Reliability of the 0-1 test for chaos

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In time series analysis, it has been considered of key importance to determine whether a complex time series measured from the system is regular, deterministically chaotic, or random. Recently, Gottwald and Melbourne have proposed an interesting test for chaos in deterministic systems. Their analyses suggest that the test may be universally applicable to any deterministic dynamical system. In order to fruitfully apply their test to complex experimental data, it is important to understand the mechanism for the test to work, and how it behaves when it is employed to analyze various types of data, including those not from clean deterministic systems. We find that the essence of their test can be described as to first constructing a random walklike process from the data, then examining how the variance of the random walk scales with time. By applying the test to three sets of data, corresponding to (i) $1/f^\alpha$ noise with long-range correlations, (ii) edge of chaos, and (iii) weak chaos, we show that the test mis-classifies (i) both deterministic and weakly stochastic edge of chaos and weak chaos as regular motions, and (ii) strongly stochastic edge of chaos and weak chaos, as well as $1/f^\alpha$ noise as deterministic chaos. Our results suggest that, while the test may be effective to discriminate regular motion from fully developed deterministic chaos, it is not useful for exploratory purposes, especially for the analysis of experimental data with little *a priori* knowledge. A few speculative comments on the future of multiscale nonlinear time series analysis are made.

I. INTRODUCTION

Complex signals with characteristics such as scaling, non-stationarity, sensitive dependence on small disturbances, long memory, and infinite variance can arise from as diverse fields as physics, geophysics, astrophysics, ecology, finance, and biology. A long-standing fundamental issue in nonlinear time series analysis is to determine whether a complex time series is regular, deterministically chaotic, or random. A steady stream of efforts has been made, and a number of effective methods [1–13] have been proposed to tackle this difficult problem. However, none of the methods has the attributes of a recent test, termed 0-1 test, for deterministic chaos, proposed by Gottwald and Melbourne [14]: (i) the test does not require phase space reconstruction, (ii) the dimension of the dynamical system and the form of the underlying equations are irrelevant, (iii) the input is the time-series data and the output is 0 or 1, depending on whether the dynamics is non-chaotic or chaotic, and (iv) the test is universally applicable to any deterministic dynamical system. These features, if they are generically true, may greatly simplify complex time series analysis, especially experimental data analysis. Therefore it is important to understand the mechanism for the test to work, and how it behaves when it is employed to analyze various types of data, including those not from clean deterministic systems. In this paper, we employ the 0-1 test to analyze three types of data, (i) edge of chaos, (ii) weak chaos, and (iii) $1/f^\alpha$ noise with long-range correlations, to assess the usefulness as well as the limitations of the test.

The paper is organized as follows. In Sec. II, we point out that the 0-1 test for chaos amounts to first constructing a random walk-type process from the data, then examining how the variance of the random walk scales with time. In Sec. III, we study $1/f^\alpha$ noise with long-range correlations. In Sec. IV, we apply the test to two types of data: (i) edge of chaos and (ii) weak chaos. Our results are largely negative. We make a few remarks in Sec. V.

II. UNDERSTANDING THE 0-1 TEST FOR CHAOS

Consider a dynamical system characterized by state variables $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]$. Let an observable be $\phi(t) = \phi(x(t))$. The 0-1 test for chaos involves computing

$$\theta(t) = ct + \int_0^t \phi(x(s))ds, \qquad (1)$$

$$p(t) = \int_0^t \phi(x(s))\cos(\theta(s))ds, \qquad (2)$$

$$M(t) = \lim_{T \to \infty} \frac{1}{T} \int_0^T [p(t+\tau) - p(\tau)]^2 d\tau, \qquad (3)$$

where c is a constant chosen more or less arbitrarily, and then examining whether

$$K = \lim_{t \to \infty} \log M(t)/\log t \qquad (4)$$

approaches 0 or 1: when K is close to 0, the motion is classified as regular, and when it is close to 1, the motion is

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classified as deterministically chaotic. Note that in the former case, $M(t)$ grows slower than t, while in the latter case, $M(t)$ grows linearly with t.

To understand the meaning of Eqs. (1)–(4), it suffices for us to recall the definition of the so-called fluctuation analysis (FA), which is a key method for fractal time series analysis. To apply FA, one tacitly assumes the time series is like a noise, and constructs a random walk. To facilitate experimental data analysis, from now on, we shall work with sampled data $\phi_1, \phi_2, \ldots$. The random walk $y(n)$ is generated by simply forming partial summations of the ϕ_i time series (with mean $\bar{\phi}$ removed),

$$y(n) = \sum_{i=1}^{n} (\phi_i - \bar{\phi}). \tag{5}$$

One then examines whether the following scaling law holds or not:

$$F(m) = \langle |y(i+m) - y(i)|^2 \rangle \sim m^{2H}, \tag{6}$$

where $\langle \rangle$ denotes average. H, called the Hurst parameter, characterizes the correlation structure of the data. When $1/2 < H < 1$, the process y is said to have persistent correlations. When $H = 1/2$, y does not have or only has short-term memory. The representative case of this is the standard Brownian motion. Its increment process is the Gaussian white noise. When $0 < H < 1/2$, y has antipersistent correlations. Note that the power spectral density for x and y is $1/f^{2H-1}$ and $1/f^{2H+1}$, respectively.

It should be clear by now that if one interprets $p(t)$ of Eq. (2) as a random walk process, then $M(t)$ of Eq. (3) plays the same role as $F(m)$ of Eq. (6). Therefore K of Eq. (4) is equivalent to H normalized by $H = 0.5$, the case of white Gaussian noise (or equivalently, the standard Brownian motion). To find out whether there exists any difference between the random walk-type process of the 0-1 test for chaos and the usual random walk process of Eq. (5), we first study $1/f^\alpha$ noise by the 0-1 test.

III. ANALYSIS OF $1/f^\alpha$ NOISE WITH LONG-RANGE-CORRELATIONS

Of the types of activity that characterize complex systems, the most ubiquitous and puzzling is perhaps the appearance of $1/f^\alpha$ noise, a form of temporal or spatial fluctuation characterized by a power-law decaying power spectral density. Some of the older literatures on this subject can be found, for example, in Press [15], Bak [16], and Wornell [17]. Some of the more recently discovered $1/f^\alpha$ processes are in traffic engineering [18–20], DNA sequence [21–24], human cognition [25], coordination [26], posture [27], dynamic images [28,29], and the distribution of prime numbers [30].

The prototypical model for the $1/f^\alpha$ process is the fractional Brownian motion (fBm) process [31]. It is a Gaussian process with mean 0, stationary increments, variance

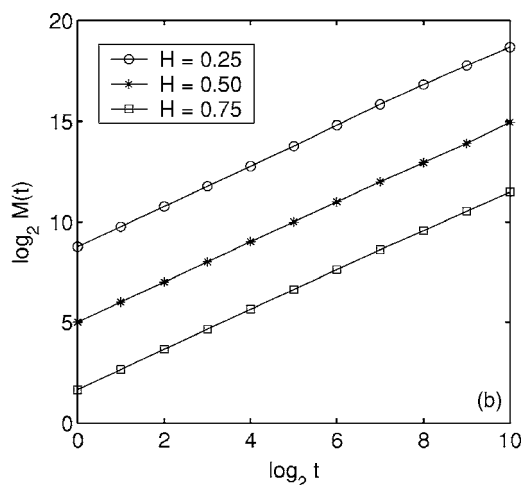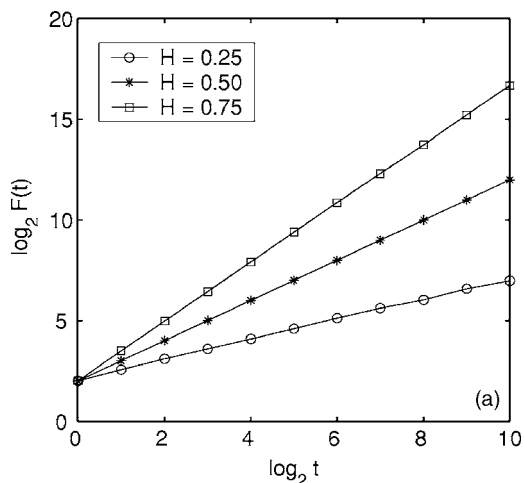$$E[(B_H(t))^2] = t^{2H}, \tag{7}$$

and covariance:

FIG. 1. (a) Fluctuation analysis of the fractional Gaussian noise with $H = 0.25$, 0.5, and 0.75; and (b) the 0-1 test yields $H = 0.5$ for all three kinds of noise.

$$E[B_H(s)B_H(t)] = \frac{1}{2}\{s^{2H} + t^{2H} - |s - t|^{2H}\}, \tag{8}$$

where H is the Hurst parameter. The increment process of the fBm, $X_i = B_H((i+1)\Delta t) - B_H(i\Delta t)$, $i \geq 1$, where Δt can be considered a sampling time, is called fractional Gaussian noise (fGn). It is a zero mean stationary Gaussian time series, with autocovariance function:

$$\gamma(k) = E(X_i X_{i+k})/E(X_i^2)$$

$$= \frac{1}{2}\{(k+1)^{2H} - 2k^{2H} + |k-1|^{2H}\}, \ k \geq 0. \tag{9}$$

Since $\gamma(k)$ is independent of Δt, without loss of generality, we can take $\Delta t = 1$. In particular, we have $\gamma(1) = \frac{1}{2}(2^{2H} - 2)$. The notions of persistent and antipersistent correlations come from the fact that $\gamma(1)$ is positive when $1/2 < H < 1$, but negative when $0 < H < 1/2$.

We now apply FA and the 0-1 test to analyze three fGn processes, with $H = 0.25$, 0.5, and 0.75. Figures 1(a) and 1(b) show the results of FA and 0-1 test for chaos, respectively.

As expected, FA is consistent with the defining Eq. (7). However, the 0-1 test for chaos simply gives $H=0.5$ (i.e., $K=1$), regardless of the value of H chosen to generate the fGn. Therefore we have to conclude that the 0-1 test for chaos constructs the random walk in such a way that long-term correlation in the original data is effectively eliminated. Using a commonly used terminology in electrical engineering, the construction resembles a whitening filter [32].

The above case study clearly indicates that there is a risk for the 0-1 test to interpret random data as deterministic chaos, if one assumes that the data comes from a deterministic system.

We have also applied the method to analyze data generated by parametric Langevin equations [33–37] and Levy motions [38–40], and have always obtained $K=1$ for the processes studied. Therefore we suspect that the method may always yield $K=1$ for all kinds of stochastic processes, independent of their defining properties (such as long-range-correlations, power-law-type tails, fractional dynamics [41–44], etc.). This implies that any kind of noise process may be interpreted as deterministic chaos by this method, if one assumes that the data come from deterministic systems.

IV. ANALYSIS OF EDGE OF CHAOS AND WEAK CHAOS

In the 1980s and early 1990s, researchers were very keen to find unambiguous evidence of deterministic chaos from apparently irregular experimental time series. A bit surprisingly, this effort was rarely fruitful. While current consensus is to attribute this fact to the noise in and the high-dimensionality of the data, another possibility is that the motion may not be simply regular, nor completely chaotic/random, but lies in between. Such a consideration motivates us to consider edge of chaos and weak chaos. Indeed, recent work of Tsallis and co-workers [45–49] has found that the dynamics of the edge of chaos is so rich that multifractal characterization is needed. In this section, we examine the deterministic and the noisy logistic and Henon map at and near the edge of chaos. Since the behaviors for the two model systems are the same, we shall only present the results for the logistic map:

$$x_{n+1} = ax_n(1-x_n) + \sigma\eta_n, \qquad (10)$$

where a is the bifurcation parameter and η_n is a white Gaussian noise with mean zero and unit variance. The parameter σ characterizes the strength of noise. For the clean system ($\sigma=0$), the edge of chaos occurs at the accumulation point, $a_\infty=3.569945672\ldots$. Besides studying a_∞, we also examine $a=a_\infty+0.001$, whose motion is weakly chaotic. Note that motions corresponding to these parameters have been examined by a new concept, power-law sensitivity to initial conditions [50].

To facilitate discussion below, we first explain a stringent dynamical test for low-dimensional chaos [9,10], which has found numerous applications in the study of the effects of noise on dynamical systems [51] and experimental time series [52]. Given a scalar time series, the test involves first reconstructing a phase space by forming vectors [53–55]:

$V_i=[x(i),x(i+L),\ldots,x(i+(m-1)L)]$, then properly choosing the embedding dimension m and the delay time L [56], and finally computing the $\Lambda(k)$ curves defined by

$$\Lambda(k) = \left\langle \ln\left(\frac{\|V_{i+k} - V_{j+k}\|}{\|V_i - V_j\|} \right) \right\rangle. \qquad (11)$$

The computation is carried out for a sequence of shells, $r_i \leqslant \|V_i - V_j\| \leqslant r_i + \Delta r_i$, where r_i and Δr_i are prescribed small distances (Δr_i is not necessarily a constant). The angle brackets denote the ensemble average of all possible (V_i, V_j) pairs, and k is called the evolution time. For true low-dimensional chaotic systems, the $\Lambda(k)$ curves for different shells form a common envelope, and the slope of the envelope accurately estimates the largest positive Lyapunov exponent. For random systems, the $\Lambda(k)$ curves corresponding to different shells do not form a common envelope, and hence the system under study cannot be interpreted as chaos [51]. For regular motions, $\Lambda(k)$ is very close to 0.

For the deterministic logistic map, the $\Lambda(k)$ curves for data corresponding to a_∞ and $a=a_\infty+0.001$ are shown in Figs. 2(a) and 2(b), respectively. The different curves in Fig. 2(b) correspond to shells of different sizes. We observe that for a_∞, the motion cannot be characterized as chaotic or regular, since $\Lambda(k)$ curves do not increase linearly to form a common envelope, and are not very close to 0. The data for $a=a_\infty+0.001$ is indeed chaotic. However, the chaos is weak, since the curves are much less smooth than those for well-developed chaos [51].

We now examine the clean data by the 0-1 test for chaos. The results are shown in Fig. 2(c), as cross and circle, respectively. We observe that in both cases, $K=0$. Therefore both types of motion are interpreted as regular. Here, one has to conclude that the 0-1 test for chaos fails to properly classify the motions.

Next we analyze the noisy logistic map with the 0-1 test. Since the difference between a_∞ and $a=a_\infty+0.001$ becomes unidentifiable when there is noise, we shall only consider the case of a_∞. When we choose the noise level $\sigma=0.001$, the variation of $M(t)$ vs t shown in Fig. 2(d) (as circle) remains very similar to that shown in Fig. 2(c). Hence the motion is again classified as regular. However, if we increase the noise level to $\sigma=0.01$, we observe the characteristic growth for a Brownian motion, as shown in Fig. 2(d), the solid line. Now, the motion would be classified as chaotic. Again, this is a misclassification. It is then clear that the 0-1 test cannot be used to study time series not simply regular nor fully chaotic.

V. CONCLUDING REMARKS

By studying three different types of data, (i) edge of chaos, (ii) weak chaos, and (iii) $1/f^\alpha$ noise with long-range correlations, we have shown that the 0-1 test for chaos misclassifies deterministic and weakly stochastic edge of chaos and weak chaos as regular motions, while strongly stochastic edge of chaos and weak chaos, as well as $1/f^\alpha$ noise, as deterministic chaos. We have to emphasize, however, that our negative results do not invalidate the 0-1 test as a proper mathematical test for distinguishing regular motion from
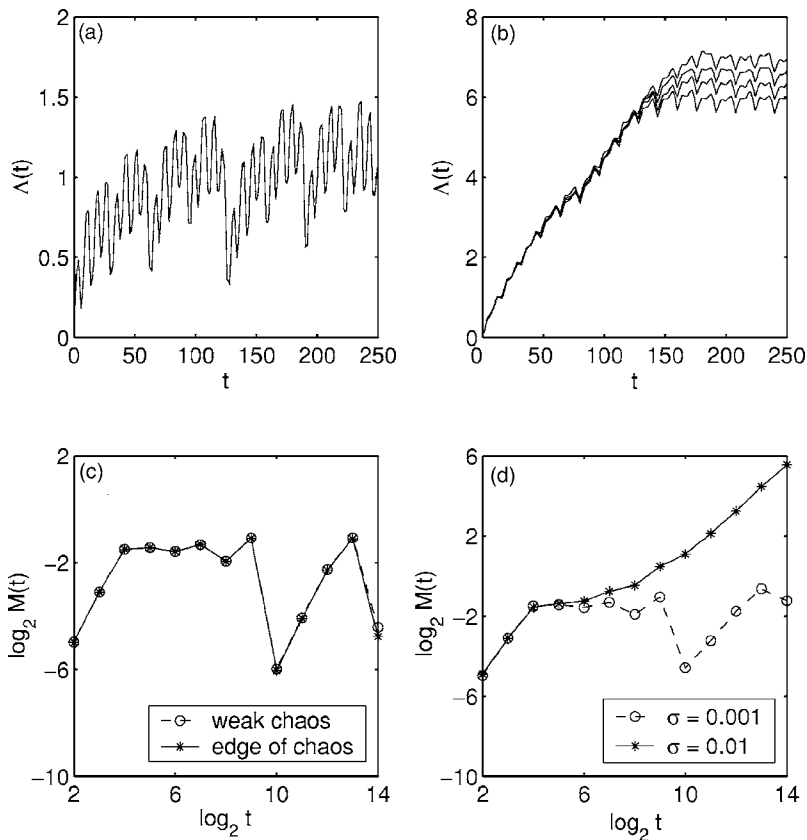
FIG. 2. (a) and (b) $\Lambda(t)$ vs t and (c) $\log_2 M(t)$ vs $\log_2 t$ for the edge of chaos and weak chaos of the logistic map without noise; and (d) $\log_2 M(t)$ vs $\log_2 t$ for the edge of chaos with different noise level.

fully developed chaotic motion. We believe that so long as the system under study is truly deterministic and the motion is far away from the boundary between chaos and regular motions, the test is valid.

It is pertinent to make a few speculative comments on the future of nonlinear time series analysis here. The continuing advances in the fields of life sciences, molecular biology, nano-sciences, and information systems are enabling the design and exploration of complex engineered and natural systems. Such systems comprise multiple subsystems that exhibit both highly nonlinear deterministic, as well as, stochastic characteristics. The Internet, for example, has been designed in a fundamentally decentralized fashion and consists of a complex web of servers and routers that cannot be controlled or analyzed by traditional tools of queuing theory or control theory, and gives rise to highly bursty and multi-scale traffic with extremely high variance [18–20], as well as complex dynamics with both deterministic and stochastic components [57,58]. Similarly, with our increasing capability to monitor and control biological activities, we have no choice but to deal with signals generated by systems that are by nature heterogeneous, massively distributed, and highly complicated. Straightforward application of deterministic chaos or random fractal theory often only gives us limited understanding of the behavior of the system. Solving the classic problem of distinguishing chaos from regular as well as stochastic motions, albeit important, may not shed much light on the complex multiscale dynamics of the system. Indeed, there exist ample examples of dynamical systems which exhibit chaoslike features on small scales but diffusive behavior on large scales [59]. When the signals to be modeled become increasingly multiscaled, chaos and random fractal theory will have to be integrated, since they may characterize different facets of the multiscale signals. To this end, it is most desirable that researchers in the chaos research community and random fractal research community can enhance communications. In some sense, categorical study on whether a signal is chaotic, regular, or random, discourages such cross talk, and hence should be paid with less attention. Instead, more efforts should be directed to develop effective methods to quantify as many different characteristics of a multiscale signal as possible.

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