Effect of absorption on the spatial coherence in scalar fields generated by statistically homogeneous and isotropic sources

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We investigate the behavior of the spectral degree of coherence of fluctuating scalar wave fields generated by statistically homogeneous and isotropic sources within absorbing media. In the limit of negligible losses, the degree of coherence is shown to be of a known universal form (sinc function), whereas in the limit of infinite losses it is shown to asymptotically approach the degree of coherence of the source. The intermediate cases are studied for fields produced by sources with Gaussian and sinc-type coherence functions. These studies show that the concept of negligible losses is meaningful only when defined with respect to specific source distribution statistics, and that even for arbitrarily small nonzero losses, the coherence length of the field can be arbitrarily short.

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I. INTRODUCTION

The relationship between the correlation properties of wave fields and the correlation properties of volume sources that generate the fields has been extensively studied [1–5]. Particular attention has been devoted to the so-called spectral degree of coherence [6] of these fields. It has been found that in statistically homogeneous and isotropic systems, the degree of coherence has a universal limiting form when the losses become negligible [2,4,5]. The studies have mainly concerned scalar fields [7], which can also describe nonoptical wave phenomena (see, for example, Ref. [8]), but recently the theory has been extended to encompass the full vectorial treatment of electromagnetic fields [5], confirming a universal behavior in that case as well.

The purpose of the present paper is to understand how the degree of coherence of the scalar fields behaves in statistically homogeneous and isotropic systems, in which the losses are not necessarily negligible. This includes an analysis of how the universal character emerges when the losses become vanishingly small as well as a study of systems where the losses become infinitely large. These considerations show that, as long as the system is even negligibly lossy, the coherence length of the field can be arbitrarily short. This is in contrast with the common perception that the coherence length of a field in an essentially lossless statistically homogeneous and isotropic system necessarily is of the order of the wavelength of the field or longer [4].

This paper is organized so that in Sec. II we derive an expression for the cross-spectral density function of the scalar wave field in terms of the cross-spectral density function of the source in statistically homogeneous and isotropic lossy systems. In Sec. III, we consider the degree of coherence in such systems and, in particular, concentrate on its behavior in completely lossless and extremely lossy systems. Further insight into these considerations is provided in Sec. IV, where the degree of coherence of the field is studied for two different model sources. Finally, in Sec. V, we draw conclusions about the degree of coherence in statistically homogeneous and isotropic lossy systems and summarize the main results of the paper. The appendices contain details that are needed for a mathematically rigorous treatment, but which are not necessary for an understanding of the results.

II. CROSS-SPECTRAL DENSITY FUNCTIONS IN ABSORBING MEDIA

The second-order spatial coherence (correlation) properties of a statistically stationary source distribution and of the scalar field it generates can be described by the cross-spectral density (covariance) functions [6]

$$W_\rho(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle\rho^*(\mathbf{r}_1,\omega)\rho(\mathbf{r}_2,\omega)\rangle, \tag{1}$$

and

$$W_U(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle U^*(\mathbf{r}_1,\omega)U(\mathbf{r}_2,\omega)\rangle, \tag{2}$$

where the functions $\rho(\mathbf{r},\omega)$ and $U(\mathbf{r},\omega)$ represent the source and field realizations at an angular frequency ω. The angle brackets and the asterisk denote ensemble averaging and complex conjugation, respectively, and the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ refer to two points in space.

When a homogeneous, isotropic, and linear medium occupies all space, the field realizations obey the inhomogeneous Helmholtz equation [9]

$$\nabla^2U(\mathbf{r},\omega)+\kappa^2U(\mathbf{r},\omega)=-4\pi\rho(\mathbf{r},\omega). \tag{3}$$

Here $\kappa=\kappa'+i\kappa''$ is the complex wave number of the field, defined as $\kappa=n(\omega)\omega/c_0$, where $n(\omega)$ is the (complex) refractive index of the medium and c_0 is the speed of light in vacuum. Hence $\kappa'=2\pi/\lambda$, where λ is the wavelength of the field in the medium. We have included ω in the equations

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above to emphasize that they hold for a single frequency component of a stationary polychromatic field, but to avoid unnecessary cluttering of the formulas, we henceforth omit ω in the notation.

Using Eq. (3), we obtain for the cross-spectral density functions in Eqs. (1) and (2) the equation

$$16\pi^2 W_\rho(\mathbf{r}_1,\mathbf{r}_2) = \left(\nabla_1^2 + \kappa^{*2}\right)\left(\nabla_2^2 + \kappa^2\right) W_U(\mathbf{r}_1,\mathbf{r}_2), \quad (4)$$

where the exchange of the order of differentiation and expectation taking is motivated by the existence of the source covariance function in Eq. (1) (cf. Sec. 53 in Ref. [10]). In the above equation, ∇_i, with $i=\{1,2\}$, operates on the position vector $\mathbf{r}_i$.

It is useful to introduce the complex degree of (spectral) coherence, defined by [6]

$$\mu_F(\mathbf{r}_1,\mathbf{r}_2) = \frac{W_F(\mathbf{r}_1,\mathbf{r}_2)}{\sqrt{W_F(\mathbf{r}_1,\mathbf{r}_1) W_F(\mathbf{r}_2,\mathbf{r}_2)}}, \quad (5)$$

where the subscript $F=\{\rho,U\}$ refers, here and henceforth, either to the source or to the field. The degree of coherence is restricted within the bounds [6]

$$0 \leqslant |\mu_F(\mathbf{r}_1,\mathbf{r}_2)| \leqslant 1, \quad (6)$$

with the upper and lower limits corresponding to complete coherence and complete incoherence, respectively.

A. Statistically homogeneous source distributions

We begin by considering fields that are generated by statistically homogeneous source distributions. The corresponding cross-spectral density (covariance) functions, as given by Eqs. (1) and (2), then depend on the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ only through their separation $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$. Hence, it is convenient to set

$$W_F(\mathbf{r}_1,\mathbf{r}_2) = W_F(\mathbf{r}). \quad (7)$$

From the properties of covariance functions (see Sec. 51 in Ref. [10]), or equivalently from Eqs. (5) and (6), we obtain the upper bound

$$|W_F(\mathbf{r})| \leqslant W_F(\mathbf{0}). \quad (8)$$

Also, the relation (7) implies that

$$\nabla_1 W_U(\mathbf{r}_1,\mathbf{r}_2) = \nabla W_U(\mathbf{r}), \quad \nabla_2 W_U(\mathbf{r}_1,\mathbf{r}_2) = -\nabla W_U(\mathbf{r}), \quad (9)$$

whereby for statistically homogeneous source distributions, Eq. (4) becomes

$$(\nabla^2 + \kappa^{*2})(\nabla^2 + \kappa^2) W_U(\mathbf{r}) = 16\pi^2 W_\rho(\mathbf{r}). \quad (10)$$

To find the solution to this fourth-order partial differential equation, we note that (at least formally) we have the Fourier-transform pair

$$W_F(\mathbf{r}) = \int \hat{W}_F(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r}) d^3k, \quad (11)$$

$$\hat{W}_F(\mathbf{k}) = \frac{1}{(2\pi)^3} \int W_F(\mathbf{r}) \exp(-i\mathbf{k}\cdot\mathbf{r}) d^3r. \quad (12)$$

If we introduce the representation (11) into Eq. (10), assume that we can exchange the order of integration and differentiation, and use the linear independence of the exponential functions (uniqueness of the Fourier transform [11]), we arrive at the equation

$$(-k^2 + \kappa^{*2})(-k^2 + \kappa^2)\hat{W}_U(\mathbf{k}) = 16\pi^2 \hat{W}_\rho(\mathbf{k}), \quad (13)$$

where $k=|\mathbf{k}|$. For lossy systems, in which $\kappa'' > 0$, we can set

$$\hat{g}(\mathbf{k}) = \frac{1}{|k^2 - \kappa^2|^2}, \quad (14)$$

so that Eq. (13) can be written as

$$\hat{W}_U(\mathbf{k}) = 16\pi^2 \hat{g}(\mathbf{k}) \hat{W}_\rho(\mathbf{k}). \quad (15)$$

The Fourier-transform convolution theorem [11] then suggests that this expression defines the solution

$$W_U(\mathbf{r}) = \frac{2}{\pi} \int g(\mathbf{R}) W_\rho(\mathbf{r}-\mathbf{R}) d^3R = \frac{2\pi}{\kappa''} \int \mathrm{sinc}(\kappa' |\mathbf{R}|) \exp(-\kappa'' |\mathbf{R}|) W_\rho(\mathbf{r}-\mathbf{R}) d^3R \quad (16)$$

to the differential equation (10). Here $\mathrm{sinc}(z)=\sin(z)/z$, and we have used the relation

$$g(\mathbf{r}) = \int \hat{g}(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r}) d^3k = \frac{\pi^2}{\kappa''} \mathrm{sinc}(\kappa' r) \exp(-\kappa'' r), \quad (17)$$

where $r=|\mathbf{r}|$ and the last expression is verified in Appendix A.

Strictly speaking, Eq. (16) only represents a formal solution to the differential equation (10). However, as is shown in Appendix B, when the cross-spectral density function W_ρ is absolutely integrable and continuous, and when $W_\rho(\mathbf{0}) < \infty$, this expression provides the actual solution. It is reasonable to assume that all cross-spectral density functions encountered in physical systems are of this kind, so henceforth we will only consider such cross-spectral density functions W_ρ.

B. Statistically homogeneous and isotropic source distributions

For a statistically homogeneous source that is also isotropic, viz.

$$W_\rho(\mathbf{r}) = W_\rho(|\mathbf{r}|), \quad (18)$$

we can develop the expression (16) further. Indeed, by changing the integration variable from $\mathbf{R}$ to $\mathbf{T}=\mathbf{r}-\mathbf{R}$ in that expression and by using Eq. (18), we get

$$W_U(\mathbf{r}) = \frac{2\pi}{\kappa''}\int \operatorname{sinc}(\kappa'|\mathbf{r}-\mathbf{T}|)\exp(-\kappa''|\mathbf{r}-\mathbf{T}|)W_\rho(|\mathbf{T}|)d^3T$$
$$= \frac{2\pi}{\kappa'\kappa''}\int_0^\infty T^2W_\rho(T)\int_0^{2\pi}\int_0^\pi \times \mathrm{Im}\left\{\sin\theta\frac{\exp(i\kappa\sqrt{r^2+T^2-2rT\cos\theta})}{\sqrt{r^2+T^2-2rT\cos\theta}}\right\}d\theta d\phi dT, \tag{19}$$

where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{T}$, and we have made use of the expansion $|\mathbf{r}-\mathbf{T}|=\sqrt{r^2+T^2-2rT\cos\theta}$. Since the integrand is everywhere regular, we arrive, after straightforward angular integrations and by changing T to R, at the expression

$$W_U(\mathbf{r}) = \frac{4\pi^2}{\kappa'\kappa''}\int_0^\infty R^2W_\rho(R) \times\mathrm{Im}\left\{\frac{1}{i\kappa rR}\bigg|_0^\pi \exp(i\kappa\sqrt{r^2+R^2-2rR\cos\theta})\right\}dR$$
$$= \frac{4\pi^2}{\kappa'\kappa''}\int_0^\infty R^2W_\rho(R)\,\mathrm{Im}\left\{\frac{1}{i\kappa r_< r_>}\times\{\exp[i\kappa(r_>+r_<)]-\exp[i\kappa(r_>-r_<)]\}\right\}dR$$
$$= \frac{8\pi^2}{\kappa'\kappa''}\int_0^\infty R^2W_\rho(R)\mathrm{Re}\{\kappa j_0(\kappa r_<)h_0^{(1)}(\kappa r_>)\}dR, \tag{20}$$

where $r_<=\min\{r,R\}$, $r_>=\max\{r,R\}$, and we have used the definitions $j_0(z)=\sin(z)/z$ and $h_0^{(1)}(z)=\exp(iz)/(iz)$ {see equations (10.1.11) and (10.1.12) in Ref. [12]}. Equation (20) describes the general form of the cross-spectral density function of a field produced by a statistically homogeneous and isotropic source distribution.

III. DEGREE OF COHERENCE IN ABSORBING MEDIA

For statistically homogeneous systems, the (spectral) degree of coherence as given by Eq. (5), assumes the form

$$\mu_F(\mathbf{r}) = \frac{W_F(\mathbf{r})}{W_F(\mathbf{0})}. \tag{21}$$

On introducing Eq. (20) into this definition, we obtain for the degree of coherence of the field in a lossy ($\kappa''>0$), statistically homogeneous, and isotropic system, the general expression

$$\mu_U(\mathbf{r}) = \frac{W_U(\mathbf{r})}{W_U(\mathbf{0})} = \frac{\int_0^\infty R^2W_\rho(R)\,\mathrm{Re}\{\kappa j_0(\kappa r_<)h_0^{(1)}(\kappa r_>)\}dR}{\int_0^\infty R^2W_\rho(R)\,\mathrm{Re}\{\kappa h_0^{(1)}(\kappa R)\}dR},$$
$$0<\kappa''<\infty, \tag{22}$$

where we have used the fact that $j_0(0)=1$.

It is of interest to note that if the denominator in Eq. (22) vanishes, i.e., if $W_U(\mathbf{0})=0$, it follows from the bound (8) that $W_U(\mathbf{r})=0$ and, hence, Eq. (10) yields $W_\rho(\mathbf{r})=0$. Thereby, Eq. (22) is valid whenever W_ρ is not the null function, in other words, whenever there is, on the average, a source present in the system.

A. Asymptotically small losses in statistically homogeneous and isotropic systems

Let us now investigate the behavior of μ_U in statistically homogeneous and isotropic systems with asymptotically small losses. Then $\kappa''\to 0^+$, so that by introducing the relation (18) into Eq. (16), we obtain

$$\kappa''W_U(\mathbf{r}) \sim 2\pi\lim_{\kappa''\to 0^+}\int \operatorname{sinc}(\kappa'|\mathbf{R}|)\exp(-\kappa''|\mathbf{R}|)W_\rho(|\mathbf{r}-\mathbf{R}|)d^3R$$
$$= 2\pi\int \operatorname{sinc}(\kappa'|\mathbf{R}|)W_\rho(|\mathbf{r}-\mathbf{R}|)d^3R$$
$$= 8\pi^2 j_0(\kappa' r)\int_0^\infty R^2W_\rho(R)j_0(\kappa' R)\mathrm{d}R,$$
$$\kappa''\to 0^+,\quad \text{unif. } \mathbf{r}, \tag{23}$$

where the uniform convergence of the first integral to the second integral with $\kappa''\to 0^+$ is proven in Appendix C 1. The last equality corresponds to the development in Eqs. (19) and (20) with $\kappa''=0$ everywhere except in the denominators $\kappa'\kappa''$, which are taken to be nonzero.

On introducing Eq. (23) into Eq. (21), we obtain for the degree of coherence of the field the expression

$$\mu_U(\mathbf{r}) = \frac{\kappa''W_U(\mathbf{r})/(8\pi^2)}{\kappa''W_U(\mathbf{0})/(8\pi^2)} \sim \frac{j_0(\kappa' r)\int_0^\infty R^2W_\rho(R)j_0(\kappa' R)dR}{\int_0^\infty R^2W_\rho(R)j_0(\kappa' R)dR}$$
$$= \operatorname{sinc}(\kappa' r),\quad \kappa''\to 0^+,\quad \text{unif. } \mathbf{r}, \tag{24}$$

where we have used the fact that $j_0(z)=\operatorname{sinc}(z)$. This expression is the universal form of the (spectral) degree of coherence of fields generated by statistically homogeneous and isotropic sources in low-loss media [2,4,5]. It must, however, be emphasized, as in Ref. [2], that this result is only valid when

$$\int_0^\infty R^2W_\rho(R)j_0(\kappa' R)dR \neq 0. \tag{25}$$

This requirement together with the bound (8) and the uniform convergence in Eq. (23) can be used to prove the uniform convergence in Eq. (24). Hence, that expression is uniformly convergent whenever its denominator does not vanish.

Equation (22) is valid for all cross-spectral density functions W_ρ under consideration, but its limiting form (24) is valid only when the restriction (25) holds. This fact, together with the fact that the restriction is not robust against small perturbations, suggests that the behavior of μ_U when $\kappa''\to 0^+$ is extremely sensitive to the exact functional form of

W_ρ. In order to understand this phenomenon better, it is useful to consider the asymptotic expression (23) in terms of the Fourier-space representation of the solution (16), as given by Eq. (B2) in Appendix B. Accordingly, setting $t=1/\kappa''$, we can compute

$$\lim_{\kappa''\to 0^+} \kappa'' W_U(\mathbf{r})$$
$$= 16\pi^2 \lim_{\kappa''\to 0^+} \int \frac{\kappa''}{|k^2-\kappa^2|^2}\hat{W}_\rho(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})d^3k$$
$$= (2\pi)^3 \lim_{t\to\infty} \int \frac{t}{\pi}\left[\frac{1}{1+t^2(k-\kappa')^2}-\frac{1}{1+t^2(k+\kappa')^2}\right] \times \frac{1}{2\kappa' k}\hat{W}_\rho(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})d^3k$$
$$= \frac{4\pi^3}{\kappa'^2}\int \delta(k-\kappa')\hat{W}_\rho(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})d^3k, \tag{26}$$

where the last step follows, since $\hat{W}_\rho$ is continuous (see Appendix B), from a definition of the δ function {see equations (8.110) and (8.115) in Ref. [13]} and by taking into account that $k+\kappa'>0$. Because the relation (18) for statistically isotropic sources implies that $\hat{W}_\rho(\mathbf{k})=\hat{W}_\rho(|\mathbf{k}|)$ (cf. "radial functions" in Ref. [11]), the peaked shape of the function $\kappa''/|k^2-\kappa^2|^2$ suggests that the universal form (24) is observed when most of the contribution to $W_U(\mathbf{r})$ is from a neighborhood around $k=\kappa'$, where $\hat{W}_\rho(\mathbf{k})$ is essentially constant. Thereby, both the amplitude of $\hat{W}_\rho(\mathbf{k})$ at $k=\kappa'$ with respect to its amplitude at all other k, as well as its detailed structure around $k=\kappa'$, strongly influence the convergence to the universal form. This explains why the convergence is sensitive to the functional form of W_ρ. Finally, a comparison of Eqs. (23) and (26) shows that the restriction (25) is equivalent to

$$\hat{W}_\rho(\kappa') \neq 0 \tag{27}$$

{compare this inequality and the inequality (25) to Eqs. (10) and (9) of Ref. [2]}. Since actual physical systems are noisy, the above restriction is seen to be satisfied almost surely and, hence, these restrictions are mostly of mathematical interest.

B. Asymptotically large losses in statistically homogeneous systems

We next consider fields generated by statistically homogeneous sources that are not necessarily isotropic. The cross-spectral density function W_U of such a field is given by Eq. (16). By changing the integration variable in that representation from $\mathbf{R}$ to $\mathbf{T}=\kappa''\mathbf{R}$, so that $d^3R=d^3T/\kappa''^3$, we obtain

$$W_U(\mathbf{r}) = \frac{2\pi}{\kappa''^4}\int \mathrm{sinc}(\kappa' T/\kappa'')\exp(-T)W_\rho(\mathbf{r}-\mathbf{T}/\kappa'')d^3T. \tag{28}$$

Because W_ρ is absolutely integrable and continuous, and since $W_\rho(\mathbf{0})<\infty$ (W_ρ bounded), the integral above converges uniformly to the integral of the limit of its integrand when $\kappa''\to\infty$. This result, which is proven in Appendix C 2, allows us to compute

$$\lim_{\kappa''\to\infty} \kappa''^4 W_U(\mathbf{r}) = 2\pi\int \mathrm{sinc}(0)\exp(-T)W_\rho(\mathbf{r}-\mathbf{0})d^3T$$
$$= 8\pi^2 W_\rho(\mathbf{r})\int_0^\infty T^2\exp(-T)dT$$
$$= 16\pi^2 W_\rho(\mathbf{r}), \quad \text{unif. } \mathbf{r}, \tag{29}$$

where we have used the fact that the last integral equals $\Gamma(3)=2$ {see equation (6.1.1) in Ref. [12]}. This result implies that

$$W_U(\mathbf{r}) \sim 16\pi^2 W_\rho(\mathbf{r})\frac{1}{\kappa''^4}, \quad \kappa''\to\infty, \quad \text{unif. } \mathbf{r}. \tag{30}$$

Hence the cross-spectral density function of the field is asymptotically proportional to the cross-spectral density function of the source, when the losses become infinite.

When we introduce the result (30) into the definition (21), we arrive at

$$\mu_U(\mathbf{r}) = \frac{W_U(\mathbf{r})}{W_U(\mathbf{0})} \sim \frac{16\pi^2 W_\rho(\mathbf{r})/\kappa''^4}{16\pi^2 W_\rho(\mathbf{0})/\kappa''^4} = \frac{W_\rho(\mathbf{r})}{W_\rho(\mathbf{0})} = \mu_\rho(\mathbf{r}),$$
$$\kappa''\to\infty, \quad \text{unif. } \mathbf{r}, \tag{31}$$

where the uniform convergence can be proven, for example, by using the uniform convergence in Eq. (30) together with the bound (8). Equation (31) implies that the degree of coherence of a field produced by a statistically homogeneous source converges uniformly to the degree of coherence of the source, when the losses become asymptotically large. This is, of course, what one might also intuitively expect.

IV. SPECIFIC EXAMPLES

A. Sources with a Gaussian cross-spectral density function

Next we consider a homogeneous and isotropic source with a Gaussian cross-spectral density function, given by

$$W_\rho(\mathbf{R}) = \exp(-|\mathbf{R}|^2/\gamma^2), \tag{32}$$

where 2γ is the $1/e$ width of the Gaussian. When we introduce this cross-spectral density function into the expression (16), we obtain

$$W_U(\mathbf{r}) = \frac{2\pi}{\kappa'\kappa''}\int \frac{\sin(\kappa'|\mathbf{R}|)\exp(\kappa''|\mathbf{R}|)}{|\mathbf{R}|}\exp(-|\mathbf{r}-\mathbf{R}|^2/\gamma^2)d^3R$$
$$= \frac{4\pi^2}{\kappa'\kappa''}\mathrm{Im}\left\{\int_0^\infty R\exp(i\kappa R)\int_0^\pi \sin\theta \times\exp[-(r^2+R^2-2rR\cos\theta)/\gamma^2]d\theta dR\right\}$$
$$= \frac{2\pi^2\gamma^2}{\kappa'\kappa'' r}\exp(-r^2/\gamma^2) \times\mathrm{Im}\left\{\int_0^\infty \exp[(i\kappa+2r/\gamma^2)R-R^2/\gamma^2]dR - \int_0^\infty \exp[(i\kappa-2r/\gamma^2)R-R^2/\gamma^2]dR\right\}, \tag{33}$$

where we have used the expansion $|\mathbf{r}-\mathbf{R}|^2=r^2+R^2-2rR\cos\theta$. The above equation can be developed further if we use the relation

$$\int_0^\infty \exp(bz-az^2)\mathrm{d}z = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right)\,\mathrm{erfc}\left(-\frac{b}{2\sqrt{a}}\right),\quad a>0, \tag{34}$$

which can be obtained by completing the square and by using the definition of the complementary error function erfc(z) {see, for example, equation (7.1.2) in Ref. [12]}. With this relation, we get from Eq. (33), after straightforward manipulations, the expression

$$W_U(\mathbf{r}) = \frac{\pi^2\sqrt{\pi}\gamma^3}{\kappa'\kappa'' r}\mathrm{Im}\{\exp(-\kappa^2\gamma^2/4)\times[\exp(i\kappa r)\,\mathrm{erfc}(-i\kappa\gamma/2-r/\gamma) - \exp(-i\kappa r)\,\mathrm{erfc}(-i\kappa\gamma/2+r/\gamma)]\}. \tag{35}$$

Using the Taylor expansions of exp(z) and erfc(z), we have, in particular,

$$W_U(\mathbf{0}) = \frac{2\pi^2\sqrt{\pi}\gamma^3}{\kappa'\kappa''}\mathrm{Im}\{i\kappa\exp(-\kappa^2\gamma^2/4)\,\mathrm{erfc}(-i\kappa\gamma/2)\}, \tag{36}$$

whereby, we obtain from Eq. (21) the expression

$$\mu_U(\mathbf{r}) = \frac{\mathrm{Im}\{\exp(-\kappa^2\gamma^2/4)[\exp(i\kappa r)\,\mathrm{erfc}(-i\kappa\gamma/2-r/\gamma) - \exp(-i\kappa r)\,\mathrm{erfc}(-i\kappa\gamma/2+r/\gamma)]\}}{2\,\mathrm{Im}\{i\kappa\exp(-\kappa^2\gamma^2/4)\,\mathrm{erfc}(-i\kappa\gamma/2)\}r} \tag{37}$$

for the degree of coherence of the field driven by a source with a Gaussian cross-spectral density function of the form (32).

The graphs in Fig. 1 display the behavior of the degree of coherence, given by Eq. (37), as a function of the normalized separation r/λ and the ratio κ''/κ'. These plots correspond to the cases $\kappa'\gamma/2=3$ and $\kappa'\gamma/2=1/3$, respectively, where $\kappa'\gamma/2$ is roughly the ratio of the width of $\exp(-r^2/\gamma^2)$ to the width of the main lobe of sinc($\kappa' r$). We note from the figures that the limit of asymptotically small losses, as described by Eq. (24), is attained when $\kappa''\lesssim 10^{-5}\kappa'$ in the first case and $\kappa''\lesssim 10^{-3}\kappa'$ in the second case. On the other hand, the degree of coherence has converged to its limit at asymptotically large losses, given by Eq. (31), when $\kappa''\gtrsim 5\kappa'$ and $\kappa''\gtrsim 10\kappa'$, respectively. This example, therefore, confirms the validity of the limiting expressions and also shows that the rates of convergence are slightly affected by the width of the Gaussian, that is, the functional form of W_ρ.

B. Sources with a "damped-sinc" cross-spectral density function

In our second example, we consider fields generated by statistically homogeneous and isotropic sources that have a cross-spectral density function of the form

$$W_\rho(\mathbf{R}) = \mathrm{sinc}(\chi' R)\exp(-\chi'' R) = \frac{1}{i2\chi' R}[\exp(i\chi R) - \exp(-i\chi^* R)], \tag{38}$$

where $\chi=\chi'+i\chi''$, and $\chi''>0$. When we introduce this function into the middle part of Eq. (20), we obtain after performing the straightforward integrations (since $\chi'',\kappa''>0$ the upper limits ∞ in the integrals do not contribute) the result

$$W_U(\mathbf{r}) = \frac{16\pi^2}{\kappa''}\frac{1}{|\chi+\kappa|^2|\chi^2-\kappa^{*2}|^2}\times\Bigg[|\chi+\kappa|^2[\chi''\,\mathrm{sinc}(\kappa' r)\exp(-\kappa'' r) + \kappa''\,\mathrm{sinc}(\chi' r)\exp(-\chi'' r)] + 4\chi''\kappa''\mathrm{Im}\left\{(\chi^*+\kappa^*)\,\mathrm{sinc}\left(\frac{\chi-\kappa}{2}r\right)\times\exp\left(i\frac{\chi+\kappa}{2}r\right)\right\}\Bigg]. \tag{39}$$

From this we can compute

$$W_U(\mathbf{0}) = \frac{16\pi^2}{\kappa''}\frac{1}{|\chi+\kappa|^2|\chi^2-\kappa^{*2}|^2}(\chi''+\kappa'')|\chi+\kappa^*|^2, \tag{40}$$

so that the degree of coherence, as given by Eq. (21), becomes

$$\mu_U(\mathbf{r}) = \left|\frac{\chi+\kappa}{\chi+\kappa^*}\right|^2\left[\frac{\chi''}{\chi''+\kappa''}\mathrm{sinc}(\kappa' r)\exp(-\kappa'' r) + \frac{\kappa''}{\chi''+\kappa''}\mathrm{sinc}(\chi' r)\exp(-\chi'' r)\right] + 4\frac{\sqrt{\chi''\kappa''}}{\chi''+\kappa''}\mathrm{Im}\left\{\frac{\sqrt{\chi''\kappa''}(\chi^*+\kappa^*)}{|\chi+\kappa^*|^2}\times\mathrm{sinc}\left(\frac{\chi-\kappa}{2}r\right)\exp\left(i\frac{\chi+\kappa}{2}r\right)\right\}. \tag{41}$$

When χ'' and κ'' are sufficiently small with respect to both χ' and κ', this expression has the asymptotic representation

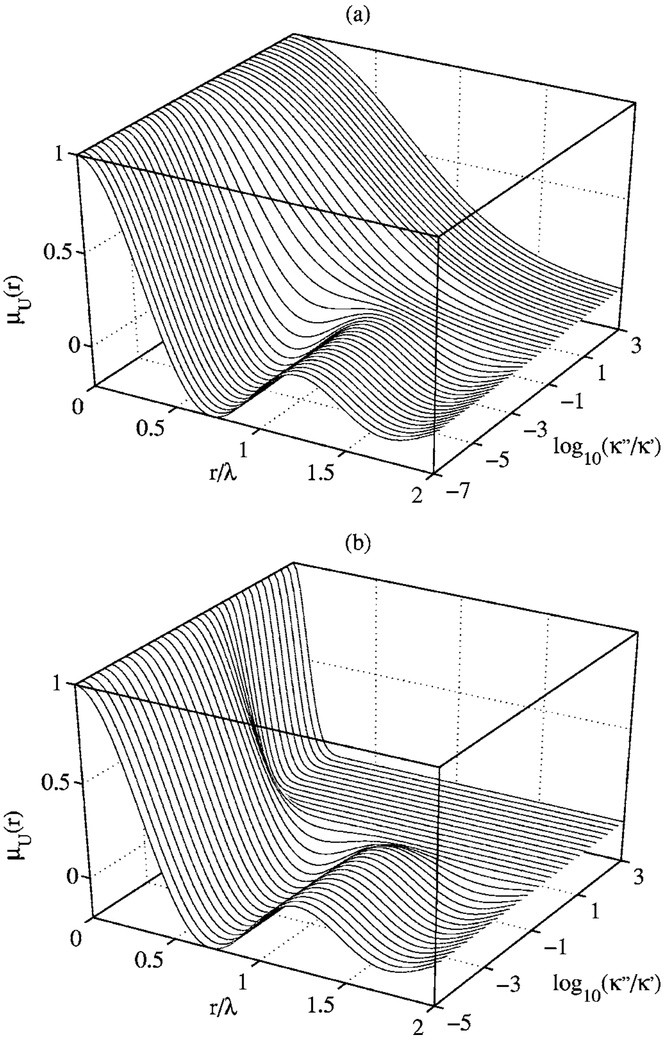

FIG. 1. Degree of coherence $\mu_U(\mathbf{r})$ as given by Eq. (37), when (a) $\kappa'\gamma/2=3$ and (b) $\kappa'\gamma/2=1/3$.

$$\mu_U(\mathbf{r}) \sim \frac{\chi''}{\chi''+\kappa''}\mathrm{sinc}(\kappa' r)\exp(-\kappa'' r) + \frac{\kappa''}{\chi''+\kappa''}\mathrm{sinc}(\chi' r)\exp(-\chi'' r), \quad \chi'',\kappa'' \to 0^+. \tag{42}$$

Since the function $\mathrm{sinc}(\kappa' r)\exp(-\kappa'' r)$ deviates only slightly from the universal form (24) for any r, when $\kappa''/\kappa' \ll 1$, the above result implies that only the ratio χ''/κ'' determines whether the degree of coherence is at its universal form or not. In fact, the universal form of μ_U is obtained in the limit $\chi''/\kappa'' \to \infty$, whereas in the limit $\chi''/\kappa'' \to 0^+$ the function μ_U converges to μ_ρ.

The degree of coherence given by Eq. (41), with $\chi'/\kappa' = 3$ and $\kappa''/\kappa' = 10^{-16}$, is plotted in Fig. 2 as a function of the normalized radial separation r/λ and the ratio χ''/κ''. The values of the parameters used in the graphs imply that the figure corresponds to the asymptotic form (42). We note that when the parameter χ'' increases, the degree of coherence of the field converges toward the universal form as expected.

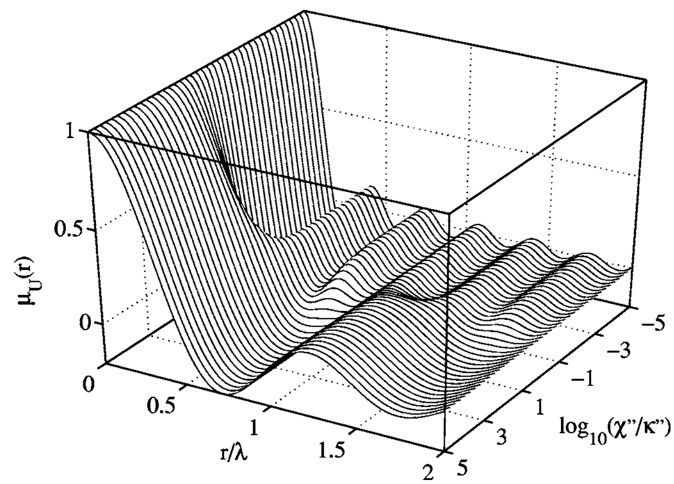

FIG. 2. Degree of coherence $\mu_U(\mathbf{r})$ as given by Eq. (41), when $\chi'/\kappa'=3$ and $\kappa''/\kappa'=10^{-16}$.

Also, when the parameter decreases, the degree of coherence converges to the degree of coherence of the source. Hence, this example shows that a fixed (and small, with respect to κ') κ'' can represent both asymptotically large losses as well as asymptotically small losses. Thereby the criterium $\kappa'' \ll \kappa'$ (used, for example, in Ref. [4]) is not sufficient for μ_U to exhibit its universal form (24).

V. DISCUSSION AND SUMMARY

In radiation by statistically homogeneous and isotropic scalar source distributions, the asymptotic limit of no losses has been studied and explained elsewhere [2,4], but it is useful to recall that the universal form of the degree of coherence of the field follows, since in a lossless system each spherical shell of sources around an observation point contributes equally to the field at that point. This means that the local features of the field are washed out; whereby, the coherence properties of the field are determined only by the contributions of sources lying infinitely far away. Thus, unless the source distribution has an infinite coherence length, the distant source points are effectively incoherent when viewed from the observation point. Indeed, it has been shown [3] that the degree of coherence of a field which consists of a uniform and uncorrelated ensemble of (propagating) plane waves has exactly the universal form of Eq. (24). Accordingly, since the degree of coherence of fields which are generated by source distributions that violate the requirement (27), or equivalently the requirement (25), is generally not of the universal form, it follows that the above interpretation is not valid for such fields. We return to this point below.

Little attention has hitherto been given to the way in which the degree of field coherence approaches its limit when the losses become vanishingly small. In the considerations of Sec. III A, we showed that the convergence is uniform in $\mathbf{r}$, but of even greater interest is the question of how small the loss, or κ'', has to be before the universal character (24) is observed. We argued that the threshold of κ'' for this to occur is sensitive to the functional form of the cross-spectral density function of the source, W_ρ, a fact evidenced by the example in Sec. IV B. That example also shows that it is, in fact, impossible to define a nonzero threshold in such a way that any smaller loss would guarantee that the degree of coherence has converged to its universal form. This leads us to conclude that "negligible losses" is an ill-defined concept, unless it is accompanied by restrictions to the cross-spectral density function of the source, for example based on physical arguments.

Furthermore, the damped-sinc example shows that for sufficiently small χ'' and κ'', the degree of coherence of the field effectively equals $\mathrm{sinc}(\chi' r)$ for any $\chi' > 0$ and κ'', when $\chi'' \ll \kappa''$. This result implies that irrespective of how small the losses in the system are, as long as they are nonzero, the coherence length of the field, which is of the order of $2\pi/\chi'$ (cf. $2\pi/\kappa' = \lambda$ and Ref. [1]), can be made arbitrarily short. Since completely lossless systems are predominantly mathematical constructs, the coherence length in actual, practically lossless systems, thereby, is not bound from below. In particular, it is independent of the wavelength λ.

We note that by setting $\kappa = \chi$, the representations (14) and (17) imply that the damped-sinc source of Eq. (38) has the Fourier transform $\hat{W}_\rho(\mathbf{k}) = \pi^{-2}\chi''/|k^2 - \chi^2|^2$. When χ'' is small, so that the discussion following Eq. (26) suggests that such a $\hat{W}_\rho$ is peaked around $k = \chi'$, and when $\chi' \neq \kappa'$, it follows that $\hat{W}_\rho(\kappa')$ is small. Hence the restriction (27) is nearly violated, so that, as mentioned in the physical explanation of lossless systems, the effect of the source distribution on the field cannot be described in terms of uncorrelated sources lying infinitely far from the observation point(s). Indeed, the exponential dampening of the field caused by nonzero losses implies that the contribution of each spherical shell of sources around an observation point to the field at that point decreases with the distance. Thereby, the local features are not (completely) washed out, and thus, the nonuniversal degree of coherence of the field is an effect which can be attributed to the interaction of fields from different mutually partially correlated source points. This suggests that the short coherence length in the damped-sinc example for small κ'' follows from the system being in a state, in which the loss and the source cross-spectral density are carefully balanced. In fact, this balance is precisely what the parameters κ'' and χ'' of that example control.

It is important to distinguish the above situation, which can occur for any $\kappa'' > 0$, from the situation in which κ'' is large and the limit (31) suggests that the degree of coherence, and hence the coherence length of the field, become that of the source. This latter situation occurs since the effective range of influence of the source on the field is decreased by the increase of the loss in the system, whereby the field realizations $U(\mathbf{r},\omega)$ must essentially be constant multiples of the corresponding source realizations $\rho(\mathbf{r},\omega)$ [cf. Eq. (3)]. Finally, we point out that both in this case and in the damped-sinc example, the coherence properties of the field are constant in all space. In particular, this means that short coherence lengths in these cases are not surface or boundary effects, in contrast to, for example, the shortness of the coherence length reported in Ref. [14] (see also the discussion in Ref. [4]).

To summarize, in this paper we have rigorously derived the general expression (21) of the degree of coherence in statistically homogeneous and lossy systems and studied its limits for completely lossless and completely lossy systems. Our results concerning the universal character of the degree of coherence of the field for asymptotically lossless systems are consistent with previous derivations [2,4]. In addition, our results show that for asymptotically lossy systems, the degree of coherence of the field converges to the degree of coherence of the source. We considered these results in the context of Gaussian and damped-sinc correlated sources. In the latter case, we observed that the field can, in fact, have an arbitrarily short coherence length in almost lossless systems. Finally, we note that although our treatment is based on scalar theory, it should be straightforward to extend the results to the complete vectorial description of electromagnetic fields (compare, for example, Refs. [4] and [5]).

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APPENDIX A: PROPERTIES OF THE FUNCTIONS g AND $\hat{g}$ OF EQS. (14) and (17)

Using the latter part of Eq. (17) to define the function g, we can compute

$$\int |g(\mathbf{r})| d^3r = \frac{\pi^2}{\kappa''}\int |\mathrm{sinc}(\kappa' r)| \exp(-\kappa'' r) d^3 r \leq \frac{4\pi^3}{\kappa''^4}\int_0^\infty z^2 \exp(-z) dz = (2\pi)^3 \frac{1}{\kappa''^4} < \infty, \quad \text{(A1)}$$

where we have used the facts that $\kappa'' > 0$ by assumption and that $|\mathrm{sinc}(z)| \leq 1$ for real z. The above result implies that the function g is absolutely integrable. Thereby, its Fourier transform is well defined and given by [11]

$$\begin{aligned}\frac{1}{(2\pi)^3}\int g(\mathbf{r}) \exp(-i\mathbf{k}\cdot\mathbf{r}) d^3 r &= \frac{1}{8\pi\kappa''}\int_0^{2\pi} d\phi \int_0^\infty r^2 \frac{\sin(\kappa' r)}{\kappa' r}\exp(-\kappa'' r) \times \int_0^\pi \sin\theta \exp(ikr\cos\theta) d\theta dr \\ &= \frac{1}{4\kappa'\kappa'' k}\mathrm{Re}\left\{\int_0^\infty \{\exp[i(\kappa - k)r] - \exp[i(\kappa + k)r]\} dr\right\} \\ &= \frac{1}{2\kappa'\kappa''}\mathrm{Re}\left\{\frac{1}{i}\frac{1}{k^2 - \kappa^2}\right\} = \frac{1}{|k^2 - \kappa^2|^2} = \hat{g}(\mathbf{k}),\end{aligned} \quad \text{(A2)}$$

where the last step follows from the definition (14) of the function $\hat{g}$. From the second part of Eq. (17), it is clear that the function g is continuous; hence, the inverse Fourier-transform relation given in the first part of that equation is valid everywhere [11].

APPENDIX B: PROOF THAT THE FORMAL SOLUTION (16) IS VALID FOR ABSOLUTELY INTEGRABLE AND CONTINUOUS FUNCTIONS W_ρ WITH $W_\rho(\mathbf{0})<\infty$

Here we assume that the cross-spectral density (covariance) function W_ρ, as defined by Eqs. (1) and (7), is an absolutely integrable, continuous function with $W_\rho(\mathbf{0})<\infty$. Since we have the bound (8), it is clear that the last condition is equivalent to the function W_ρ being bounded. These properties of the function W_ρ ensure that the Fourier-transform pair given by Eqs. (11) and (12) is valid everywhere for both W_ρ and $\hat{W}_\rho$ [11].

Let us now *define* the function W_U by Eq. (16). Since both g and W_ρ are absolutely integrable functions, it follows that so is W_U. Furthermore, we can then compute

$$\begin{aligned}|W_U(\mathbf{r}+\mathbf{d})-W_U(\mathbf{r})| &= \frac{2}{\pi}\left|\int g(\mathbf{R})[W_\rho(\mathbf{r}+\mathbf{d}-\mathbf{R})-W_\rho(\mathbf{r}-\mathbf{R})]d^3R\right| \\ &\leqslant \frac{2}{\pi}\int |g(\mathbf{r}-\mathbf{T})||W_\rho(\mathbf{T}+\mathbf{d})-W_\rho(\mathbf{T})|d^3T \\ &\leqslant \frac{2\pi}{\kappa''}\left[\int_{\overline{B(\mathbf{0},T_0)}}|W_\rho(\mathbf{T}+\mathbf{d})-W_\rho(\mathbf{T})|d^3T \right.\\ &\quad \left. +2\int_{\overline{B(\mathbf{0},T_0-d_0)}^{\mathsf{C}}}|W_\rho(\mathbf{T})|d^3T\right],\end{aligned} \tag{B1}$$

where $B(\mathbf{r}_0,R)$ denotes an open ball centered at $\mathbf{r}_0$ with radius R. The first inequality follows from the triangle inequality and by setting $\mathbf{T}=\mathbf{r}-\mathbf{R}$. The second inequality in turn results when, to begin with, the integration over all space is split into an integration over the closed ball $\overline{B(\mathbf{0},T_0)}$ and an integration over its complement $\overline{B(\mathbf{0},T_0)}^{\mathsf{C}}$. Following that, the latter integral is overestimated by using the triangle inequality and by overestimating the integration region by $\overline{B(\mathbf{0},T_0-d_0)}^{\mathsf{C}}$, where it is assumed that $d<d_0$ for some fixed $d_0<\infty$. This facilitates a combination of the two terms obtained from the triangle inequality. Finally, both integrals are overestimated by using the bound $|g(\mathbf{r})|\leqslant\pi^2/\kappa''$, which follows from Eq. (17). Because the function W_ρ is absolutely integrable, we can make the latter integral in the last expression of Eq. (B1) arbitrarily small by choosing the radius $T_0 \in (d_0,\infty)$ sufficiently large. After that, since $\overline{B(\mathbf{0},T_0)}$ is a compact region and since W_ρ is continuous, we can in turn make the first integral arbitrary small by choosing a sufficiently small $d\in(0,d_0)$. Hence, Eq. (B1) shows that the function W_U is continuous. As W_U is also absolutely integrable, it follows that the Fourier-transform pair of Eqs. (11) and (12) is valid everywhere for both W_U and $\hat{W}_U$ [11]. Thus, we may introduce the solution, given by Eqs. (14) and (15), into the inverse transform relation (11) to obtain the representation

$$W_U(\mathbf{r})=\int 16\pi^2\frac{1}{|k^2-\kappa^2|^2}\hat{W}_\rho(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})d^3k. \tag{B2}$$

Since both g and W_ρ are absolutely integrable and bounded, this expression agrees everywhere with the solution (16) [11] and, therefore, can be used to prove the validity of that solution.

As $\hat{W}_\rho$ is the Fourier transform of an absolutely integrable function W_ρ, it is continuous [15]. Since the function W_ρ is related to a covariance function by the representation (7), it is positive definite (see Secs. 51 and 54 in Ref. [10]) and Hermitian in the sense that $W_\rho^{\,*}(-\mathbf{r})=W_\rho(\mathbf{r})$. Thereby, its boundedness implies that it, in particular, is a "positive-definite function" in the terminology of Ref. [15]. The inequality following the inequality (8.24) on page 327 of that reference then suggests that $\hat{W}_\rho(\mathbf{k})\geqslant 0$ for all $\mathbf{k}$, so that specifically we have

$$\int|\hat{W}_\rho(\mathbf{k})|d^3k=\int\hat{W}_\rho(\mathbf{k})d^3k=W_\rho(\mathbf{0})<\infty, \tag{B3}$$

which shows that $\hat{W}_\rho$ is absolutely integrable. The last equality follows from Eq. (11). On the other hand, since $\kappa''>0$, the functions $k^q/|k^2-\kappa^2|^2$ are continuous when $k>0$. Because these functions also remain bounded at $k=0$ and $k=\infty$ when $0\leqslant q\leqslant 4$, it follows that there exist finite upper bounds $C_q<\infty$, such that

$$\frac{k^q}{|k^2-\kappa^2|^2}\leqslant C_q, \quad 0\leqslant q\leqslant 4, \quad k\geqslant 0. \tag{B4}$$

We then consider the integral

$$\begin{aligned}&\int\frac{\partial^{l+m+n}}{\partial r_1^l\,\partial r_2^m\,\partial r_3^n}\frac{1}{|k^2-\kappa^2|^2}\hat{W}_\rho(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})d^3k \\ &=\int(-ik_1)^l(-ik_2)^m(-ik_3)^n\frac{1}{|k^2-\kappa^2|^2}\hat{W}_\rho(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})d^3k,\end{aligned} \tag{B5}$$

where $\mathbf{r}=(r_1,r_2,r_3)$, $\mathbf{k}=(k_1,k_2,k_3)$, and $l,m,n\geqslant 0$ and $l+m+n\leqslant 4$. Using the bounds (B3) and (B4), we can compute the estimate

$$\begin{aligned}&\left|\int_{\overline{B(\mathbf{0},K)}^{\mathsf{C}}}\frac{\partial^{l+m+n}}{\partial r_1^l\,\partial r_2^m\,\partial r_3^n}\frac{1}{|k^2-\kappa^2|^2}\hat{W}_\rho(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})d^3k\right| \\ &\leqslant\int_{\overline{B(\mathbf{0},K)}^{\mathsf{C}}}\frac{k^{(l+m+n)}}{|k^2-\kappa^2|^2}|\hat{W}_\rho(\mathbf{k})|d^3k \\ &\leqslant C_{l+m+n}\int_{\overline{B(\mathbf{0},K)}^{\mathsf{C}}}|\hat{W}_\rho(\mathbf{k})|d^3k \xrightarrow{K\to\infty} 0,\end{aligned} \tag{B6}$$

which shows (see Sec. 13.15 in Ref. [16]) that the integral in Eq. (B5) is uniformly convergent with respect to $\mathbf{r}$. Since, in addition, the integrand of that integral is continuous for all $\mathbf{k}$ and $\mathbf{r}$, it follows that the order of integration and differentiation may be exchanged (see Sec. 13.15 in Ref. [16]), when the latter operation is at most of order 4. In particular, the representation (B2) then yields

$$(\nabla^2+\kappa^{*2})(\nabla^2+\kappa^2)W_U(\mathbf{r}) = \int 16\pi^2\frac{(-k^2+\kappa^{*2})(-k^2+\kappa^2)}{|k^2-\kappa^2|^2}\hat{W}_\rho(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})d^3k = 16\pi^2\int \hat{W}_\rho(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r})d^3k = 16\pi^2 W_\rho(\mathbf{r}). \tag{B7}$$

This result shows that the function W_U, as given by Eq. (16), is the solution to the differential equation (10), when the cross-spectral density function W_ρ, as defined by Eqs. (1) and (7), is a bounded, absolutely integrable, continuous function.

APPENDIX C: PROOFS OF UNIFORM CONVERGENCE

1. Integrand in Eq. (23)

Let us investigate how the integral in Eq. (23) behaves as $\kappa''\to 0^+$. Since W_ρ is assumed continuous, it follows that the integrand is continuous, and hence, we have the estimate

$$\left|\int \mathrm{sinc}(\kappa'|\mathbf{R}|)\exp(-\kappa''|\mathbf{R}|)W_\rho(|\mathbf{r}-\mathbf{R}|)d^3R - \int \mathrm{sinc}(\kappa'|\mathbf{R}|)W_\rho(|\mathbf{r}-\mathbf{R}|)d^3R\right| \leq W_\rho(\mathbf{0})\int_{B(\mathbf{0},R_0)}[1-\exp(-\kappa''|\mathbf{R}|)]d^3R + 2\int_{B(\mathbf{0},R_0)^{\mathsf{C}}}|W_\rho(|\mathbf{r}-\mathbf{R}|)||\mathrm{sinc}(\kappa'|\mathbf{R}|)|d^3R \leq \frac{4}{3}\pi R_0^3 W_\rho(\mathbf{0})[1-\exp(-\kappa''R_0)] + \frac{2}{\kappa' R_0}\int |W_\rho(|\mathbf{R}|)|d^3R. \tag{C1}$$

Here the first inequality follows from the bound (8) as well as the bounds $|\exp(-\kappa''|\mathbf{R}|)|\leq 1$ and $|\mathrm{sinc}(\kappa'|\mathbf{R}|)|\leq 1$. The second inequality in turn follows when the integrand in the first term and the integral in the second term of the preceding expression are overestimated. For that purpose, we have applied the bound $|\mathrm{sinc}(\kappa'|\mathbf{R}|)|\leq 1/(\kappa' R)$. As W_ρ is absolutely integrable, the second term in the last expression in Eq. (C1) is finite for $R_0>0$ and can be made arbitrarily small by choosing $R_0<\infty$ sufficiently large. After that, since $W_\rho(\mathbf{0})<\infty$, the first term can be made arbitrarily small by choosing $\kappa''>0$ sufficiently small. Thereby, the left-hand side of Eq. (C1) can be made arbitrarily small for every $\mathbf{r}$ by choosing $\kappa''>0$ sufficiently small. This shows that the integral in Eq. (23) converges uniformly (with respect to $\mathbf{r}$) to the integral of the limit of its integrand when $\kappa''\to 0^+$.

2. Integrand in Eq. (28)

Let us assume, as before, that the cross-spectral density function W_ρ, given by Eqs. (1) and (7), is absolutely integrable and continuous, with $W_\rho(\mathbf{0})<\infty$. From the considerations in Appendix B, it thus follows that W_ρ is defined everywhere as the (inverse) Fourier transform (11) of an absolutely integrable function $\hat{W}_\rho$. Thereby, it has the limit [11]

$$\lim_{r\to\infty} W_\rho(\mathbf{r}) = 0. \tag{C2}$$

On the other hand, the continuity of W_ρ implies that the integrand in Eq. (28) is continuous, and we can investigate the convergence of the corresponding integral when $\kappa''\to\infty$ by considering the estimate

$$\left|\int \mathrm{sinc}(\kappa' T/\kappa'')\exp(-T)W_\rho(\mathbf{r}-\mathbf{T}/\kappa'')d^3T - \int \exp(-T)W_\rho(\mathbf{r})d^3T\right| \leq 2\int_{B(\mathbf{0},T_0)^{\mathsf{C}}}[|\mathrm{sinc}(\kappa' T/\kappa'')W_\rho(\mathbf{r}-\mathbf{T}/\kappa'')|+|W_\rho(\mathbf{r})|]\exp(-T)d^3T + \int_{B(\mathbf{0},T_0)}|\mathrm{sinc}(\kappa' T/\kappa'')W_\rho(\mathbf{r}-\mathbf{T}/\kappa'')-W_\rho(\mathbf{r})|\exp(-T)d^3T \leq 2\int_{B(\mathbf{0},T_0)^{\mathsf{C}}}\exp(-T)d^3T\,W_\rho(\mathbf{0}) + \begin{cases} 8\pi\max_{T\leq T_0}\{|W_\rho(\mathbf{r}-\mathbf{T}/\kappa'')|+|W_\rho(\mathbf{r})|\}, & r>r_0,\\ 8\pi\max_{T\leq T_0}\{|\mathrm{sinc}(\kappa' T/\kappa'')W_\rho(\mathbf{r}-\mathbf{T}/\kappa'')-W_\rho(\mathbf{r})|\}, & r\leq r_0,\end{cases} \tag{C3}$$

where we have used the bound (8) and the fact that $\int\exp(-T)d^3T=4\pi\int T^2\exp(-T)dT=8\pi$ {cf. equation (6.1.1) in Ref. [12]}. This expression implies that the first term on the right-hand side can be made arbitrarily small by choosing $T_0<\infty$ sufficiently large. After that, in view of the limiting behavior (C2) and since we may here assume $\kappa''>\kappa''_0$ for some $\kappa''_0>0$, we note that once T_0 is fixed, the case $r>r_0$ in the second term can be made arbitrarily small by choosing $r_0<\infty$ sufficiently large. Finally, since the function $\mathrm{sinc}(\kappa' T/\kappa'')W_\rho(\mathbf{r}-\mathbf{T}/\kappa'')$ is continuous, it is uniformly continuous in the compact region defined by $T\leq T_0$ and $r\leq r_0$. Thereby, once T_0 and r_0 are fixed, we can make the case $r\leq r_0$ in the second term arbitrary small by choosing $\kappa''<\infty$ sufficiently large. Hence, the expression on the left-hand side of Eq. (C3) can be made arbitrarily small for all $\mathbf{r}$ by choosing a sufficiently large $\kappa''<\infty$. This proves that the integral in Eq. (28) is uniformly convergent (with respect to $\mathbf{r}$) to the integral of the limit of its integrand when $\kappa''\to\infty$.

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