), is mainly due to the fluctuations in the displacements u of all the sources of the force (in this term the average over x is only a smoothing operation), while the latter, given by the second sum, is determined basically by the fluctuations created by the stochastic displacement x of the particle initially in the origin on which we evaluate the force (in this term is the averages over u to play a role of simple smoothing).

It is interesting and useful in applications to calculate all the above expressions by evaluating all the terms in the sums in Eq. (20) to second order in $(u\pm x)/na$. In order to do this, we use the following second-order Taylor expansion for $B \ll A$:

$$(A+B)^{-\gamma} = A^{-\gamma}\left(1+\frac{B}{A}\right)^{-\gamma} = A^{-\gamma}\left[1-\gamma\frac{B}{A} + \frac{\gamma(\gamma+1)}{2}\left(\frac{B}{A}\right)^2 + o\left(\frac{B}{A}\right)^2\right].$$

From this, substituting, respectively, A with na and B with $u\pm x$, we have that

$$(u \pm x + na)^{-\gamma} \simeq (na)^{-\gamma}\left[1 - \gamma\frac{u \pm x}{na} + \frac{\gamma(\gamma+1)}{2}\left(\frac{u \pm x}{na}\right)^2\right].$$

Moreover, we have that $\langle\langle(u\pm x)^{2n+1}\rangle_u\rangle_x=0$ for any integer n due to the symmetry $p(u)=p(-u)$, while we have that $\langle\langle(u\pm x)^2\rangle_u\rangle_x=2\sigma^2$ where $\sigma^2=\langle u^2\rangle_u=\int_{-u^*}^{u^*}du u^2 p(u)$ is the variance of the single displacement. Therefore we can write

$$\langle\langle(u \pm x + na)^{-\gamma}\rangle_u\rangle_x \simeq (na)^{-\gamma}\left[1 + \gamma(\gamma+1)\frac{\sigma^2}{(na)^2}\right]$$

and

$$\langle\langle(u \pm x + na)^{-\alpha}\rangle_u^2\rangle_x \simeq (na)^{-2\alpha}\left[1 + \alpha(3\alpha+2)\frac{\sigma^2}{(na)^2}\right].$$

Henceforth,

$$\langle\langle(u \pm x + na)^{-2\alpha}\rangle_u - \langle(u \pm x + na)^{-\alpha}\rangle_u^2\rangle_x = \frac{\alpha^2\sigma^2}{(na)^{2(\alpha+1)}}.$$

Moreover,

$$\begin{aligned}&\langle\langle(u - x + na)^{-\alpha}\rangle_u\langle(u \pm x + la)^{-\alpha}\rangle_u\rangle_x \\ &\simeq (na)^{-\alpha}(la)^{-\alpha}\left[1 \mp \frac{\alpha^2\sigma^2}{(la)(na)} + \alpha(\alpha+1)\sigma^2\left(\frac{1}{(na)^2} + \frac{1}{(la)^2}\right)\right],\end{aligned}$$

from which

$$\begin{aligned}&\langle\langle(u - x + na)^{-\alpha}\rangle_u[\langle(u - x + la)^{-\alpha}\rangle_u - \langle(u + x + la)^{-\alpha}\rangle_u]\rangle_x \\ &\simeq \frac{2\alpha^2\sigma^2}{(la)^{\alpha+1}(na)^{\alpha+1}}.\end{aligned} \tag{21}$$

Using all these results in all the terms of Eq. (20), we obtain

$$\frac{\overline{\mathcal{F}^2}}{2} \simeq \alpha^2\sigma^2\left\{\sum_{n=1}^{N}\frac{1}{(na)^{2(\alpha+1)}} + 2\left[\sum_{n=1}^{N}\frac{1}{(na)^{\alpha+1}}\right]^2\right\}. \tag{22}$$

It is simple to verify that both sums in Eq. (22) are converging for $N\to+\infty$ for all $\alpha>0$, for which we can then rewrite

$$\frac{\overline{F^2}}{2} \simeq \frac{\alpha^2\sigma^2}{a^{2(\alpha+1)}}[\zeta(2\alpha+2) + 2\zeta^2(\alpha+1)], \tag{23}$$

where $\zeta(t)$ is the *Riemann zeta function* [note that for $t\to1^+$ we have $\zeta(t)\simeq1/(t-1)$]. Again the first term is due to the fluctuations in the position of the sources, while the second one is due to the fluctuations in the position of the particle on which we are calculating the force. In particular in Eq. (22) the generic term of the first sum give the relative weight of the nth-nearest-neighbor particles in determining the force on the particle in X_0. At last we can say that, in the case of displacements limited within a box well contained in a unitary cell around the initial lattice position, we can approximate $W(F)$ with a Gaussian PDF with zero mean and variance given by Eq. (23). However, as already pointed out, there is no constraint, in the limit $N\to\infty$, toward rigorous Gaussianity and non-Gaussian corrections are in general present.

B. Case (ii): Power-law-tailed $W(F)$

As shown above, this is the case in which the OC is satisfied; i.e., particles are permitted to jump out of their initial lattice positions beyond the limit of the related unitary cell in such a way to be found arbitrarily close to some other particle. Note that this is always the case when the support of $p(u)$ is unlimited—i.e., if $p(u)>0$, $\forall\ u\in\mathbb{R}$. However, the same kind of $W(F)$ is also obtained if $\exists\ u^*>a/2$ such that $p(u)>0$, $\forall\ u\in[-u^*,u^*]$ and zero outside. The difference between these two subcases is only in the amplitude of the power-law tail of $W(F)$ but not in its exponent. In general, if the particle initially at the lattice site $x=na$ is permitted, through displacements, to be found arbitrarily close to the particle initially at $x=0$, it will contribute to the product (10) through a factor of the type (15). If instead this is not permitted, it will contribute to (10) through a factor of the form (13) or (14) depending respectively on whether $n>0$ or $n<0$. In any case, if $M=[1/\alpha]$, in order to find the main terms of the small k expansion of $\hat{W}(k)$ [so as to determine the large-F tail of $W(F)$], it is sufficient to truncate all the small-k expansion of the different factors in Eq. (10) at most to the order $M+1$. For the sake of simplicity, let us limit the discussion to the case in which strictly $\alpha M<1$ in such a way to exclude logarithmic corrections in k. We can write

$$\begin{aligned}\hat{P}(k;x_0) \simeq &\prod_n^{(\mathrm{OC})}\left[\sum_{m=0}^{M}\frac{(ik)^m}{m!}\left\langle\left(\frac{x}{|x|^{\alpha+1}}\right)^m\right\rangle_{n,x_0} + A(n,x_0,\alpha)k^{1/\alpha}\right] \\ &\times\prod_l^{(\mathrm{NOC})}\left[\sum_{m=0}^{M+1}\frac{(ik)^m}{m!}\left\langle\left(\frac{x}{|x|^{\alpha+1}}\right)^m\right\rangle_{l,x_0}\right],\end{aligned} \tag{24}$$

where $A(n,x_0,\alpha)$ is the coefficient of the term $\sim k^{1/\alpha}$ in Eq. (16) and the first product on n is on the particles in X_n with $n\neq0$ which can be found arbitrarily near to the particle in X_0 (i.e., satisfying the OC with respect to the particle in X_0), while the product on l is on the particles in X_l with $l\neq0$ which have a positive minimal distance to the same particle (i.e., satisfying the NOC with respect to the particle in X_0). If $p(u)>0\ \forall\ u\in\mathbb{R}$, all system particles with $n\neq0$ are included in the first product. If instead $p(u)>0$ for $u\in[-u^*,u^*]$ with $u^*>a/2$ and zero outside, the first product includes only contributions from the particles with $-2u^*/a<n<2u^*/a$ and $n\neq0$, while the others are included in the second product. The large-F behavior of $P(F;x_0)$ and consequently of $W(F)$ is completely determined by the singular term of order $k^{1/\alpha}$ in the small-k expansion of $\hat{P}(k;x_0)$. It is simple to see that, up to the order $k^{1/\alpha}$

$$\hat{P}(k;x_0) = \sum_{m=0}^{M}c_m(x_0)k^m + c_{1/\alpha}(x_0)k^{1/\alpha},$$

where the the coefficients $c_m(x_0)$ can be deduced by counting from Eq. (24) [in particular $c_m(x_0)=i^m\widetilde{\mathcal{F}^m}(x_0)/m!$, where $\widetilde{\mathcal{F}^m}(x_0)=\int_{-\infty}^{+\infty}dFP(F;x_0)F^m$ is the mth moment of $P(F;x_0)$ and $m\leqslant M$] and

$$c_{1/\alpha} = \sum_{-2u^*/a<n<2u^*/a}^{n\neq 0} A(n;x_0),$$

where the formula includes also the case $u^* \to \infty$. The small-k expansion up to the order $k^{1/\alpha}$ of $W(F)$ will be, consequently,

$$\hat{W}(k) \simeq \sum_{m=0}^{M} b_m k^m + b_{1/\alpha} k^{1/\alpha},$$

where

$$b_m = \int_{-\infty}^{+\infty} dx_0 p(x_0) c_m(x_0) = \frac{i^m}{m!}\overline{\mathcal{F}^m} \quad \text{with } m \leq M,$$

$$b_{1/\alpha} = \int_{-\infty}^{+\infty} dx_0 p(x_0) c_{1/\alpha}(x_0),$$

where $\overline{\mathcal{F}^m} = \int_{-\infty}^{+\infty} dF W(F) F^m = \int_{-\infty}^{+\infty} dx_0 p(x_0) \widetilde{\mathcal{F}^m}(x_0) \equiv \langle \widetilde{\mathcal{F}^m}(x_0) \rangle$. It is possible to evaluate explicitly $b_{1/\alpha}$ by using Eq. (16).

We are now in the situation to connect the singular term $b_{1/\alpha}k^{1/\alpha}$ of the small-k expansion of $\hat{W}(k)$ to the large-F tail of $W(F)$ by using directly the arguments in Appendix A. This gives simply

$$W(F) \simeq B F^{-1-1/\alpha},$$

with

$$B = \frac{1}{\alpha}\int_{-\infty}^{+\infty} dx_0 p(x_0) \sum_{-2u^*/a<n<2u^*/a}^{n\neq 0} p(x_0 - na). \qquad (25)$$

Note that if the support of $p(x_0 - na)$ is much larger than a and $p(u)$ is smooth (i.e., approximately constant) on the scale a, we can approximate Eq. (25) with

$$B = \frac{1 - p(0)a}{\alpha a}.$$

Note that this last approximated expression is not dependent on the details of $p(u)$ for $u \neq 0$. Finally, we can observe that we have obtained a power-law-tailed $W(F)$ characterized by the same exponent of the case of a homogeneous Poisson particle distribution presented in Appendix B. The only differences are the two following.

(i) The amplitude of this power-law tail is reduced in the shuffled lattice with respect to that of the Poisson particle distribution, given by Eq. (B5), of a factor

$$\int_{-\infty}^{+\infty} dx_0 p(x_0) \sum_{-2u^*/a<n<2u^*/a}^{n\neq 0} a p(x_0 - na) \simeq 1 - p(0)a.$$

(ii) In the shuffled lattice we have this power-law tail for each $\alpha > 0$, while in the Poisson case the problem is not well defined for $\alpha \leq 1/2$ (see Appendix B).

VI. CONCLUSIONS

We have presented a detailed study of the PDF $W(F)$ of the stochastic force $\mathcal{F}$ generated by a randomly perturbed lattice of sources of a scale invariant attractive pair interaction field $f(x) = -Cx/|x|^{\alpha+1}$ with $\alpha > 0$ at distance x from the source.

In general we distinguish two cases.

(i) The NOC is satisfied and no pair of particles can be found at an arbitrarily small reciprocal distance.

(ii) The OC is satisfied and it exists at least one of such pairs of particles.

In the first case we have a fast decreasing $W(F)$ similar to a Gaussian PDF at large F, even though no constraint toward an exact Gaussian central limit theorem is found. In the second case a power-law-tailed $W(F)$ is found. The unique exponent of such power law is directly related to the pair interaction exponent α, while its amplitude depends also on the lattice spacing a [with respect to the unit distance through which we measure x in $f(x)$] and in general on the shape of the perturbations PDF $p(u)$. In particular in this case $W(F)$ has a power-law tail with the same exponent as the stable Lévy distribution found in the Poisson case (see Appendix B) but with a reduced amplitude, even though, analogously to the case (i), no constraint has been found toward the stable Lévy distribution.

Some further general considerations have can now be done.

(i) In the case in which the probability of finding arbitrarily close to one each other, the large-F behavior of $W(F)$ is basically determined by the small-x behavior of $f(x)$ and not at all by the the fact if it is long range or not. Therefore, if we considered a fast decreasing $f(x)$ but with the same divergence in $x=0$, we would have deduced the same conclusions about the exponent of the large-F tail of $W(F)$.[2]

(ii) For this reason, even if we consider a lattice perturbed by *correlated* displacements, we expect to obtain the partition into the two cases (i) and (ii) above considered depending on the possibility or not to find pair of particles arbitrarily close.

(iii) Actually different cases for the large-F tail of $W(F)$ between the Gaussian-like "fast decreasing" and Lévy-like "power-law"-tailed PDF with exponent $\beta = (\alpha+1)/\alpha$ are possible in very particular cases. These cases correspond to the choice of $p(u)$ such that $p(u) > 0$ exactly for $u \in [-a/2, a/2]$ and zero outside. By changing the limit behaviors of $p(u)$ when $u \to \pm a/2$ we can obtain different large F behaviors of $W(F)$. In particular, if $p(a/2) > 0$ and finite [we consider $p(u) = p(-u)$], we have the same case as (ii) described above with $\beta = (\alpha+1)/\alpha$. If instead $p(a/2) = 0$, depending on the behavior of $p(u)$ for $u \to (a/2)^-$, we will have different values of β but in general larger than $(\alpha+1)/\alpha$. If, finally, $p(a/2) = +\infty$ [in such a way that $p(u)$ remains anyway integrable], in general we obtain β smaller than $(\alpha+1)/\alpha$ [but always >1 so that $W(F)$ remains integrable].

[2]It is possible to show [19] instead that the large-x behavior of the pair interaction $f(x)$ determines the large-$(|x-y|)$ behavior of the field-field correlation function $\langle \mathcal{F}(x)\mathcal{F}(y) \rangle$.

APPENDIX A: FOURIER TRANSFORM OF POWER-LAW-TAILED PDF's

We are interested in the small-k behavior of the characteristic function $\hat{f}(k)$ of a given power-law-tailed PDF $f(x)$ which for large $|x|$ behaves as $A|x|^{-\alpha}$ with $\alpha>1$. Let us call $[\alpha]=n\geqslant 1$ the integer part of α. In this hypothesis $\hat{f}(k)$ has a regular Taylor expansion up to the order $n-1$ followed by a singular term proportional to $k^{\alpha-1}$:

$$\hat{f}(k) \equiv \int_{-\infty}^{+\infty} dx f(x) e^{ikx} = \sum_{m=0}^{n-1} \frac{(ik)^m}{m!} \overline{x^m} + \hat{f}_s(k), \qquad \text{(A1)}$$

where $\overline{x^m}=\int_{-\infty}^{+\infty}dx\, x^m p(x)$ and $\hat{f}_s(k)$ contains the singular part of $\hat{f}(k)$ and at small k is an infinitesimal of order $\alpha-1$ in k (if α is an integer it contains also logarithmic corrections).

Now we show that effectively at sufficiently small k, $\hat{f}_s(k)\sim Bk^{\alpha-1}$ (where now $a(k)\sim b(k)$ means that $\lim_{k\to 0}[a(k)/b(k)]=1$), giving an explicit expression for B as a function of both the amplitude A and the exponent α.

First of all, let us study the case of a function $h(x)$ that can be written as

$$h(x) = B|x|^{-\beta} + h_0(x), \qquad \text{(A2)}$$

where $B>0$, $0<\beta<1$ and $h_0(x)$ is a smooth function, integrable in $x=0$ and such that $x^\beta h_0(x)\to 0$ for $|x|\to\infty$. This means that $h(x)$ presents an even power-law tail. In this case the small-k behavior of $\hat{h}(k)=\int_{-\infty}^{+\infty}dx\, h(x)e^{ikx}$ is completely determined by the Fourier transform of $B|x|^{-\beta}$—i.e,

$$\hat{h}(k) \sim B\int_{-\infty}^{+\infty} dx |x|^{-\beta} e^{ikx}.$$

In order to perform this Fourier transform we introduce the integral representation

$$|x|^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty dz z^{\beta-1} e^{-|x|z}, \qquad \text{(A3)}$$

where $\Gamma(\beta)$ is the Euler gamma function. Using Eq. (A3) we can write

$$\int_{-\infty}^{+\infty} dx |x|^{-\beta} e^{ikx} = \frac{1}{\Gamma(\beta)} \int_0^\infty dz z^{\beta-1} \int_{-\infty}^{+\infty} dx e^{ikx-|x|z}$$
$$= \frac{2k^{\beta-1}}{\Gamma(\beta)} \int_0^\infty dq \frac{q^\beta}{1+q^2}.$$

Using the general relation valid for $0<\beta<1$,

$$\int_0^\infty dq \frac{q^\beta}{1+q^2} = \frac{\pi}{2\cos\left(\frac{\beta}{2}\pi\right)}, \qquad \text{(A4)}$$

we can conclude

$$\hat{h}(k) = \frac{B\pi}{\Gamma(\beta)\cos\left(\frac{\beta}{2}\pi\right)} k^{\beta-1} + o(k^{\beta-1}). \qquad \text{(A5)}$$

If instead of using the integral representation (A3) one used

$$|x|^{-\beta} = \frac{1}{\Gamma\left(\frac{\beta}{2}\right)} \int_0^\infty dz z^{\beta/2-1} e^{-x^2 z},$$

one should obtain an alternative expression containing only gamma functions:

$$\hat{h}(k) = \frac{B\sqrt{\pi}\,\Gamma\left(\frac{1-\beta}{2}\right)}{2^\beta \Gamma\left(\frac{\beta}{2}\right)} k^{\beta-1} + o(k^{\beta-1}).$$

Note that the coefficient of the term $k^{\beta-1}$ is real and positive. Another important case is when the function $h(x)$ has an odd nonintegrable power-law tail—i.e.,

$$h(x) = B|x|^{-\beta}[2\theta(x)-1] + h_0(x), \qquad \text{(A6)}$$

where $\theta(x)$ is the usual Heaviside step function, $B>0$, $0<\beta<1$, and $h_0(x)$ the same as in Eq. (A2). By using the same integral transformation leading to Eq. (A5), in this case we obtain

$$\hat{h}(k) = i\frac{B\pi}{\Gamma(\beta)\sin\left(\frac{\beta}{2}\pi\right)} k^{\beta-1} + o(k^{\beta-1}).$$

At this point we can go back to the problem of finding the dominant small-k contribution of the term $\hat{f}_s(k)$ in Eq. (A1) for the PDF $f(x)$ decaying at large $|x|$ as $A|x|^{-\alpha}$. Note that now we cannot apply directly the argument we have used for the above function $h(x)$. In fact if, from one side, also in this case we can write

$$f(x) = A|x|^{-\alpha} + f_0(x),$$

with $|x|^\alpha f_0(x)\to 0$ for $|x|\to\infty$, from the other side $\alpha>1$ (for definiteness of probability) and $f_0(x)$ contains a nonintegrable singularity at $x=0$ so that to cancel the nonintegrable contribution of the $A|x|^{-\alpha}$ term at small x. In order to circumvent this difficulty we introduce the function

$$g(x) = x^n f(x),$$

where n is the integer part of α. In this way $g(x)$ is similar to the function $h(x)$ of Eq. (A2) if n is even and to the $h(x)$ of Eq. (A6) if n is odd. Therefore, by defining as usual $\hat{g}(k)=\int_{-\infty}^{+\infty}dx\, g(x)e^{ikx}$, we can say that

$$\hat{g}(k) = \frac{A\pi}{\Gamma(\alpha-n)\cos\left(\frac{\alpha-n}{2}\pi\right)} k^{\alpha-n-1} + o(k^{\alpha-n-1})$$

if n is even and

$$\hat{g}(k) = i\frac{A\pi}{\Gamma(\alpha-n)\sin\left(\frac{\alpha-n}{2}\pi\right)}k^{\alpha-n-1} + o(k^{\alpha-n-1})$$

if n is odd. Now in order to find the singular part $\hat{f}_s(k)$ of $\hat{f}(k)$ it is sufficient to integrate n times $\hat{g}(k)$ [the integration constants giving rise to the finite moments terms of $\hat{f}(k)$ in Eq. (A1)]. In this way we obtain

$$\hat{f}_s(k) = \begin{cases} \dfrac{(-1)^{n/2}A\pi}{\Gamma(\alpha)\cos\left(\dfrac{\alpha-n}{2}\pi\right)}k^{\alpha-1} + o(k^{\alpha-1}) & \text{for even } n, \\ \dfrac{(-1)^{(n+1)/2}A\pi}{\Gamma(\alpha)\sin\left(\dfrac{\alpha-n}{2}\pi\right)}k^{\alpha-1} + o(k^{\alpha-1}) & \text{for odd } n, \end{cases} \tag{A7}$$

where we have used the following property of the gamma function: $\Gamma(x+1)=x\Gamma(x)$, hence $[(\alpha-1)\cdots(\alpha-n)\Gamma(\alpha-n)] = \Gamma(\alpha)$. Note that in both cases the coefficient of the term $k^{\alpha-1}$ is real.

APPENDIX B: FORCE PDF IN A POISSON PARTICLE DISTRIBUTION

Let us consider the case in which the particles are distributed on the line interval $(-L/2, L/2]$ of length L following a spatially stationary Poisson process with average density $\rho_0 > 0$. We want to know the PDF $W_P(F)$ of the field

$$F = \sum_{i=1}^{N} \frac{x_i}{|x_i|^{\alpha+1}} \tag{B1}$$

generated at the origin of the space by all the N system particles (we can consider $N=\rho_0 L$ as the fluctuations of order $\sqrt{\rho_0 L}$, due to the Poisson statistics, are completely unimportant for this problem in the large-L limit). We will follow the procedure to find $W_P(F)$ in three dimensions used by Chandrasekhar in [1] for the gravitational force. Note that, as the positions of different particles are uncorrelated, the joint PDF $p(x_1,\ldots,x_N)$ of the positions of the N system particles is simply

$$p_N(x_1,\ldots,x_N) = \prod_{i=1}^{N} p(x_i),$$

where $p(x_i)=1/L$. Therefore we can write

$$W_P(F) = \int \int_{-L/2}^{L/2} \left[\prod_{i=1}^{N} \frac{dx_i}{L}\right] \delta\left(F - \sum_{j=1}^{N} \frac{x_i}{|x_j|^{\alpha+1}}\right). \tag{B2}$$

Let us now study the characteristic function

$$\hat{W}_P(k) = \mathcal{F}[W_P(F)] = \int_{-\infty}^{+\infty} dF W_P(F) e^{ikF}.$$

By taking the FT of Eq. (B2) we obtain

$$\hat{W}_P(k) = \left[\int_{-L/2}^{L/2} \frac{dx}{L} \exp\left(ik\frac{x}{|x|^{\alpha+1}}\right)\right]^N.$$

By adding and subtracting 1 inside the square brackets and taking the limit $L\to\infty$ with $N=\rho_0 L$ we arrive at the final expression:

$$\hat{W}_P(k) = \exp\left\{-\rho_0 k^{1/\alpha}\int_{-\infty}^{+\infty} dt\left[1 - \exp\left(i\frac{t}{|t|^{\alpha+1}}\right)\right]\right\}.$$

Note that

$$\int_{-\infty}^{+\infty} dt\left[1 - \exp\left(i\frac{t}{|t|^{\alpha+1}}\right)\right] = 2\int_0^\infty dt[1-\cos(t^{-\alpha})]$$
$$= \frac{2}{\alpha}\int_0^\infty du u^{-1-1/\alpha}(1-\cos u),$$

where the last passage is due to the change of variable $t^{-\alpha}=u$. Let us now use, as in the previous appendix, the integral representation

$$u^{-1-1/\alpha} = \frac{1}{\Gamma\left(\frac{\alpha+1}{\alpha}\right)}\int_0^\infty dz z^{1/\alpha} e^{-uz}.$$

Through this transformation we arrive finally at the relation

$$\int_{-\infty}^{+\infty} dt\left[1 - \exp\left(i\frac{t}{|t|^{\alpha+1}}\right)\right] = \frac{2}{\alpha\Gamma\left(\frac{\alpha+1}{\alpha}\right)}\int_0^\infty dz\frac{z^{-1+1/\alpha}}{1+z^2}. \tag{B3}$$

Note that the last integral is diverging for $\alpha \leq 1/2$, indicating that the problem is not well defined for these values of α as F is not a well-defined stochastic quantity. This means that the sum in Eq. (B1) needs an L-dependent normalization to become a well-defined stochastic variable. In fact, differently to the shuffled lattice case, where the typical mass fluctuation on regions of size R is proportional to R^0, in the Poisson case this is due to the fact that such a fluctuation is proportional to $R^{1/2}$. The field due to the mass fluctuation in a sphere of radius R on the origin of the sphere is of order $R^{-\alpha}$ for the shuffled lattice and $R^{-\alpha+1/2}$ in the Poisson point process. This explains why for a shuffled lattice the problem is well defined for any $\alpha>0$ and not only for $\alpha>1/2$. This also says that for $\alpha<1/2$, in order to have a well-defined stochastic field also in the Poisson case, we have to divide the field in Eq. (B1) by $L^{-\alpha+1/2}$ where L is the system size. The same argument can be used in d dimensions, for which the same mass fluctuations are, respectively, proportional to $R^{(d-1)/2}$ in the shuffled lattice and $R^{d/2}$ in the Poisson case. This says that the problem is well defined, without L-dependent normalization of the field, for $\alpha>(d-1)/2$ for the shuffled lattice and $\alpha>d/2$ for the Poisson case (in the gravitational case in $d=3$ faced by Chandrasekhar [1] we have $\alpha=2$ and the field is a well-defined stochastic quantity in both cases).

Let us now go back to Eq. (B3) for $\alpha>1/2$. By using Eq. (A4) we can rewrite it as

$$\int_{-\infty}^{+\infty} dt\left[1-\exp\left(i\frac{t}{|t|^{\alpha+1}}\right)\right]=\frac{\pi}{\alpha\Gamma\left(\frac{\alpha+1}{\alpha}\right)\sin\left(\frac{\pi}{2\alpha}\right)},$$

where we have also used $\cos[\pi(1/\alpha-1)/2]=\sin[\pi/(2\alpha)]$. Consequently,

$$\hat{W}_P(k)=\exp\left[-\frac{\pi\rho_0}{\alpha\Gamma\left(\frac{\alpha+1}{\alpha}\right)\sin\left(\frac{\pi}{2\alpha}\right)}k^{1/\alpha}\right], \qquad \text{(B4)}$$

which is exactly of the form of the characteristic function of a Lévy stable PDF [17]. By expanding Eq. (B4) to the first nonvanishing order larger than zero, we have

$$\hat{W}_P(k)=1-\frac{\pi\rho_0}{\alpha\Gamma\left(\frac{\alpha+1}{\alpha}\right)\sin\left(\frac{\pi}{2\alpha}\right)}k^{1/\alpha}+o(k^{1/\alpha}).$$

If we now invert this Fourier transform as explained in the preceding appendix, we can conclude that

$$W_P(F)\simeq BF^{-(\alpha+1)/\alpha} \quad \text{for large } F,$$

with

$$B=\frac{\rho_0}{\alpha}. \qquad \text{(B5)}$$

[1] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
[2] A. Del Popolo and M. Gambera, Astron. Astrophys. **342**, 34 (1998).
[3] A. Gabrielli, F. Sylos Labini, and S. Pellegrini, Europhys. Lett. **46**, 127 (1999).
[4] J. H. Wesenberg and K. Mølmer, Phys. Rev. Lett. **93**, 143903 (2004).
[5] V. Beato, L. Pietronero, and S. Zapperi, J. Stat. Mech.: Theory Exp. P04011 (2005).
[6] G. Løvoll, K. J. Måløy, and E. G. Flekkøy, Phys. Rev. E **60**, 5872 (1999).
[7] J. H. Snoeijer, M. van Hecke, E. Somfai, and W. van Saarloos, Phys. Rev. E **67**, 030302(R) (2003).
[8] A. Gabrielli, A. P. Masucci, and F. Sylos Labini, Phys. Rev. E **69**, 031110 (2004).
[9] A. Gabrielli, F. Sylos Labini, M. Joyce, and L. Pietronero, *Statistical Physics for Cosmic Structures* (Springer-Verlag, Berlin, 2004).
[10] G. Efstathiou, M. Davis, C. Frenk, and S. D. M. White, Astrophys. J., Suppl. Ser. **57**, 241 (1985).
[11] A. Gabrielli, T. Baertshiger, M. Joyce, B. Marcos, and F. Sylos Labini (unpublished).
[12] M. Baus and J.-P. Hansen, Phys. Rep. **59**, 1 (1980).
[13] D. Pines, *Elementary Excitations in Solids* (Benjamin, New York, 1964).
[14] J. Binney and S. Tremaine, *Galactic Dynamics*, Princeton Series in Astrophysics, edited by D. N. Spergel (Princeton University Press, Princeton, 1987).
[15] P. J. E. Peebls, *The Large-Scale Structure of the Universe* (Princeton University Press, Princeton, 1980).
[16] A. Gabrielli, Phys. Rev. E **70**, 066131 (2004).
[17] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, Reading, MA, 1954).
[18] W. Feller, *An Introduction to Probability Theory and its Applications 2*, 2nd ed. (Wiley, New York, 1971).
[19] A. Gabreilli (unpublished).